DISCRETE MECHANICS AND OPTIMAL CONTROL

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Abstract: A new approach to the solution of optimal control problems for mechanical systems is proposed. It is based on a direct discretization of the Lagrange-d'Alembert principle for the system (as opposed to using, for example, collocation or multiple shooting to enforce the equations of motion as constraints). The resulting forced discrete Euler-Lagrange equations then serve as constraints for the optimization of a given cost functional. We numerically illustrate the method by optimizing a low thrust satellite orbit transfer as well as the reconfiguration of a group of hovercraft in the plane.

Keywords: discrete mechanics, variational analysis, optimal control

1. INTRODUCTION

Consider the following optimal control problem: a mechanical system with configuration space Q is to be moved on a curve $q(t) \in Q$ during a time interval, say $[0, 1]$, from a state $(q^0, \dot{q}^0)$ to a state $(q^1, \dot{q}^1)$ under the influence of a force f chosen such that a given *cost functional*

$$J(q, f) = \int_0^1 C(q(t), \dot{q}(t), f(t))\, dt \qquad (1)$$

is minimized. At the same time, the motion $q(t)$ of the system is to satisfy the *Lagrange-d'Alembert principle*, which requires that

$$\delta \int_0^1 L(q(t), \dot{q}(t))\, dt + \int_0^1 f(t) \cdot \delta q(t)\, dt = 0 \quad (2)$$

for all variations δq with $\delta q(0) = \delta q(1) = 0$, where $L : TQ \to \mathbb{R}$ is the Lagrangian of the mechanical system.

Abstractly, one is faced with an equality constrained optimization problem: one seeks to minimize a functional

$$f \mapsto J(q, f), \qquad (3)$$

subject to a constraint of the form

$$\mathcal{L}(q, f) = 0. \qquad (4)$$

In this paper, we propose to exploit the variational structure directly, without first deriving the equations of motions (the Euler-Lagrange equations) for the system. This is in contrast to other methods like, e.g., *shooting* (Hicks and Ray, 1971; Kraft, 1985; Stoer and Bulirsch, 2002), *multiple shooting* (Deuflhard, 1974; Bock and Plitt, 1984; Leineweber *et al.*, 2003) or *collocation* methods (Biegler, 1984; von Stryk, 1993), which rely on a direct integration or a fulfillment at certain grid points of the associated ordinary differential equations (see also (Binder *et al.*, 2001) for an overview of the current state of the art). Using a global discretization of the states and the controls one directly obtains, via the *discrete Lagrange-d'Alembert principle*, equality constraints for the resulting finite dimensional nonlinear optimization problem, which is solved by standard methods like *sequential quadratic programming* (Schittkowski, 1980; Gill *et al.*, 1999; Gill *et al.*, 2001).

Advantages of the Method. A full analysis of the advantages of the proposed method over some of the standard approaches is currently under investigation. However, we expect several specific benefits and the examples indicate that this should be the case. For example, in variational integrators (Marsden and

West, 2001), one respects the energy of forced and damped systems much better than with standard algorithms. Correspondingly, we expect the energy budget of a control system to be more accurately computed using the presented metho, and this will be especially so for long duration simulations, such as for low thrust spacecraft missions. In addition, one expects the presented method to be more robust to modeling errors.

2. DISCRETIZATION

Following (Marsden and West, 2001), we replace the state space TQ of the system by $Q \times Q$ and a path $q : [0,1] \to Q$ by a *discrete path* $q_d : \{0, h, 2h, \ldots, Nh = 1\} \to Q$, $N \in \mathbb{N}$, where we view $q_k = q_d(kh)$ as an approximation to $q(kh)$. Analogously, the continuous force $f : [0,1] \to T^*Q$ is approximated by a discrete force $f_d : \{0, h, 2h, \ldots, Nh = 1\} \to T^*Q$ (writing $f_k = f_d(kh)$).

2.1 Discrete Lagrange-d'Alembert principle

Based on this discretization, the action integral in (2) is approximated on a time slice $[kh, (k+1)h]$ by a *discrete Lagrangian* $L_d : Q \times Q \to \mathbb{R}$,

$$L_d(q_k, q_{k+1}) \approx \int_{kh}^{(k+1)h} L(q(t), \dot{q}(t))\, dt,$$

and likewise the virtual work by an expression of the form

$$f_k^- \cdot \delta q_k + f_k^+ \cdot \delta q_{k+1} \approx \int_{kh}^{(k+1)h} f(t) \cdot \delta q(t)\, dt,$$

where $f_k^-, f_k^+ \in T^*Q$ will be called the *left* and *right discrete forces*, respectively.

The discrete version of the Lagrange-d'Alembert principle (2) requires one to find discrete paths $\{q_k\}_{k=0}^N$ such that for all variations $\{\delta q_k\}_{k=0}^N$ with $\delta q_0 = \delta q_N = 0$, one has

$$\delta \sum_{k=0}^{N-1} L_d(q_k, q_{k+1}) + \sum_{k=0}^{N-1} f_k^- \cdot \delta q_k + f_k^+ \cdot \delta q_{k+1} = 0. \tag{5}$$

The discrete Lagrange-d'Alembert principle is equivalent to the system

$$D_2 L_d(q_{k-1}, q_k) + D_1 L_d(q_k, q_{k+1}) + f_{k-1}^+ + f_k^- = 0, \tag{6}$$

where $k = 1, \ldots, N-1$, called the *forced discrete Euler-Lagrange equations*.

2.2 Discrete Cost Function

We approximate the cost functional (1) on the time slice $[kh, (k+1)h]$

$$C_d(q_k, q_{k+1}, f_k, f_{k+1}) \approx \int_{kh}^{(k+1)h} C(q, \dot{q}, f)\, dt,$$

yielding the *discrete cost functional*

$$J_d(q_d, f_d) = \sum_{k=0}^{N-1} C_d(q_k, q_{k+1}, f_k, f_{k+1}).$$

2.3 Boundary Conditions

Finally, one needs to incorporate the boundary conditions $q(0) = q^0$, $\dot{q}(0) = \dot{q}^0$ and $q(1) = q^1$, $\dot{q}(1) = \dot{q}^1$ into the discrete description. To this end, the description in $Q \times Q$ is linked to one in TQ using the *discrete Legendre transforms* $\mathbb{F}^{f+}L_d : Q \times Q \to T^*Q$ and $\mathbb{F}^{f-}L_d : Q \times Q \to T^*Q$ for forced systems, defined as follows:

$$\begin{aligned} &\mathbb{F}^{f+}L_d : (q_{k-1}, q_k) \mapsto (q_k, p_k), \\ &\quad p_k = D_2 L_d(q_{k-1}, q_k) + f_{k-1}^+ \quad \text{and} \\ &\mathbb{F}^{f-}L_d : (q_{k-1}, q_k) \mapsto (q_{k-1}, p_{k-1}), \\ &\quad p_{k-1} = -D_1 L_d(q_{k-1}, q_k) - f_{k-1}^-. \end{aligned}$$

Using the *standard Legendre transform* $\mathbb{F}L : TQ \to T^*Q$

$$\mathbb{F}L : (q, \dot{q}) \mapsto (q, p) = (q, D_2 L(q, \dot{q})),$$

this leads to the two *discrete boundary conditions*

$$\begin{aligned} D_2 L(q_0, \dot{q}_0) + D_1 L_d(q_0, q_1) + f_0^- &= 0, \\ -D_2 L(q_N, \dot{q}_N) + D_2 L_d(q_{N-1}, q_N) + f_{N-1}^+ &= 0. \end{aligned}$$

2.4 The Discrete Constrained Optimization Problem

To summarize, after performing the above discretization steps, one is faced with the following equality constrained nonlinear optimization problem: Minimize

$$J_d(q_d, f_d) = \sum_{k=0}^{N-1} C_d(q_k, q_{k+1}, f_k, f_{k+1})$$

with respect to f_d, subject to the constraints $q_0 = q^0$, $q_N = q^1$ and

$$\begin{aligned} D_2 L(q_0, \dot{q}_0) + D_1 L_d(q_0, q_1) + f_0^- &= 0, \\ D_2 L_d(q_{k-1}, q_k) + D_1 L_d(q_k, q_{k+1}) + f_{k-1}^+ + f_k^- &= 0, \\ -D_2 L(q_N, \dot{q}_N) + D_2 L_d(q_{N-1}, q_N) + f_{N-1}^+ &= 0, \end{aligned}$$

$k = 1, \ldots, N-1$.

2.5 Implementation

In the example computation in the following section we employ the Midpoint Rule for approximating the relevant integrals, i.e., we set

$$\begin{aligned} &C_d(q_k, q_{k+1}, f_k, f_{k+1}) \\ &\quad = hC\left(\tfrac{q_{k+1}+q_k}{2}, \tfrac{q_{k+1}-q_k}{h}, \tfrac{f_{k+1}+f_k}{2}\right), \end{aligned}$$

$$L_d(q_k, q_{k+1}) = hL\left(\tfrac{q_{k+1}+q_k}{2}, \tfrac{q_{k+1}-q_k}{h}\right),$$

as well as

$$\int_{kh}^{(k+1)h} f(t)\cdot\delta q(t)\,dt \approx h\frac{f_{k+1}+f_k}{2}\cdot\frac{\delta q_{k+1}+\delta q_k}{2}$$
$$=\frac{h}{4}(f_{k+1}+f_k)\cdot\delta q_k+\frac{h}{4}(f_{k+1}+f_k)\cdot\delta q_{k+1},$$

i.e., $f_k^- = f_k^+ = \frac{h}{4}(f_{k+1}+f_k)$ were used as the left and right discrete forces.

3. LOW THRUST ORBITAL TRANSFER

Since we expect our approach to be particularly useful for systems which are nearly conservative, we investigate as a first example the problem of optimally transferring a satellite with a continuously acting propulsion system from one circular orbit around the Earth to another one.

3.1 The Problem

Consider a satellite with mass m which moves in the gravitational field of the Earth (mass M). The satellite is to be transferred from one circular orbit to one in the same plane with a larger radius, while the number of revolutions around the Earth is fixed. In 2d-polar coordinates $q=(r,\varphi)$, the Lagrangian of the system has the form

$$L(q,\dot q)=\frac{1}{2}m(\dot r^2+r^2\dot\varphi^2)+\gamma\frac{Mm}{r}.$$

Assume that the propulsion system continuously exhibits a force u in the direction of motion of the satellite, so that the corresponding term in (2) is given by

$$f=\begin{pmatrix}0\\ r\,u\end{pmatrix}.$$

Assume further that the satellite initially moves on a circular orbit of radius r_0. Let $(r(0),\varphi(0))=(r_0,0)$ be its position at $t=0$, then its initial velocity is given by $\dot r(0)=0$ and $\dot\varphi(0)=\sqrt{\gamma M/r_0^3}$. Using its thruster, the satellite is required to reach the point $(r_1,0)$ at time $T=p(T_0+T_1)/2$ and, without any further control input, continue to move on the circle with radius r_1. Here p is a prescribed number of revolutions around the Earth and T_0 and T_1 are the orbital periods of the initial and the final circle, respectively. Thus, the boundary values at $t=T$ are given by $(r(T),\varphi(T))=(r_1,0)$ and $(\dot r(T),\dot\varphi(T))=(0,\sqrt{\gamma M/r_1^3})$.

During this transfer, our goal is to minimize the control effort, correspondingly the cost function is given by

$$J(q,u)=\int_0^T u(t)^2\,dt.$$

The computations were performed with the following parameter values:

$$m=100,\ M=6\cdot10^{24},$$
$$\gamma=6.673\cdot10^{-26},\ r_0=5,\ r_1=6,$$
$$T_0=2\pi\sqrt{r_0^3/(\gamma M)},\ T_1=2\pi\sqrt{r_1^3/(\gamma M)}.$$

3.2 Results

We compare our method to a simple finite difference approach, where the dynamical constraints are discretized by applying a one-step method to the associated ordinary differential equations of the system (i.e. the forced Euler-Lagrange equations). For demonstration purposes Euler's scheme is used, in which case the constraints read

$$x_{k+1}-x_k-h\,F(x_k)=0,\quad k=0,\dots,N-1,$$

(where $x_k=(q_k,\dot q_k)$ and F denotes the vector field of the forced Euler-Lagrange equations), as well as the Midpoint Rule, yielding the constraints

$$x_{k+1}-x_k-h\,F\left(\frac{x_{k+1}+x_k}{2}\right)=0,$$

$k=0,\dots,N-1$.

We consider $p=1$ and $p=2$ revolutions around the Earth and solve the problem for various N. In Figure 1 the dependence of the resulting cost on N for all methods as well as for $p=1$ (top) and $p=2$ (bottom) is shown. It is intriguing to see that the cost is almost constant for the variational method, even for very large stepsizes, whereas the cost of the Euler-based method seems to converge towards this "benchmark value" for larger N. The Midpoint Rule performs, as one might have expected, almost equally well as the variational discretization. These results are consistent with the well known conservation properties of variational integrators. However, note that for the variational discretization the resulting optimiziation problem is only half as large as for the finite difference schemes.

As a second numerical test, we investigate how well the computed open loop control performs for "the real solution". To this end, the forced Euler-Lagrange equations were integrated using the classical fourth order Runge-Kutta scheme with very small constant step size $h=10^{-3}$, interpolating the computed control values by a cubic spline. In Figure 2 the deviation of the resulting final state (at $t=T$) from the requested one is shown.

4. A GROUP OF HOVERCRAFT

As a more demanding application, we consider a group of (identical) hovercraft which, starting from an arbitrary initial state, are required to attain a given final *formation* with minimal control effort. The final formation is defined by a set of equality constraints on the final configurations of the individual hovercraft and a fixed final velocity.

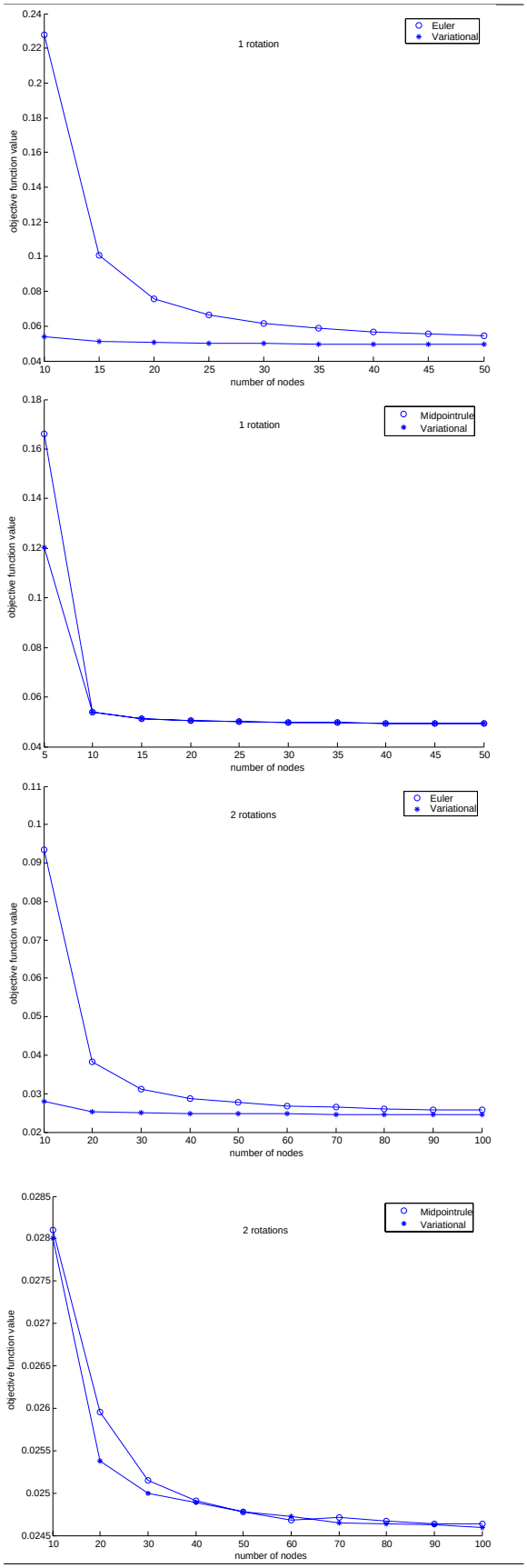

Fig. 1. Variational approach vs. a finite difference based discretization (Euler's scheme and Midpoint Rule): approximated cost of the orbital transfer in dependence of the number N of discretization points for one (top) and two (bottom) revolutions around the Earth.

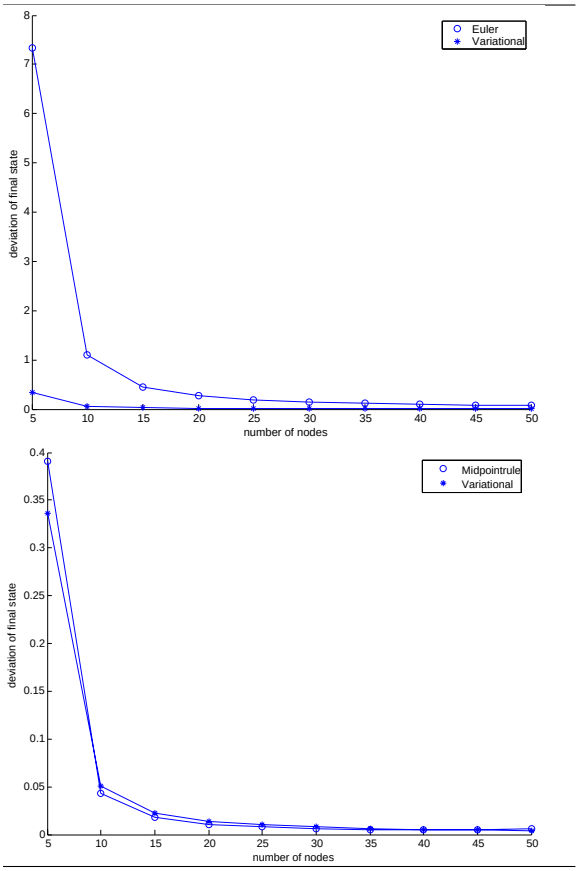

Fig. 2. Comparison of the accuracy of the computed open loop control for the variational, the Euler-based approach and the Midpoint Rule: Deviation of the actual final state of the satellite from the requested one in dependence of the number of discretization points.

4.1 The Problem

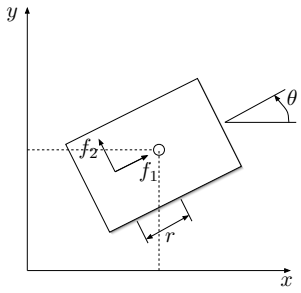

Fig. 3. Hovercraft

The configuration of a hovercraft is described by three degrees of freedom: its position $(x, y) \in \mathbb{R}^2$ and its orientation $\theta \in S^1$, i.e., its configuration manifold is $Q = \mathbb{R}^2 \times S^1$. It is actuated by two control forces f_1 and f_2, applied at a distance r from the center of mass (cf. Figure 3). The force f_1 acts in the direction of motion of the body, while f_2 acts orthogonally to it. The system is underactuated, but still configuration controllable (Ober-Blöbaum, 2004), i.e. each point in configuration space can be reached by applying suitably chosen forces $f_1(t)$ and $f_2(t)$.

The Lagrangian of the system consists only of its kinetic energy,

$$L(q,\dot{q}) = \frac{1}{2}(m\dot{x}^2 + m\dot{y}^2 + J\dot{\theta}^2),$$

where $q = (x, y, \theta)$, m is the mass of the hovercraft and J its moment of inertia. The forces acting in x-, y- and θ-direction resulting from f_1 and f_2 are

$$f(t) = \begin{pmatrix} \cos\theta(t) f_1(t) - \sin\theta(t) f_2(t) \\ \sin\theta(t) f_1(t) + \cos\theta(t) f_2(t) \\ -r f_2(t) \end{pmatrix}.$$

The resulting forced discrete Euler-Lagrange equations read

$$\begin{aligned} &\frac{1}{h} M\,(-q_{k-1} + 2q_k - q_{k+1}) + \\ &\quad \frac{h}{2}\left(\frac{f_{k-1} + f_k}{2} + \frac{f_k + f_{k+1}}{2}\right) = 0, \qquad (7) \\ &k = 1, ..., N-1, \end{aligned}$$

where $M = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J \end{pmatrix}$. We denote by $q^i = (x^i, y^i, \theta^i)$ the configuration of the i-th hovercraft and by $f^i = (f_1^i, f_2^i)$ the corresponding forces.

The goal is to minimize the control effort while, at the same time, attaining the desired final formation. As a suitable cost function for each hovercraft we again choose a measure of the control effort

$$J(q^i, f^i) = \int_0^1 \left(f_1^i(t)\right)^2 + \left(f_2^i(t)\right)^2 \, dt, \qquad (8)$$

while the cost function for the entire group is given by the sum of these.

The final configuration of the group has certain degrees of freedom: In the case of more than one hovercraft the final formation has an overall rotational degree of freedom; it is determined by the following conditions:

(a) a fixed final orientation φ^i of each hovercraft:

$$\theta^i(1) = \varphi^i, \quad i = 1, 2, 3,$$

(b) equal distances r between the final positions:

$$(x^i(1) - x^j(1))^2 + (y^i(1) - y^j(1))^2 = r^2,$$

$1 \le i, j \le 3, i \ne j$;

(c) the center $M = (M_x, M_y)$ of the formation is prescribed:

$$\begin{aligned} (x^1(1) + x^2(1) + x^3(1))/3 &= M_x, \\ (y^1(1) + y^2(1) + y^3(1))/3 &= M_y, \end{aligned}$$

(d) fixed final velocities:

$$\dot{x}^i = v_x^i,\ \dot{y}^i = v_y^i,\ \dot{\theta}^i = v_\theta^i, \quad i = 1, 2, 3,$$

where v_x^i, v_y^i and v_θ^i are given.

The boundary conditions for the case of a group of six hovercraft are determined analogously, i.e., the craft are required to form a regular hexagon.

5. CONCLUSION

This paper proposes a new approach to the solution of optimal control problems for mechanical systems: The given cost functional is extremized subject to constraints defined via the discrete Lagrange-d'Alembert principle. The numerical investigations support the conjecture that this approach has some numerical advantages, in particular for systems which are nearly conservative.

When applying a local solver to the resulting finite dimensional constrained optimization problem like, e.g., an SQP-method, one will typically only find a local (and not the global) optimum. As with any other discretization scheme, the extremum found will strongly depend on the initial guess that is provided to the solver. This phenomenon is particularly important in the hovercraft example, where the ordering of the craft in the final configuration is not prescribed and one would actually be interested in the global optimum. These issues will be addressed in future investigations.

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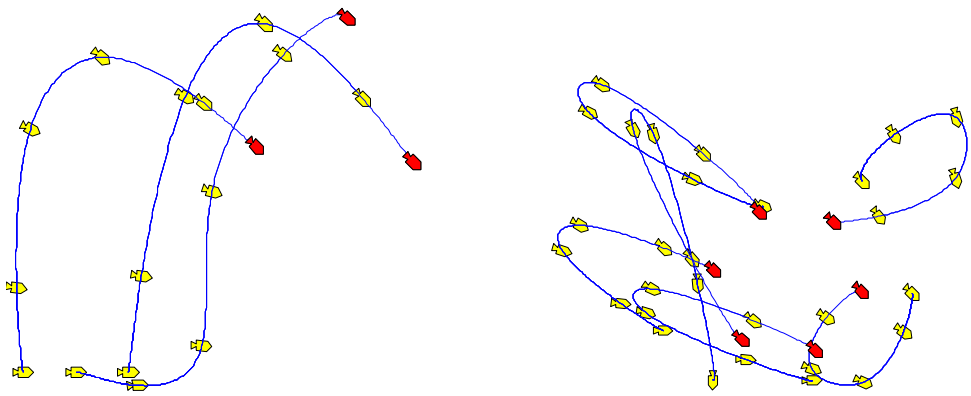

Fig. 4. Left: Optimal rearrangement of a group of three hovercraft from an initial configuration along a line into a triangle; Right: Optimal rearrangement of a group of six hovercraft from a random initial configuration into a hexagon.

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