Infinite–Dimensional Symmetry Groups

Peter J. Olver

University of Minnesota

http://www.math.umn.edu/~olver

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Sur la théorie, si importante sans doute, mais pour nous si obscure, des «groupes de Lie infinis», nous ne savons rien que ce qui trouve dans les mémoires de Cartan, première exploration à travers une jungle presque impénétrable; mais celle-ci menace de se refermer sur les sentiers déjà tracés, si l'on ne procède bientôt à un indispensable travail de défriissement.

— André Weil, 1947
What’s the Deal with Infinite–Dimensional Groups?

• Lie invented Lie groups to study symmetry and solution of differential equations.

◊ In Lie’s time, there were no abstract Lie groups. All groups were realized by their action on a space.

♦ Therefore, Lie saw no essential distinction between finite-dimensional and infinite-dimensional group actions.

However, with the advent of abstract Lie groups, the two subjects have gone in radically different directions.

❤ The general theory of finite-dimensional Lie groups has been rigorously formalized and applied.

♣ But there is still no generally accepted abstract object that represents an infinite-dimensional Lie pseudo-group!
Ehresmann’s Trinity

1953:
Ehresmann’s Trinity

1953:

• Lie Pseudo-groups
Ehresmann’s Trinity

1953:

- Lie Pseudo-groups
- Jets
Ehresmann’s Trinity

1953:

• Lie Pseudo-groups
• Jets
• Groupoids
Lie Pseudo-groups in Action

- Lie — Medolaghi — Vessiot
- Cartan
- Ehresmann
- Kuranishi, Spencer, Goldschmidt, Guillemin, Sternberg, Kumpera, ...

- Relativity
- Noether’s (Second) Theorem
• Gauge theory and field theories:
  Maxwell, Yang–Mills, conformal, string, . . .
• Fluid mechanics, metereology: Euler, Navier–Stokes, boundary layer, quasi-geostropic, . . .
• Solitons (in 2 + 1 dimensions):
  K–P, Davey-Stewartson, . . .
• Kac–Moody
• Linear and linearizable PDEs
• *Lie groups*!
Moving Frames

In collaboration with Juha Pohjanpelto and Jeongoo Cheh, I have recently established a moving frame theory for infinite-dimensional Lie pseudo-groups mimicking the earlier equivariant approach for finite-dimensional Lie groups developed with Mark Fels and others.

The finite-dimensional theory and algorithms have had a very wide range of significant applications, including differential geometry, differential equations, calculus of variations, computer vision, Poisson geometry and solitons, numerical methods, relativity, classical invariant theory, . . .
What’s New?

In the infinite-dimensional case, the moving frame approach provides new constructive algorithms for:

- Invariant Maurer–Cartan forms
- Structure equations
- Moving frames
- Differential invariants
- Invariant differential operators
- Basis Theorem
- Syzygies and recurrence formulae
• Further applications:

⇒ Symmetry groups of differential equations
⇒ Vessiot group splitting; explicit solutions
⇒ Gauge theories
⇒ Calculus of variations
⇒ Numerical methods
Symmetry Groups — Review

System of differential equations:

$$\Delta_\nu(x, u^{(n)}) = 0, \quad \nu = 1, 2, \ldots, k$$

By a symmetry, we mean a transformation that maps solutions to solutions.

**Lie:** To find the symmetry group of the differential equations, work infinitesimally.

The vector field

$$v = \sum_{i=1}^{p} \xi^i(x, u) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x, u) \frac{\partial}{\partial u^\alpha}$$

is an infinitesimal symmetry if its flow $\exp(t \, v)$ is a one-parameter symmetry group of the differential equation.
To find the infinitesimal symmetry conditions, we prolong $\mathbf{v}$
to the jet space whose coordinates are the derivatives appearing
in the differential equation:

$$
\mathbf{v}^{(n)} = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{\# J=0}^{n} \varphi^J_{\alpha} \frac{\partial}{\partial u^\alpha_J}
$$

where

$$
\varphi^J_{\alpha} = D_J \left( \varphi^\alpha - \sum_{i=1}^{p} u_i^\alpha \xi^i \right) + \sum_{i=1}^{p} u^\alpha_{J,i} \xi^i
$$

$$
\equiv \Phi^J_{\alpha}(x, u^{(n)}; \xi^{(n)}, \varphi^{(n)})
$$

Infinitesimal invariance criterion:

$$
\mathbf{v}^{(n)}(\Delta_{\nu}) = 0 \quad \text{whenever} \quad \Delta = 0.
$$

Infinitesimal determining equations:

$$
\mathcal{L}(x, u; \xi^{(n)}, \varphi^{(n)}) = 0
$$
The Korteweg–deVries equation

\[ u_t + u_{xxx} + uu_x = 0 \]

Symmetry generator:

\[ \mathbf{v} = \tau(t, x, u) \frac{\partial}{\partial t} + \xi(t, x, u) \frac{\partial}{\partial x} + \varphi(t, x, u) \frac{\partial}{\partial u} \]

Prolongation:

\[ \mathbf{v}^{(3)} = \mathbf{v} + \varphi^t \frac{\partial}{\partial u_t} + \varphi^x \frac{\partial}{\partial u_x} + \cdots + \varphi^{xxx} \frac{\partial}{\partial u_{xxx}} \]

where

\[ \varphi^t = \varphi_t + u_t \varphi_u - u_t \tau_t - u_t^2 \tau_u - u_x \xi_t - u_t u_x \xi_u \]

\[ \varphi^x = \varphi_x + u_x \varphi_u - u_t \tau_x - u_t u_x \tau_u - u_x \xi_x - u_x^2 \xi_u \]

\[ \varphi^{xxx} = \varphi_{xxx} + 3 u_x \varphi_u + \cdots \]
Infinitesimal invariance:

\[ v^{(3)}(u_t + u_{xxx} + uu_x) = \varphi_t + \varphi^{xxx} + u\varphi^x + u_x \varphi = 0 \]
on solutions

Infinitesimal determining equations:

\[ \tau_x = \tau_u = \xi_u = \varphi_t = \varphi_x = 0 \]
\[ \varphi = \xi_t - \frac{2}{3} u \tau_t \quad \varphi_u = -\frac{2}{3} \tau_t = -2 \xi_x \]
\[ \tau_{tt} = \tau_{tx} = \tau_{xx} = \cdots = \varphi_{uu} = 0 \]

General solution:

\[ \tau = c_1 + 3c_4 t, \quad \xi = c_2 + c_3 t + c_4 x, \quad \varphi = c_3 - 2c_4 u. \]
Basis for symmetry algebra $\mathfrak{g}_{KdV}$:

\[
\begin{align*}
\mathbf{v}_1 &= \partial_t, \\
\mathbf{v}_2 &= \partial_x, \\
\mathbf{v}_3 &= t \partial_x + \partial_u, \\
\mathbf{v}_4 &= 3t \partial_t + x \partial_x - 2u \partial_u.
\end{align*}
\]

The symmetry group $\mathcal{G}_{KdV}$ is four-dimensional

\[
(x, t, u) \longmapsto (\lambda^3 t + a, \lambda x + ct + b, \lambda^{-2} u + c)
\]
\[ \mathbf{v}_1 = \partial_t, \quad \mathbf{v}_2 = \partial_x, \]
\[ \mathbf{v}_3 = t \partial_x + \partial_u, \quad \mathbf{v}_4 = 3t \partial_t + x \partial_x - 2u \partial_u. \]

Commutator table:

\[
\begin{array}{cccc}
& \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \\
\hline
\mathbf{v}_1 & 0 & 0 & 0 & \mathbf{v}_1 \\
\mathbf{v}_2 & 0 & 0 & \mathbf{v}_1 & 3 \mathbf{v}_2 \\
\mathbf{v}_3 & 0 & -\mathbf{v}_1 & 0 & -2 \mathbf{v}_3 \\
\mathbf{v}_3 & -\mathbf{v}_1 & -3 \mathbf{v}_2 & 2 \mathbf{v}_3 & 0 \\
\end{array}
\]

Entries: \[ [\mathbf{v}_i, \mathbf{v}_j] = \sum_k C_{ij}^k \mathbf{v}_k \quad C_{ij}^k \quad \text{structure constants of } \mathfrak{g} \]
Navier–Stokes Equations

\[
\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p + \nu \Delta u, \quad \nabla \cdot u = 0.
\]

Symmetry generators:

\[
v_\alpha = \alpha(t) \cdot \partial_x + \alpha'(t) \cdot \partial_u - \alpha''(t) \cdot x \partial_p
\]

\[
v_0 = \partial_t
\]

\[
s = x \cdot \partial_x + 2t \partial_t - u \cdot \partial_u - 2p \partial_p
\]

\[
r = x \wedge \partial_x + u \wedge \partial_u
\]

\[
w_h = h(t) \partial_p
\]
Kadomtsev–Petviashvili (KP) Equation

\[
( u_t + \frac{3}{2} u u_x + \frac{1}{4} u_{xxx} )_x \pm \frac{3}{4} u_{yy} = 0
\]

Symmetry generators:

\[
v_f = f(t) \partial_t + \frac{2}{3} y f'(t) \partial_y + \left( \frac{1}{3} x f'(t) + \frac{2}{9} y^2 f''(t) \right) \partial_x
\]

\[
+ \left( -\frac{2}{3} u f'(t) + \frac{2}{9} x f''(t) + \frac{4}{27} y^2 f'''(t) \right) \partial_u,
\]

\[
w_g = g(t) \partial_y + \frac{2}{3} y g'(t) \partial_x + \frac{4}{9} y g''(t) \partial_u,
\]

\[
z_h = h(t) \partial_x + \frac{2}{3} h'(t) \partial_u.
\]

\[\Rightarrow \] Kac–Moody loop algebra \( A_4^{(1)} \)
Main Goals

Given a system of partial differential equations:

• Find the structure of its symmetry (pseudo-) group.
• Find a discussion of its invariance.
• Use symmetry reduction or group splitting to construct explicit solutions — both invariant and non-invariant.
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• Find and classify its differential invariants.

• Use symmetry reduction or group splitting to construct explicit solutions.
Pseudo-groups

$$M \quad \text{— smooth (analytic) manifold}$$

**Definition.** A pseudo-group is a collection of local diffeomorphisms $\varphi : M \to M$ such that

- **Identity:** $1_M \in \mathcal{G}$,
- **Inverses:** $\varphi^{-1} \in \mathcal{G}$,
- **Restriction:** $U \subset \text{dom } \varphi \implies \varphi \mid U \in \mathcal{G}$,
- **Continuation:** $\text{dom } \varphi = \bigcup U_\kappa$ and $\varphi \mid U_\kappa \in \mathcal{G} \implies \varphi \in \mathcal{G}$,
- **Composition:** $\text{im } \varphi \subset \text{dom } \psi \implies \psi \circ \varphi \in \mathcal{G}$. 
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- **Composition:** $\text{im} \varphi \subset \text{dom} \psi \implies \psi \circ \varphi \in \mathcal{G}$.

$\implies$ small category with inverses.
Lie Pseudo-groups

Definition. A Lie pseudo-group $\mathcal{G}$ is a pseudo-group whose transformations are the solutions to an involutive system of partial differential equations:

$$F(z, \varphi^{(n)}) = 0.$$ 

called the nonlinear determining equations.

$$\implies \text{analytic (Cartan–Kähler)}$$

★ ★ Key complication: $\nexists$ abstract object $\mathcal{G}$ ★ ★
A Non-Lie Pseudo-group

Acting on $M = \mathbb{R}^2$:

$$X = \varphi(x) \quad Y = \varphi(y)$$

where $\varphi \in \mathcal{D}(\mathbb{R})$ is any local diffeomorphism.

♠ Cannot be characterized by a system of partial differential equations

$$\Delta(x, y, X^{(n)}, Y^{(n)}) = 0$$
**Theorem.** (Johnson, Itskov) Any non-Lie pseudo-group can be completed to a Lie pseudo-group with the same differential invariants.

Completion of previous example:

\[ X = \varphi(x), \quad Y = \psi(y) \]

where \( \varphi, \psi \in D(\mathbb{R}) \).
Infi nitesimal Generators

\[ \mathfrak{g} \] — Lie algebra of infi nitesimal generators of the pseudo-group \( \mathcal{G} \)

\[ z = (x, u) \] — local coordinates on \( M \)

Vector field:

\[ \mathbf{v} = \sum_{a=1}^{m} \zeta^a(z) \frac{\partial}{\partial z^a} = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha \frac{\partial}{\partial u^\alpha} \]

Vector field jet:

\[ j_n \mathbf{v} \mapsto \zeta^{(n)} = ( \ldots \zeta_b^A \ldots ) \]

\[ \zeta_b^A = \frac{\partial^k \zeta^b}{\partial z_a^1 \cdots \partial z_a^k} \]
The infinitesimal generators of $\mathcal{G}$ are the solutions to the

*Infinitesimal (Linearized) Determining Equations*

\[ \mathcal{L}(z, \zeta^{(n)}) = 0 \] (*

Remark: If $\mathcal{G}$ is the symmetry group of a system of differential equations $\Delta(x, u^{(n)}) = 0$, then (*) is the (involutive completion of) the usual Lie determining equations for the symmetry group.
The Diffeomorphism Pseudo-group

\[ M \quad \text{— smooth } m\text{-dimensional manifold} \]

\[ \mathcal{D} = \mathcal{D}(M) \quad \text{— pseudo-group of all local diffeomorphisms} \]

\[ Z = \varphi(z) \]

\[ \begin{cases} 
  z = (z^1, \ldots, z^m) \quad \text{— source coordinates} \\
  Z = (Z^1, \ldots, Z^m) \quad \text{— target coordinates}
\end{cases} \]
Jets

Jets are a fancy name for Taylor polynomials/series.

For $0 \leq n \leq \infty$:

Given a smooth map $\varphi : M \to M$, written in local coordinates as $Z = \varphi(z)$, let $j_n \varphi|_z$ denote its $n$–jet at $z \in M$, i.e., its $n^{th}$ order Taylor polynomial or series based at $z$.

$J^n(M, M)$ is the $n^{th}$ order jet bundle, whose points are the jets.

Local coordinates on $J^n(M, M)$:

$$(z, Z^{(n)}) = (\ldots z^a \ldots Z^b_A \ldots), \quad Z^b_A = \frac{\partial^k Z^b}{\partial z^{a_1} \ldots \partial z^{a_k}}$$
Diffeomorphism Jets

The $n^{th}$ order diffeomorphism jet bundle is the subbundle
\[ \mathcal{D}^{(n)} = \mathcal{D}^{(n)}(M) \subset J^n(M, M) \]
consisting of $n^{th}$ order jets of local diffeomorphisms $\varphi : M \to M$.

The Inverse Function Theorem tells us it is defined by the non-vanishing of the Jacobian determinant:
\[ \det( Z^a_b ) = \det( \partial Z^a / \partial z^b ) \neq 0 \]

A Lie pseudo-group $\mathcal{G} \subset \mathcal{D}$ defines the subbundle
\[ \mathcal{G}^{(n)} = \{ F(z, Z^{(n)}) = 0 \} \subset \mathcal{D}^{(n)} \]
consisting of the jets of pseudo-group diffeomorphisms, and therefore characterized by the pseudo-group’s nonlinear determining equations.
\[ G^{(n)} = \{ F(z, Z^{(n)}) = 0 \} \subset D^{(n)} \]

♥ Local coordinates on \( G^{(n)} \), e.g., the restricted diffeomorphism jet coordinates \( z^c, Z^a_B \), are viewed as the pseudo-group parameters, playing the same role as the local coordinates on a Lie group \( G \).

♠ The pseudo-group jet bundle \( G^{(n)} \) does not form a group, but rather a groupoid.
Groupoid Structure

Double fibration:

\[ \begin{align*}
\sigma^{(n)}(z, Z^{(n)}) &= z & \text{— source map} \\
\tau^{(n)}(z, Z^{(n)}) &= Z & \text{— target map}
\end{align*} \]

You are only allowed to multiply \( h^{(n)} \cdot g^{(n)} \) if

\[ \sigma^{(n)}(h^{(n)}) = \tau^{(n)}(g^{(n)}) \]

\( \star \) Composition of Taylor polynomials/series is well-defined only when the source of the second matches the target of the first.
One-dimensional case: \( M = \mathbb{R} \)

Source coordinate: \( x \) \hspace{1cm} Target coordinate: \( X \)

Local coordinates on \( D^n(\mathbb{R}) \)

\[ g^{(n)} = (x, X, X_x, X_{xx}, X_{xxx}, \ldots, X_n) \]

Jet:

\[ X[h] = X + X_x h + \frac{1}{2} X_{xx} h^2 + \frac{1}{6} X_{xxx} h^3 + \cdots \]

\[ \Rightarrow \text{ Taylor polynomial/series at a source point } x \]
Groupoid multiplication of diffeomorphism jets:

\[(X, X, X_X, X_{XX}, \ldots) \cdot (x, X, X_x, X_{xx}, \ldots)\]

\[= (x, X, X_X X_x, X_X X_{xx} + X_{XX} X_x^2, \ldots)\]

\[\implies \text{Composition of Taylor polynomials/series}\]

The higher order terms are expressed in terms of Bell polynomials according to the general Fàa–di–Bruno formula.

- The groupoid multiplication (or Taylor composition) is only defined when the source coordinate \(X\) of the first multiplicand matches the target coordinate \(X\) of the second.
Structure of Lie Pseudo-groups

The structure of a finite-dimensional Lie group $G$ is specified by its Maurer–Cartan forms — a basis $\mu^1, \ldots, \mu^r$ for the right-invariant one-forms:

$$d\mu^k = \sum_{i<j} C^k_{ij} \mu^i \wedge \mu^j$$
What should be the Maurer–Cartan forms of a Lie pseudo-group?
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**Cartan**: Use exterior differential systems and prolongation to determine the structure equations.
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What should be the Maurer–Cartan forms of a Lie pseudo-group?

**Cartan**: Use exterior differential systems and prolongation to determine the structure equations.

I propose a direct approach based on the following observation:

The Maurer–Cartan forms for a pseudo-group can be identified with the right-invariant one-forms on the jet groupoid $G^{(\infty)}$.

The structure equations can be determined immediately from the infinitesimal determining equations.
The Variational Bicomplex

★ The differential one-forms on an infinite jet bundle split into two types:
  - horizontal forms
  - contact forms

Definition. A contact form $\theta$ is a differential form that vanishes on all jets: $\theta | j_n\varphi = 0$ for all local diffeomorphisms $\varphi \in \mathcal{D}$. 
For the diffeomorphism jet bundle

$$\mathcal{D}(\infty) \subset J(\infty)(M, M)$$

Local coordinates:

$$z^1, \ldots, z^m, \quad Z^1, \ldots, Z^m, \quad \ldots, Z^b_A, \ldots$$

source \hspace{2cm} target \hspace{2cm} jet

Horizontal forms:

$$dz^1, \ldots, dz^m$$

Basis contact forms:

$$\Theta^b_A = d_G Z^b_A = dZ^b_A - \sum_{a=1}^{m} Z^a_{A,a} dz^a$$
One-dimensional case: \( M = \mathbb{R} \)

Local coordinates on \( \mathcal{D}^{(\infty)}(\mathbb{R}) \)

\[ (x, X, X_x, X_{xx}, X_{xxx}, \ldots, X_n, \ldots) \]

Horizontal form:

\[ dx \]

Contact forms:

\[ \Theta = dX - X_x \, dx \]
\[ \Theta_x = dX_x - X_{xx} \, dx \]
\[ \Theta_{xx} = dX_{xx} - X_{xxx} \, dx \]
\[ \vdots \]

- the contact forms vanish when \( X = \varphi(x) \)
The Variational Bicomplex

\[ \Rightarrow \text{Vinogradov, Tsujishita, I. Anderson} \]

Infinite jet space

\[ J^\infty = \lim_{n \to \infty} J^n \]

Local coordinates

\[ z^{(\infty)} = (x, u^{(\infty)}) = (\ldots x^i \ldots u^\alpha_J \ldots) \]

Horizontal one-forms

\[ dx^1, \ldots, dx^p \]

Contact (vertical) one-forms

\[ \theta_J^\alpha = du^\alpha_J - \sum_{i=1}^{p} u^\alpha_{J,i} \, dx^i \]
Bigrading of the differential forms on $J^\infty$:

$$\Omega^* = \bigoplus_{r,s} \Omega^{r,s}$$

$r = \# \text{ of } dx^i$

$s = \# \text{ of } \theta^\alpha_j$

Vertical and Horizontal Differentials

$$d = d_H + d_V$$
The Variational Bicomplex

\[ d_H : \Omega^{r,s} \longrightarrow \Omega^{r+1,s} \]
\[ d_V : \Omega^{r,s} \longrightarrow \Omega^{r,s+1} \]

\[ d_H F = \sum_{i=1}^{p} (D_i F) \, dx^i \quad \text{— total differential} \]
\[ d_V F = \sum_{\alpha,J} \frac{\partial F}{\partial u^\alpha_J} \, \theta^\alpha_J \quad \text{— "variation"} \]

\[ \pi : \Omega^{p,k} \longrightarrow \mathcal{F}^k = \Omega^{p,k} / d_H [\Omega^{p-1,k}] \quad \text{— integration by parts} \]
The Variational Bicomplex

- conservation laws
- Lagrangians
- PDEs (Euler–Lagrange)
- Helmholtz conditions
The Simplest Example. \( M = \mathbb{R}^2 \), \( x, u \in \mathbb{R} \)

Horizontal form

\[ dx \]

Contact (vertical) forms

\[ \theta = du - u_x \, dx \]
\[ \theta_x = D_x \theta = du_x - u_{xx} \, dx \]
\[ \theta_{xx} = D^2_x \theta = du_{xx} - u_{xxx} \, dx \]
\[ \vdots \]
Differential \[ F = F(x, u, u_x, u_{xx}, \ldots) \]

\[ dF = \frac{\partial F}{\partial x} \, dx + \frac{\partial F}{\partial u} \, du + \frac{\partial F}{\partial u_x} \, du_x + \frac{\partial F}{\partial u_{xx}} \, du_{xx} + \cdots \]

\[ = (D_x F) \, dx + \frac{\partial F}{\partial u} \, \theta + \frac{\partial F}{\partial u_x} \, \theta_x + \frac{\partial F}{\partial u_{xx}} \, \theta_{xx} + \cdots \]

\[ = d_H F \quad + \quad d_V F \]

Total derivative

\[ D_x F = \frac{\partial F}{\partial u} \, u_x + \frac{\partial F}{\partial u_x} \, u_{xx} + \frac{\partial F}{\partial u_{xx}} \, u_{xxx} + \cdots \]
Lagrangian form: \[ \lambda = L(x, u^{(n)}) \, dx \in \Omega^{1,0} \]  
Vertical derivative — variation:
\[
d\lambda = d_V \lambda = d_V L \wedge dx \\
= \left( \frac{\partial L}{\partial u} \theta + \frac{\partial L}{\partial u_x} \theta_x + \frac{\partial L}{\partial u_{xx}} \theta_{xx} + \cdots \right) \wedge dx \in \Omega^{1,1} 
\]
Integration by parts — compute modulo im \( d_H \):
\[
d\lambda \sim \delta \lambda = \left( \frac{\partial L}{\partial u} - D_x \frac{\partial L}{\partial u_x} + D_x^2 \frac{\partial L}{\partial u_{xx}} - \cdots \right) \theta \wedge dx \\
= \mathcal{E}(L) \theta \wedge dx \\
\implies \text{Euler-Lagrange source form.}
Maurer–Cartan Forms

The Maurer–Cartan forms for the diffeomorphism pseudo-group are the right-invariant one-forms on the diffeomorphism jet groupoid $\mathcal{D}^{(\infty)}$.

Key observation:
The target coordinate functions $Z^a$ are right-invariant. Thus, when we decompose

$$dZ^a = \sigma^a + \mu^a$$

the two constituents are also right-invariant.
Invariant horizontal forms:

\[ \sigma^a = d_M Z^a = \sum_{b=1}^{m} Z_b^a d z^b \]

Invariant total differentiation (dual operators):

\[ \mathbb{D}_{Z^a} = \sum_{b=1}^{m} (Z_b^a)^{-1} \mathbb{D}_{z^b} \]

Invariant contact forms:

\[ \mu^b = d_G Z^b = \Theta^b = dZ^b - \sum_{a=1}^{m} Z_b^a dZ^a \]

\[ \mu^b_A = \mathbb{D}_{Z^a} \mu^b = \mathbb{D}_{Z^a_1} \cdots \mathbb{D}_{Z^a_m} \Theta^b \]

\[ b = 1, \ldots, m, \# A \geq 0 \]
One-dimensional case: \( M = \mathbb{R} \)

Contact forms:

\[
\Theta = dX - X_x \, dx
\]

\[
\Theta_x = \mathbb{D}_x \Theta = dX_x - X_{xx} \, dx
\]

\[
\Theta_{xx} = \mathbb{D}_x^2 \Theta = dX_{xx} - X_{xxx} \, dx
\]

Right-invariant horizontal form:

\[
\sigma = d_M X = X_x \, dx
\]

Invariant differentiation:

\[
\mathbb{D}_X = \frac{1}{X_x} \mathbb{D}_x
\]
Invariant contact forms:

$$\mu = \Theta = dX - X_x \, dx$$

$$\mu_X = \mathbb{D}_X \mu = \frac{\Theta_x}{X_x} = \frac{dX_x - X_{xx} \, dx}{X_x}$$

$$\mu_{XX} = \mathbb{D}_X^2 \mu = \frac{X_x \Theta_{xx} - X_{xx} \Theta_x}{X_x^3}$$

$$= \frac{X_x dX_{xx} - X_{xx} \, dX_x + (X_{xx}^2 - X_x X_{xxx}) \, dx}{X_x^3}$$

$$\vdots$$

$$\mu_n = \mathbb{D}_X^n \mu$$
The Structure Equations for the Diffeomorphism Pseudo–group

Maurer–Cartan series:

\[ \mu^b[H] = \sum_A \frac{1}{A!} \mu^b_A H^A \]

\[ H = (H^1, \ldots, H^m) \] — formal parameters

\[ d\mu[H] = \nabla \mu[H] \wedge (\mu[H] - dZ) \]

\[ d\sigma = -d\mu[0] = \nabla \mu[0] \wedge \sigma \]
One-dimensional case: \( M = \mathbb{R} \)

Structure equations:

\[
d\sigma = \mu_X \wedge \sigma \quad \frac{d\mu}{dH}[H] = \frac{d\mu}{dH} [H] \wedge (\mu[H] - dZ)
\]

where

\[
\sigma = X_x \, dx = dX - \mu
\]

\[
\mu[H] = \mu + \mu_X H + \frac{1}{2} \mu_{XX} H^2 + \cdots
\]

\[
\mu[H] - dZ = -\sigma + \mu_X H + \frac{1}{2} \mu_{XX} H^2 + \cdots
\]

\[
\frac{d\mu[H]}{dH} = \mu_X + \mu_{XX} H + \frac{1}{2} \mu_{XXX} H^2 + \cdots
\]
In components:

\[ d\sigma = \mu_1 \wedge \sigma \]

\[ d\mu_n = -\mu_{n+1} \wedge \sigma + \sum_{i=0}^{n-1} \binom{n}{i} \mu_{i+1} \wedge \mu_{n-i} \]

\[ = \sigma \wedge \mu_{n+1} - \sum_{j=1}^{\left[ \frac{n+1}{2} \right]} \frac{n - 2j + 1}{n + 1} \binom{n+1}{j} \mu_j \wedge \mu_{n+1-j}. \]

\[ \implies \text{Cartan} \]
The Maurer–Cartan Forms for a Lie Pseudo-group

The Maurer–Cartan forms for $\mathcal{G}$ are obtained by restricting the diffeomorphism Maurer–Cartan forms $\sigma^a, \mu^b_A$ to $\mathcal{G}^{(\infty)} \subset \mathcal{D}^{(\infty)}$.

⋆ The resulting one-forms are no longer linearly independent.
**Theorem.** The Maurer–Cartan forms on $G^{(\infty)}$ satisfy the invariant infinitesimal determining equations

\[ \mathcal{L}( \ldots Z^a \ldots \mu^b_A \ldots ) = 0 \quad (\star \star) \]

obtained from the infinitesimal determining equations

\[ \mathcal{L}( \ldots z^a \ldots \zeta^b_A \ldots ) = 0 \quad (\star) \]

by replacing

- source variables $z^a$ by target variables $Z^a$
- derivatives of vector field coefficients $\zeta^b_A$ by right-invariant Maurer–Cartan forms $\mu^b_A$
The Structure Equations
for a Lie Pseudo-group

**Theorem.** The structure equations for the pseudo-group \( \mathcal{G} \) are obtained by restricting the universal diffeomorphism structure equations

\[
d\mu[ H ] = \nabla \mu[ H ] \wedge ( \mu[ H ] - dZ )
\]

to the solution space of the linearized involutive system

\[
\mathcal{L}( \ldots Z^a, \ldots \mu^b_A, \ldots ) = 0.
\]
The Korteweg–deVries Equation

\[ u_t + u_{xxx} + uu_x = 0 \]

Diffeomorphism Maurer–Cartan forms:

\[ \mu^t, \mu^x, \mu^u, \mu^t_T, \mu^x_T, \mu^u_T, \mu^U, \mu^t_U, \mu^X, \mu^t_X, \mu^u_X, \ldots \]
Maurer–Cartan determining equations:

\[
\begin{align*}
\mu^t_X &= \mu^t_U = \mu^x_U = \mu^u_T = \mu^u_X = 0, \\
\mu^u &= \mu^x_T - \frac{2}{3} U \mu^t_T, \quad \mu^u_U = -\frac{2}{3} \mu^t_T = -2 \mu^x_X, \\
\mu^t_{TT} &= \mu^t_{TX} = \mu^t_{XX} = \cdots = \mu^u_{UU} = \cdots = 0.
\end{align*}
\]

Basis (dim \(G_{KdV} = 4\)):

\[
\mu^1 = \mu^t, \quad \mu^2 = \mu^x, \quad \mu^3 = \mu^u, \quad \mu^4 = \mu^t_T.
\]
Structure equations:

\[ d\mu^1 = -\mu^1 \wedge \mu^4, \]

\[ d\mu^2 = -\mu^1 \wedge \mu^3 - \frac{2}{3} U \mu^1 \wedge \mu^4 - \frac{1}{3} \mu^2 \wedge \mu^4, \]

\[ d\mu^3 = \frac{2}{3} \mu^3 \wedge \mu^4, \]

\[ d\mu^4 = 0. \]

\[ d\mu^i = C^i_{jk} \mu^j \wedge \mu^k \]
The structure equations are on the principal bundle $\mathcal{G}^{(\infty)}$; if $G$ is a finite-dimensional Lie group, then $\mathcal{G}^{(\infty)} \simeq M \times G$, and the usual Lie group structure equations are found by restriction to the target fibers $\{Z = c\} \simeq G$. 
Lie–Kumpera Example

\[
X = f(x) \quad U = \frac{u}{f'(x)}
\]

Linearized determining system

\[
\xi_x = -\frac{\varphi}{u} \quad \xi_u = 0 \quad \varphi_u = \frac{\varphi}{u}
\]
Maurer–Cartan forms:

\[
\sigma = \frac{u}{U} \, dx = f_x \, dx, \quad \tau = U_x \, dx + \frac{U}{u} \, du = -u \frac{f_{xx} \, dx + f_x \, du}{f_x^2} \\
\mu = dX - \frac{U}{u} \, dx = df - f_x \, dx, \quad \nu = dU - U_x \, dx - \frac{U}{u} \, du = -\frac{u}{f_x^2} \left( df_x - f_{xx} \, dx \right) \\
\mu_X = \frac{du}{u} - \frac{dU - U_x \, dx}{U} = \frac{df_x - f_{xx} \, dx}{f_x}, \quad \mu_U = 0 \\
\nu_X = \frac{U}{u} \left( dU_x - U_{xx} \, dx \right) - \frac{U_x}{u} \left( dU - U_x \, dx \right) \\
\quad \quad \quad = -\frac{u}{f_x^3} \left( df_{xx} - f_{xxx} \, dx \right) + \frac{u f_{xx}}{f_x^4} \left( df_x - f_{xx} \, dx \right) \\
\nu_U = -\frac{du}{u} + \frac{dU - U_x \, dx}{U} = -\frac{df_x - f_{xx} \, dx}{f_x}
\]
Right-invariant linearized system:

\[ \mu_X = - \frac{\nu}{U} \quad \mu_U = 0 \quad \nu_U = \frac{\nu}{U} \]

First order structure equations:

\[ d\mu = -d\sigma = \frac{\nu \wedge \sigma}{U}, \quad d\nu = -\nu_X \wedge \sigma - \frac{\nu \wedge \tau}{U} \]

\[ d\nu_X = -\nu_{XX} \wedge \sigma - \frac{\nu_X \wedge (\tau + 2\nu)}{U} \]
Action of Pseudo-groups on Submanifolds
a.k.a. Solutions of Differential Equations

\( \mathcal{G} \) — Lie pseudo-group acting on \( p \)-dimensional submanifolds:

\[
N = \{ u = f(x) \} \subset M
\]

For example, \( \mathcal{G} \) may be the symmetry group of a system of differential equations

\[
\Delta(x, u^{(n)}) = 0
\]

and the submanifolds the graphs of solutions \( u = f(x) \).
Prolongation

\( J^n = J^n(M, p) \) — \( n \)th order submanifold jet bundle

Local coordinates:

\[ z^{(n)} = (x, u^{(n)}) = (\ldots x^i \ldots u^\alpha J \ldots) \]

Prolonged action of \( \mathcal{G}^{(n)} \) on submanifolds:

\[ (x, u^{(n)}) \quad \mapsto \quad (X, \hat{U}^{(n)}) \]

Coordinate formulae:

\[ \hat{U}^\alpha_J = F^\alpha_J(x, u^{(n)}, g^{(n)}) \]

\( \implies \) Implicit differentiation.
Differential Invariants

A differential invariant is an invariant function $I : J^n \rightarrow \mathbb{R}$ for the prolonged pseudo-group action

$$I(g^{(n)} \cdot (x, u^{(n)})) = I(x, u^{(n)})$$

$\Rightarrow$ curvature, torsion, ...

Invariant differential operators:

$\mathcal{D}_1, \ldots, \mathcal{D}_p$

$\Rightarrow$ arc length derivative

$I(\mathcal{G})$ — the algebra of differential invariants
The Basis Theorem

**Theorem.** The differential invariant algebra \( \mathbb{I}(G) \) is locally generated by a finite number of differential invariants

\[
I_1, \ldots , I_\ell
\]

and \( p = \dim S \) invariant differential operators

\[
\mathcal{D}_1, \ldots , \mathcal{D}_p
\]

meaning that *every* differential invariant can be locally expressed as a function of the generating invariants and their invariant derivatives:

\[
\mathcal{D}_j I_\kappa = \mathcal{D}_{j_1} \mathcal{D}_{j_2} \cdots \mathcal{D}_{j_n} I_\kappa.
\]

\[\Rightarrow\] Lie groups: *Lie, Ovsiannikov*

\[\Rightarrow\] Lie pseudo-groups: *Tresse, Kumpera, Pohjanpelto–O*
Key Issues

- **Minimal basis** of generating invariants: $I_1, \ldots, I_\ell$
- **Commutation formulae** for

  the invariant differential operators:

  \[
  [\mathcal{D}_j, \mathcal{D}_k] = \sum_{i=1}^{p} Y_{jk}^i \mathcal{D}_i
  \]

  $\implies$ Non-commutative differential algebra

- **Syzygies** (functional relations) among

  the differentiated invariants:

  \[
  \Phi(\ldots \mathcal{D}_j I_\kappa \ldots) \equiv 0
  \]

  $\implies$ Codazzi relations
Examples of Differential Invariants

Euclidean Group on $\mathbb{R}^3$

$$G = \text{SE}(3) = \text{SO}(3) \ltimes \mathbb{R}^3$$

$\implies$ group of rigid motions

$$z \mapsto Rz + b \quad R \in \text{SO}(3)$$

• Induced action on curves and surfaces.
Euclidean Curves \( C \subset \mathbb{R}^3 \)

- \( \kappa \) — curvature: order = 2
- \( \tau \) — torsion: order = 3
Euclidean Curves $C \subset \mathbb{R}^3$

- $\kappa$ — curvature: order = 2
- $\tau$ — torsion: order = 3
- $\kappa_s, \tau_s, \kappa_{ss}, \ldots$ — derivatives w.r.t. arc length $ds$
Euclidean Curves $C \subset \mathbb{R}^3$

- $\kappa$ — curvature: order = 2
- $\tau$ — torsion: order = 3
- $\kappa_s, \tau_s, \kappa_{ss}, \ldots$ — derivatives w.r.t. arc length $ds$

**Theorem.** Every Euclidean differential invariant of a space curve $C \subset \mathbb{R}^3$ can be written

$$I = H(\kappa, \tau, \kappa_s, \tau_s, \kappa_{ss}, \ldots)$$
**Euclidean Curves** \( C \subset \mathbb{R}^3 \)

- \( \kappa \) — curvature: order = 2
- \( \tau \) — torsion: order = 3
- \( \kappa_s, \tau_s, \kappa_{ss}, \ldots \) — derivatives w.r.t. arc length \( ds \)

**Theorem.** Every Euclidean differential invariant of a space curve \( C \subset \mathbb{R}^3 \) can be written

\[
I = F(\kappa, \tau, \kappa_s, \tau_s, \kappa_{ss}, \ldots)
\]

Thus, \( \kappa \) and \( \tau \) generate the differential invariants of space curves under the Euclidean group.
Euclidean Surfaces \( S \subset \mathbb{R}^3 \)

- \( H = \frac{1}{2} (\kappa_1 + \kappa_2) \) — mean curvature: order = 2
- \( K = \kappa_1 \kappa_2 \) — Gauss curvature: order = 2
Euclidean Surfaces  \( S \subset \mathbb{R}^3 \)

- \( H = \frac{1}{2} (\kappa_1 + \kappa_2) \) — mean curvature: order = 2
- \( K = \kappa_1 \kappa_2 \) — Gauss curvature: order = 2
- \( D_1 H, D_2 H, D_1 K, D_2 K, D_1^2 H, \ldots \) — derivatives with respect to the equivariant Frenet frame on \( S \)
Euclidean Surfaces \( S \subset \mathbb{R}^3 \)

- \( H = \frac{1}{2} (\kappa_1 + \kappa_2) \) — mean curvature: order = 2
- \( K = \kappa_1 \kappa_2 \) — Gauss curvature: order = 2
- \( \mathcal{D}_1 H, \mathcal{D}_2 H, \mathcal{D}_1 K, \mathcal{D}_2 K, \mathcal{D}_1^2 H, \ldots \) — derivatives with respect to the equivariant Frenet frame on \( S \)

**Theorem.** Every Euclidean differential invariant of a non-umbilic surface \( S \subset \mathbb{R}^3 \) can be written

\[
I = F(H, K, \mathcal{D}_1 H, \mathcal{D}_2 H, \mathcal{D}_1 K, \mathcal{D}_2 K, \mathcal{D}_1^2 H, \ldots)
\]
Euclidean Surfaces $S \subset \mathbb{R}^3$

- $H = \frac{1}{2} (\kappa_1 + \kappa_2)$ — mean curvature: order = 2
- $K = \kappa_1 \kappa_2$ — Gauss curvature: order = 2
- $\mathcal{D}_1 H, \mathcal{D}_2 H, \mathcal{D}_1 K, \mathcal{D}_2 K, \mathcal{D}_1^2 H, \ldots$ — derivatives with respect to the equivariant Frenet frame on $S$

**Theorem.** Every Euclidean differential invariant of a non-umbilic surface $S \subset \mathbb{R}^3$ can be written

$$I = F(H, K, \mathcal{D}_1 H, \mathcal{D}_2 H, \mathcal{D}_1 K, \mathcal{D}_2 K, \mathcal{D}_1^2 H, \ldots)$$

Thus, $H, K$ generate the differential invariants of (generic) Euclidean surfaces.
Euclidean Surfaces

Theorem.

The algebra of Euclidean differential invariants for a non-degenerate surface is generated by the mean curvature through invariant differentiation.

\[ K = \Phi(H, D_1H, D_2H, \ldots) \]
Euclidean Surfaces

Theorem.

The algebra of Euclidean differential invariants for a non-degenerate surface is generated by the mean curvature through invariant differentiation.

\[ K = \Phi(H, D_1H, D_2H, \ldots) \]
Applications of Differential Invariants

Every (regular) $\mathcal{G}$-invariant system of differential equations can be expressed in terms of the differential invariants:

$$F(\ldots \mathcal{D}_j I_\kappa \ldots ) = 0$$

Every $\mathcal{G}$-invariant variational problem can be expressed in terms of the differential invariants and an invariant volume form:

$$\mathcal{I}[u] = \int L(\ldots \mathcal{D}_j I_\kappa \ldots ) \Omega$$

**Question:** How to go directly from the differential invariant form of the variational problem to the differential invariant form of the Euler–Lagrange equations? (See Kogan–O.)
• Characterization of moduli spaces

• Integration of invariant ordinary differential equations.

• Symmetry reduction and group splitting (Vessiot’s method) for finding explicit solutions to partial differential equations.

• Equivalence and symmetry of solutions/submanifolds — differential invariant signatures.

• Image processing.

• Design of symmetry-preserving numerical algorithms.
Computing Differential Invariants

♠ The infinitesimal method:
\[ v(I) = 0 \quad \text{for every infinitesimal generator} \quad v \in \mathfrak{g} \]
\[ \implies \quad \text{Requires solving differential equations.} \]

♥ Moving frames. (Cartan; PJO–Fels–Pohjanpelto–⋯)

• Completely algebraic.
• Can be adapted to arbitrary group and pseudo-group actions.
• Describes the complete structure of the differential invariant algebra \( \mathfrak{H}(\mathcal{G}) \) — using only linear algebra & differentiation!
• Prescribes differential invariant signatures for equivalence and symmetry detection.
**Moving Frames for Pseudo–Groups**

In the finite-dimensional Lie group case, a moving frame is **defined** as an equivariant map

$$\rho^{(n)} : J^n \longrightarrow G$$

$$\implies$$ All classical moving frames can be thus interpreted.
However, we do not have an appropriate abstract object to represent our pseudo-group $\mathcal{G}$.

Consequently, the moving frame will be an equivariant section

$$\rho^{(n)} : J^n \longrightarrow \mathcal{H}^{(n)}$$

of the pulled-back pseudo-group jet groupoid:

$$\begin{array}{ccc}
\mathcal{G}^{(n)} & & \mathcal{H}^{(n)} \\
\downarrow & & \downarrow \\
M & \longleftrightarrow & J^n.
\end{array}$$
Moving Frames for Pseudo–Groups

Definition. A (right) moving frame of order \( n \) is a right-equivariant section \( \rho^{(n)} : V^n \to \mathcal{H}^{(n)} \) defined on an open subset \( V^n \subset J^n \).

\[ \Rightarrow \quad \text{Groupoid action.} \]

Proposition. A moving frame of order \( n \) exists if and only if \( \mathcal{G}^{(n)} \) acts \textit{freely} and regularly.
Freeness

For Lie group actions, freeness means no isotropy. For infinite-dimensional pseudo-groups, this definition cannot work, and one must restrict to the transformation jets of order $n$, using the $n^{th}$ order isotropy subgroup:

$$\mathcal{G}^{(n)}_{z(n)} = \{ g^{(n)} \in \mathcal{G}^{(n)}_z \mid g^{(n)} \cdot z^{(n)} = z^{(n)} \}$$

**Definition.** At a jet $z^{(n)} \in J^n$, the pseudo-group $\mathcal{G}$ acts

- freely if $\mathcal{G}^{(n)}_{z(n)} = \{ 1^{(n)}_z \}$
- locally freely if
  - $\mathcal{G}^{(n)}_{z(n)}$ is a discrete subgroup of $\mathcal{G}^{(n)}_z$
  - the orbits have $\dim = r_n = \dim \mathcal{G}^{(n)}_z$
Freeness Theorem

Theorem. If $n \geq 1$ and $G^{(n)}$ acts locally freely at $z^{(n)} \in J^n$, then it acts locally freely at any $z^{(k)} \in J^k$ with $\tilde{\pi}^k_n(z^{(k)}) = z^{(n)}$ for all $k > n$. 
The Normalization Algorithm

To construct a moving frame:

I. Compute the prolonged pseudo-group action

\[ u_\alpha^K \mapsto U_\alpha^K = F_\alpha^K(x, u^{(n)}, g^{(n)}) \]

by implicit differentiation.

II. Choose a cross-section to the pseudo-group orbits:

\[ u_\alpha^K \xi = c_\kappa, \quad \kappa = 1, \ldots, r_n = \text{fiber dim } G^{(n)} \]
III. Solve the normalization equations

\[ U_{J_{\kappa}}^{\alpha_{\kappa}} = F_{J_{\kappa}}^{\alpha_{\kappa}}(x, u^{(n)}, g^{(n)}) = c_{\kappa} \]

for the \( n^{\text{th}} \) order pseudo-group parameters

\[ g^{(n)} = \rho^{(n)}(x, u^{(n)}) \]

IV. Substitute the moving frame formulas into the un-normalized jet coordinates \( u_{K}^{\alpha} = F_{K}^{\alpha}(x, u^{(n)}, g^{(n)}) \). The resulting functions form a complete system of \( n^{\text{th}} \) order differential invariants

\[ I_{K}^{\alpha}(x, u^{(n)}) = F_{K}^{\alpha}(x, u^{(n)}, \rho^{(n)}(x, u^{(n)})) \]
Invariantization

A moving frame induces an invariantization process, denoted \( \iota \), that projects functions to invariants, differential operators to invariant differential operators; differential forms to invariant differential forms, etc.

Geometrically, the invariantization of an object is the unique invariant version that has the same cross-section values.

Algebraically, invariantization amounts to replacing the group parameters in the transformed object by their moving frame formulas.
**Invariantization**

In particular, invariantization of the jet coordinates leads to a complete system of functionally independent differential invariants:

\[ \iota(x^i) = H^i \quad \iota(u^\alpha_J) = I^\alpha_J \]

- Phantom differential invariants: \( I^\alpha_J\kappa = c_\kappa \)
- The non-constant invariants form a functionally independent generating set for the differential invariant algebra \( \mathcal{I}(\mathcal{G}) \)
• Replacement Theorem

\[ I( \ldots x^i \ldots u_j^\alpha \ldots ) = \iota( I( \ldots x^i \ldots u_j^\alpha \ldots ) ) = I( \ldots H^i \ldots I_j^\alpha \ldots ) \]

◊ Differential forms \( \implies \) invariant differential forms

\[ \iota(dx^i) = \omega^i \quad i = 1, \ldots, p \]

◊ Differential operators \( \implies \) invariant differential operators

\[ \iota(Dx^i) = D_i \quad i = 1, \ldots, p \]
Recurrence Formulae

Invariantization and differentiation do not commute

The recurrence formulae connect the differentiated invariants with their invariantized counterparts:

\[ \mathcal{D}_i I_J^\alpha = I_{J,i}^\alpha + M_{J,i}^\alpha \]

\[ \Rightarrow M_{J,i}^\alpha \text{ — correction terms} \]
Once established, the recurrence formulae completely prescribe the structure of the differential invariant algebra \( \mathbb{I}(\mathcal{G}) \) — thanks to the functional independence of the non-phantom normalized differential invariants.

The recurrence formulae can be explicitly determined using only the infinitesimal generators and linear differential algebra!
Korteweg–deVries Equation

Prolonged Symmetry Group Action:

\[ T = e^{3\lambda_4}(t + \lambda_1) \]

\[ X = e^{\lambda_4}(\lambda_3 t + x + \lambda_1 \lambda_3 + \lambda_2) \]

\[ U = e^{-2\lambda_4}(u + \lambda_3) \]

\[ U_T = e^{-5\lambda_4}(u_t - \lambda_3 u_x) \]

\[ U_X = e^{-3\lambda_4}u_x \]

\[ U_{TT} = e^{-8\lambda_4}(u_{tt} - 2\lambda_3 u_{tx} + \lambda_3^2 u_{xx}) \]

\[ U_{TX} = D_X D_T U = e^{-6\lambda_4}(u_{tx} - \lambda_3 u_{xx}) \]

\[ U_{XX} = e^{-4\lambda_4}u_{xx} \]

\[ \vdots \]
Cross Section:

\[ T = e^{3\lambda_4}(t + \lambda_1) = 0 \]
\[ X = e^{\lambda_4}(\lambda_3 t + x + \lambda_1 \lambda_3 + \lambda_2) = 0 \]
\[ U = e^{-2\lambda_4}(u + \lambda_3) = 0 \]
\[ U_T = e^{-5\lambda_4}(u_t - \lambda_3 u_x) = 1 \]

Moving Frame:

\[ \lambda_1 = -t, \quad \lambda_2 = -x, \quad \lambda_3 = -u, \quad \lambda_4 = \frac{1}{5} \log(u_t + uu_x) \]
Moving Frame:
\[
\lambda_1 = -t, \quad \lambda_2 = -x, \quad \lambda_3 = -u, \quad \lambda_4 = \frac{1}{5} \log(u_t + uu_x)
\]

Invariantization:
\[
\iota(u_K) = U_K \mid_{\lambda_1 = -t, \lambda_2 = -x, \lambda_3 = -u, \lambda_4 = \log(u_t + uu_x)/5}
\]

Phantom Invariants:
\[
H^1 = \iota(t) = 0
\]
\[
H^2 = \iota(x) = 0
\]
\[
I_{00} = \iota(u) = 0
\]
\[
I_{10} = \iota(u_t) = 1
\]
Normalized differential invariants:

\[ I_{01} = \iota(u_x) = \frac{u_x}{(u_t + uu_x)^{3/5}} \]

\[ I_{20} = \iota(u_{tt}) = \frac{u_{tt} + 2uu_{tx} + u^2u_{xx}}{(u_t + uu_x)^{8/5}} \]

\[ I_{11} = \iota(u_{tx}) = \frac{u_{tx} + uu_{xx}}{(u_t + uu_x)^{6/5}} \]

\[ I_{02} = \iota(u_{xx}) = \frac{u_{xx}}{(u_t + uu_x)^{4/5}} \]

\[ I_{03} = \iota(u_{xxx}) = \frac{u_{xxx}}{u_t + uu_x} \]

\[ \vdots \]
Invariantization:

\[ \iota( F(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \ldots ) ) \]

\[ = F(\iota(t), \iota(x), \iota(u), \iota(u_t), \iota(u_x), \iota(u_{tt}), \iota(u_{tx}), \iota(u_{xx}), \ldots ) \]

\[ = F(H^1, H^2, I_{00}, I_{10}, I_{01}, I_{20}, I_{11}, I_{02}, \ldots ) \]

\[ = F(0, 0, 0, 1, I_{01}, I_{20}, I_{11}, I_{02}, \ldots ) \]

Replacement Theorem:

\[ 0 = \iota(u_t + uu_x + u_{xxx}) = 1 + I_{03} = \frac{u_t + uu_x + u_{xxx}}{u_t + uu_x}. \]

Invariant horizontal one-forms:

\[ \omega^1 = \iota(dt) = (u_t + uu_x)^{3/5} dt, \]

\[ \omega^2 = \iota(dx) = -u(u_t + uu_x)^{1/5} dt + (u_t + uu_x)^{1/5} dx. \]
Invariant differential operators:

\[ \mathcal{D}_1 = \iota(D_t) = (u_t + uu_x)^{-3/5} D_t + u(u_t + uu_x)^{-3/5} D_x, \]

\[ \mathcal{D}_2 = \iota(D_x) = (u_t + uu_x)^{-1/5} D_x. \]

Commutation formula:

\[ [\mathcal{D}_1, \mathcal{D}_2] = I_{01} \mathcal{D}_1 \]

Recurrence formulae:

\[ \mathcal{D}_1 I_{01} = I_{11} - \frac{3}{5} I_{01}^2 - \frac{3}{5} I_{01} I_{20}, \]

\[ \mathcal{D}_1 I_{20} = I_{30} + 2I_{11} - \frac{8}{5} I_{01} I_{20} - \frac{8}{5} I_{20}^2, \]

\[ \mathcal{D}_1 I_{11} = I_{21} + I_{02} - \frac{6}{5} I_{01} I_{11} - \frac{6}{5} I_{11} I_{20}, \]

\[ \mathcal{D}_1 I_{02} = I_{12} - \frac{4}{5} I_{01} I_{02} - \frac{4}{5} I_{02} I_{20}, \]

\[ \vdots \]

\[ \mathcal{D}_2 I_{01} = I_{02} - \frac{3}{5} I_{01}^3 - \frac{3}{5} I_{01} I_{11}, \]

\[ \mathcal{D}_2 I_{20} = I_{21} + 2I_{01} I_{11} - \frac{8}{5} I_{01}^2 I_{20} - \frac{8}{5} I_{11} I_{20}, \]

\[ \mathcal{D}_2 I_{11} = I_{12} + I_{01} I_{02} - \frac{6}{5} I_{01}^2 I_{11} - \frac{6}{5} I_{11}^2, \]

\[ \mathcal{D}_2 I_{02} = I_{03} - \frac{4}{5} I_{01}^2 I_{02} - \frac{4}{5} I_{02} I_{11}, \]

\[ \vdots \]
Generating differential invariants:

\[ I_{01} = \iota(u_x) = \frac{u_x}{(u_t + uu_x)^{3/5}}, \quad I_{20} = \iota(u_{tt}) = \frac{u_{tt} + 2uu_{tx} + u^2u_{xx}}{(u_t + uu_x)^{8/5}}. \]

Fundamental syzygy:

\[ \mathcal{D}_1^2 I_{01} + \frac{3}{5} I_{01} \mathcal{D}_1 I_{20} - \mathcal{D}_2 I_{20} + \left( \frac{1}{5} I_{20} + \frac{19}{5} I_{01} \right) \mathcal{D}_1 I_{01} \]

\[ -\mathcal{D}_2 I_{01} - \frac{6}{25} I_{01} I_{20}^2 - \frac{7}{25} I_{01}^2 I_{20} + \frac{24}{25} I_{01}^3 = 0. \]
Lie–Tresse–Kumpera Example

\[
X = f(x), \quad Y = y, \quad U = \frac{u}{f'(x)}
\]

Horizontal coframe

\[
d_H X = f_x \, dx, \quad d_H Y = dy,
\]

Implicit differentiations

\[
D_X = \frac{1}{f_x} D_x, \quad D_Y = D_y.
\]
Prolonged pseudo-group transformations on surfaces $S \subset \mathbb{R}^3$

$$X = f \quad Y = y \quad U = \frac{u}{f_x}$$

$$U_X = \frac{u_x}{f_x^2} - \frac{u f_{xx}}{f_x^3} \quad U_Y = \frac{u_y}{f_x}$$

$$U_{XX} = \frac{u_{xx}}{f_x^3} - \frac{3 u_x f_{xx}}{f_x^4} - \frac{u f_{xxx}}{f_x^4} + \frac{3 u f_{xx}^2}{f_x^5}$$

$$U_{XY} = \frac{u_{xy}}{f_x^2} - \frac{u_y f_{xx}}{f_x^3} \quad U_{YY} = \frac{u_{yy}}{f_x}$$

$\implies$ action is free at every order.

Coordinate cross-section

$$X = f = 0, \quad U = \frac{u}{f_x} = 1, \quad U_X = \frac{u_x}{f_x^2} - \frac{u f_{xx}}{f_x^3} = 0, \quad U_{XX} = \cdots = 0.$$
Moving frame

\[ f = 0, \quad f_x = u, \quad f_{xx} = u_x, \quad f_{xxx} = u_{xx}. \]

Differential invariants

\[ U_Y \mapsto - \frac{u_y}{u} \]

\[ U_{XY} \mapsto J_1 = \frac{u u_{xy} - u_x u_y}{u^3} \]

\[ U_{YY} \mapsto J_2 = \frac{u_{yy}}{u} \]

Invariant horizontal forms

\[ d_H X = f_x \, dx \mapsto u \, dx, \quad d_H Y = dy \mapsto dy, \]

Invariant differentiations

\[ \mathcal{D}_1 = \frac{1}{u} \, D_x \quad \mathcal{D}_2 = D_y \]
Higher order differential invariants: $\mathcal{D}_1^m \mathcal{D}_2^n J$

\[ J,1 = \mathcal{D}_1 J = \frac{u u_{xy} - u_x u_y}{u^3} = J_1, \]
\[ J,2 = \mathcal{D}_2 J = \frac{u u_{yy} - u_y^2}{u^2} = J_2 - J^2. \]

Recurrence formulae:

\[ \mathcal{D}_1 J = J_1, \quad \mathcal{D}_2 J = J_2 - J^2, \]
\[ \mathcal{D}_1 J_1 = J_3, \quad \mathcal{D}_2 J_1 = J_4 - 3 J J_1, \]
\[ \mathcal{D}_1 J_2 = J_4, \quad \mathcal{D}_2 J_2 = J_5 - J J_2, \]
The Master Recurrence Formula

\[ d_H I^\alpha_J = \sum_{i=1}^{p} (D_i I^\alpha_J) \omega^i = \sum_{i=1}^{p} I^\alpha_{J,i} \omega^i + \hat{\psi}^\alpha_J \]

where
\[ \hat{\psi}^\alpha_J = \iota(\hat{\varphi}^\alpha_J) = \Phi^\alpha_J( \ldots H^i \ldots I^\alpha_J \ldots ; \ldots \gamma^b_A \ldots ) \]
are the invariantized prolonged vector field coefficients, which are particular linear combinations of
\[ \gamma^b_A = \iota(\zeta^b_A) \quad \text{— invariantized Maurer–Cartan forms prescribed by the invariantized prolongation map.} \]

- The invariantized Maurer–Cartan forms are subject to the invariantized determining equations:
\[ \mathcal{L}(H^1, \ldots, H^p, I^1, \ldots, I^q, \ldots, \gamma^b_A, \ldots) = 0 \]
\[ d_H I^\alpha_J = \sum_{i=1}^{p} I^\alpha_{J,i} \omega^i + \hat{\psi}^\alpha_J( \ldots \gamma^b_A \ldots ) \]

**Step 1:** Solve the phantom recurrence formulas

\[ 0 = d_H I^\alpha_J = \sum_{i=1}^{p} I^\alpha_{J,i} \omega^i + \hat{\psi}^\alpha_J( \ldots \gamma^b_A \ldots ) \]

for the invariantized Maurer–Cartan forms:

\[ \gamma^b_A = \sum_{i=1}^{p} J^b_{A,i} \omega^i \]  

**Step 2:** Substitute (*) into the non-phantom recurrence formulae to obtain the explicit correction terms.
◊ Only uses linear differential algebra based on the specification of cross-section.

♡ Does not require explicit formulas for the moving frame, the differential invariants, the invariant differential operators, or even the Maurer–Cartan forms!
The Korteweg–deVries Equation (continued)

Recurrence formula:
\[
dI_{jk} = I_{j+1,k} \omega_1 + I_{j,k+1} \omega_2 + \iota(\varphi^{jk})
\]

Invariantized Maurer–Cartan forms:
\[
\iota(\tau) = \lambda, \quad \iota(\xi) = \mu, \quad \iota(\varphi) = \psi = \nu, \quad \iota(\tau_t) = \psi^t = \lambda_t, \quad \ldots
\]

Invariantized determining equations:
\[
\begin{align*}
\lambda_x &= \lambda_u = \mu_u = \nu_t = \nu_x = 0 \\
\nu &= \mu_t \quad \nu_u = -2 \mu_x = -\frac{2}{3} \lambda_t \\
\lambda_{tt} &= \lambda_{tx} = \lambda_{xx} = \cdots = \nu_{uu} = \cdots = 0
\end{align*}
\]

Invariantizations of prolonged vector field coefficients:
\[
\begin{align*}
\iota(\tau) &= \lambda, \quad \iota(\xi) = \mu, \quad \iota(\varphi) = \nu, \quad \iota(\varphi^t) = -I_{01} \nu - \frac{5}{3} \lambda_t, \\
\iota(\varphi^x) &= -I_{01} \lambda_t, \quad \iota(\varphi^{tt}) = -2I_{11} \nu - \frac{8}{3} I_{20} \lambda_t, \quad \ldots
\end{align*}
\]
Phantom recurrence formulae:
\[ 0 = d_H H^1 = \omega^1 + \lambda, \]
\[ 0 = d_H H^2 = \omega^2 + \mu, \]
\[ 0 = d_H I_{00} = I_{10}\omega^1 + I_{01}\omega^2 + \psi = \omega^1 + I_{01}\omega^2 + \nu, \]
\[ 0 = d_H I_{10} = I_{20}\omega^1 + I_{11}\omega^2 + \psi^t = I_{20}\omega^1 + I_{11}\omega^2 - I_{01}\nu - \frac{5}{3} \lambda_t, \]
\[ \Rightarrow \quad \text{Solve for} \quad \lambda = -\omega^1, \quad \mu = -\omega^2, \quad \nu = -\omega^1 - I_{01}\omega^2, \]
\[ \lambda_t = \frac{3}{5} (I_{20} + I_{01})\omega^1 + \frac{3}{5} (I_{11} + I_{01})\omega^2. \]

Non-phantom recurrence formulae:
\[ d_H I_{01} = I_{11}\omega^1 + I_{02}\omega^2 - I_{01}\lambda_t, \]
\[ d_H I_{20} = I_{30}\omega^1 + I_{21}\omega^2 - 2I_{11}\nu - \frac{8}{3} I_{20}\lambda_t, \]
\[ d_H I_{11} = I_{21}\omega^1 + I_{12}\omega^2 - I_{02}\nu - 2I_{11}\lambda_t, \]
\[ d_H I_{02} = I_{12}\omega^1 + I_{03}\omega^2 - \frac{4}{3} I_{02}\lambda_t, \]
\[ \vdots \]
\[ \mathcal{D}_1 I_{01} = I_{11} - \frac{3}{5} I_{01}^2 - \frac{3}{5} I_{01} I_{20}, \]
\[ \mathcal{D}_1 I_{20} = I_{30} + 2 I_{11} - \frac{8}{5} I_{01} I_{20} - \frac{8}{5} I_{20}^2, \]
\[ \mathcal{D}_1 I_{11} = I_{21} + I_{02} - \frac{6}{5} I_{01} I_{11} - \frac{6}{5} I_{11} I_{20}, \]
\[ \mathcal{D}_1 I_{02} = I_{12} - \frac{4}{5} I_{01} I_{02} - \frac{4}{5} I_{02} I_{20}, \]
\[ \vdots \]
\[ \mathcal{D}_2 I_{01} = I_{02} - \frac{3}{5} I_{01}^3 - \frac{3}{5} I_{01} I_{11}, \]
\[ \mathcal{D}_2 I_{20} = I_{21} + 2 I_{01} I_{11} - \frac{8}{5} I_{01}^2 I_{20} - \frac{8}{5} I_{11} I_{20}, \]
\[ \mathcal{D}_2 I_{11} = I_{12} + I_{01} I_{02} - \frac{6}{5} I_{01}^2 I_{11} - \frac{6}{5} I_{11}^2, \]
\[ \mathcal{D}_2 I_{02} = I_{03} - \frac{4}{5} I_{01}^2 I_{02} - \frac{4}{5} I_{02} I_{11}, \]
\[ \vdots \]
Lie–Tresse–Kumpera Example (continued)

\[
X = f(x), \quad Y = y, \quad U = \frac{u}{f'(x)}
\]

Phantom recurrence formulae:

\[
0 = dH = \varpi^1 + \gamma, \quad 0 = dI_{10} = J_1 \varpi^2 + \varphi_1 - \gamma_2,
\]

\[
0 = dI_{00} = J \varpi^2 + \varphi - \gamma_1, \quad 0 = dI_{20} = J_3 \varpi^2 + \varphi_3 - \gamma_3,
\]

Solve for pulled-back Maurer–Cartan forms:

\[
\gamma = -\varpi^1, \quad \gamma_2 = J_1 \varpi^2 + \varphi_1,
\]

\[
\gamma_1 = J \varpi^2 + \varphi, \quad \gamma_3 = J_3 \varpi^2 + \varphi_3,
\]

Recurrence formulae:

\[
dy = \varpi^2
\]

\[
dJ = J_1 \varpi^1 + (J_2 - J^2) \varpi^2 + \varphi_2 - J \varphi,
\]

\[
dJ_1 = J_3 \varpi^1 + (J_4 - 3JJ_1) \varpi^2 + \varphi_4 - J \varphi_1 - J_1 \varphi,
\]

\[
dJ_2 = J_4 \varpi^1 + (J_5 - JJ_2) \varpi^2 + \varphi_5 - J_2 \varphi,
\]