Mathematics 1c, Spring 2008. Practice Midterm Examination

J. Marsden

Print Your Name:
Your Section:

- This exam has five questions.
- This exam should take about 3.5 to 4 hours to complete. The real midterm will be constructed so that students who are well versed in the material will require 3 hours. Thus, the real exam will be a 3 hour exam. There is no credit for overtime work.
- No aids (including notes, books, calculators etc.) are permitted.
- The real exam must be turned in by ***noon on Wednesday, May 7.***
- All 5 questions should be answered on this exam, using the backs of the sheets or appended pages if necessary.
- Show all your work and justify all claims using plain and proper English.
- Each question is worth 20 points.
- **Good Luck !!**

1	
2	
3	
4	
5	
	/100

1. Consider the matrix

$$A = \frac{1}{2}\begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix}$$

   (a) Is A diagonalizable?

   (b) Is A invertible?

   (c) Find the eigenvalues and eigenvectors of A

   (d) Compute $(A^{-1})^6$

2. Let A be a 3×3 orthogonal matrix.

   (a) Show that the determinant of A is ± 1.

   (b) Let λ be a real eigenvalue of A. Show that $\lambda = \pm 1$

   (c) Suppose that A is orthogonal and $\det A = 1$. Show that at least one eigenvalue of A equals 1. Must one of them equal -1 as well?

   (d) Give an example of a 3×3 orthogonal matrix that is not diagonalizable (as a real matrix).

3. Do each of the following calculations

   (a) Find the tangent vector to the curve

   $$\boldsymbol{\sigma}(t) = e^{1-t}\mathbf{i} - t^2\mathbf{j} + \sin(\pi t/2)\mathbf{k}$$

   at the point $t = 1$.

   (b) If a particle following the curve in (a) flies off on a tangent at $t = 1$, where is it at $t = 2$?

   (c) Find the gradient of the function

   $$f(x, y, z) = yz\sin(\pi x) - xyz$$

   at the point $(1, 1, 2)$.

   (d) Find the equation of the tangent plane to the graph of

   $$z = 3y^2 - x^3 + 2$$

   at the point on the graph with $x = 1$ and $y = -1$.

   (e) Calculate the gradient of the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = -3r^{-2}$, where $r = \sqrt{x^2 + y^2 + z^2}$. Verify that the gradient is orthogonal to the level sets of f.

4. Let $f(x, y, z)$ be a given (smooth) function defined on the whole of $\mathbb{R}^3$.

(a) Let $\boldsymbol{\sigma}(t) = t\,\mathbf{i} - e^t\,\mathbf{j} - t^3\,\mathbf{k}$. Write the chain rule for $f \circ \boldsymbol{\sigma}$ in gradient and component notation.

(b) If the gradient ∇f has a positive z component in the half space $z \geq 0$, must $f(1, 2, 1)$ be larger than $f(1, 2, 0)$? Must it be larger than $f(0, -1, 0)$? Prove or find a counter example.

(c) If $f(x, y, z) = x^2 - y^2 + z^4$, find the derivative of f in the direction of the vector $\mathbf{i} - \mathbf{j}$ at the point $(1, -1, 1)$.

(d) In what direction is the function in (c) increasing the fastest at $(1, -1, 1)$?

(e) In what directions is the function in (c) increasing at half its maximum rate at the point $(1, -1, 1)$?

5. This problem has three main parts.

   (a) Consider the two functions defined on $\mathbb{R}^3$ by

   $$U = x^2 - y^2 + \sin z \quad \text{and} \quad V = xy\cos(xz).$$

      i. Suppose that (x, y, z) are functions of new variables (u, v). Write out the chain rule for this situation, giving the derivatives of U and V as functions of (u, v) in matrix notation.

      ii. Consider the specific situation in which $x = u - v$, $y = u + v$ and $z = u$. Calculate the derivative matrix of the resulting map of (u, v) to (U, V) evaluated at $u = 1$, $v = 0$.

   (b) Find the extreme points of $f(x, y, z) = x + y + z$ subject to the *two constraints* $x^2 + y^2 = 5$ and $y + 2z = 3$.

   (c) Let $f(x, y) = x^2 + 3xy + y^2 + 16$. Calculate the eigenvalues of the matrix of second partial derivatives of f at the origin. Using this information alone, determine if the origin is a maximum, a minimum or a saddle point.