Metamorphoses for Pattern Matching

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- Finding correspondences between images, surfaces, curves . . .
- Comparing deformable structures (with distances).
- Building Riemannian manifolds on shapes.
- Important applications in medical imaging (computational anatomy).

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1. Equip the group of diffeomorphisms with a right-invariant Riemannian metric.
2. Assume that an object n_{temp} is given (the template)and consider the orbit $N = \mathrm{Diff}.n_{temp}$.

- Equip the orbit with the Riemannian projection of the metric on Diff.

Optimal matching with deformable templates

- Geodesics provide optimal correspondences: with fixed n_0 and n_1, minimize
$$\int_0^1 \|u_t\|^2 dt$$
with constraints $\dot{n}_t - u_t . n_t = 0$.

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with $\dot{g}_t - u_t g_t = 0$.
- But the metric aspect is lost.

Metamorphoses

Definition

Let N be a manifold acted upon by a Lie group G. A metamorphosis is a pair of curves $(g_t, \eta_t) \in G \times N$ parameterized by t, with $g_0 = \mathrm{id}_G$.
Its **image** is the curve $n_t \in N$ defined by the action $n_t = g_t.\eta_t$.

Invariant Lagrangian minimization

- Given n_0 and n_1, we will optimize, over metamorphoses (g_t, η_t), the integrated Lagrangian

$$\int_0^1 \mathcal{L}(g_t, \dot{g}_t, \eta_t, \dot{\eta}_t) dt$$

with end-point conditions $g_0 = \mathrm{id}_G$, $\eta_0 = n_0$ and $g_1 \eta_1 = n_1$.

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- We assume that $\mathcal{L}$ is invariant by the right action of G on $G \times N$ defined by $(g, \eta)h = (gh, h^{-1}\eta)$.

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- Thus, there exists a function ℓ defined on $\mathfrak{g} \times TN$ such that

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- For a metamorphosis (g_t, η_t), let $u_t = \dot{g}_t g_t^{-1}$, $n_t = g_t \eta_t$ and $\nu_t = g_t \dot{\eta}_t$, We have

$$\mathcal{L}(g_t, \dot{g}_t, \eta_t, \dot{\eta}_t) = \ell(u_t, n_t, \nu_t).$$

Notation

- The pairing between a linear form p and a vector u is denoted $\left(p \mid u\right)$. Duality with respect to this pairing is denoted with a $*$ exponent:

$$\left(p \mid Au\right) = \left(A^{*}p \mid u\right).$$

- When G acts on a manifold $\tilde{N}$, define the $\diamond$ operator on $T\tilde{N}^{*} \times \tilde{N}$, taking values in $\mathfrak{g}^{*}$ by

$$\left(\delta \diamond \tilde{n} \mid u\right) = -\left(\delta \mid u\tilde{n}\right).$$

Euler Equations

- Compute the first variation for

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- Consider a variation δu and $\omega = \delta n$.
- From $n = g\eta$ and $\nu = g\dot{\eta}$ we get $\dot{n} = \nu + un$ and $\dot{\omega} = \delta\nu + u\omega + \delta un$. (Derivatives are assumed to be taken in a chart.)

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- Write

$$\int_0^1 \left(\left(\frac{\delta\ell}{\delta u} \,\middle|\, \delta u \right) + \left(\frac{\delta\ell}{\delta n} \,\middle|\, \omega \right) + \left(\frac{\delta\ell}{\delta\nu} \,\middle|\, \dot{\omega} - u\omega - \delta u\, n \right) \right) dt = 0.$$

- From the δu term

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- From the ω term:

$$\frac{\partial}{\partial t}\frac{\delta \ell}{\delta \nu} + u \star \frac{\delta \ell}{\delta \nu} - \frac{\delta \ell}{\delta n} = 0$$

with

$$\Big(\frac{\delta \ell}{\delta \nu} \,\Big|\, u\omega\Big) = \Big(u \star \frac{\delta \ell}{\delta \nu} \,\Big|\, \omega\Big).$$

- We therefore obtain the system of equations

$$\begin{cases} \dfrac{\delta \ell}{\delta u} + \dfrac{\delta \ell}{\delta \nu} \diamond n = 0 \\[2ex] \dfrac{\partial}{\partial t} \dfrac{\delta \ell}{\delta \nu} + u \star \dfrac{\delta \ell}{\delta \nu} = \dfrac{\delta \ell}{\delta n} \\[2ex] \dot{n} = \nu + un \end{cases}$$

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- Note that $\frac{\delta \ell}{\delta u} + \frac{\delta \ell}{\delta \nu} \diamond n$ associated to the invariance of the Lagrangian. The special form of the boundary conditions ensure that this momenum is zero.

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- We have $\delta u = \dot{\xi} + [\xi, u]$ (EP reduction on the group, cf. Holm, Marsden, Ratiu).
- We also have $\delta n = \delta(g\eta) = \varpi + \xi n$. Using $\nu = g\dot{\eta}$, yielding $\delta\nu = g\delta\dot{\eta} + \xi\nu$ and $\varpi = g\delta\eta$, yielding $\dot{\varpi} = u\varpi + g\dot{\eta}$. we get

$$\delta\nu = \dot{\varpi} + \xi\nu - u\varpi.$$

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- At $t = 1$, $g_1\eta_1 = \mathrm{cst}$ so that $\xi_1 n_1 + \omega_1 = 0$.

- Write

$$\int_0^1 \left(\left(\frac{\delta \ell}{\delta u} \,|\, \dot{\xi} - \mathrm{ad}_u \, \xi \right) + \left(\frac{\delta \ell}{\delta n} \,|\, \varpi + \xi n \right) + \left(\frac{\delta \ell}{\delta \nu} \,|\, \dot{\varpi} + \xi \nu - u \varpi \right) \right) dt = 0$$

- Using integration by parts we get the boundary equation

$$\left(\frac{\delta \ell}{\delta u} \right)_1 + \left(\frac{\delta \ell}{\delta \nu} \right)_1 \diamond n_1 = 0.$$

- Evolution from ξ:

$$\frac{\partial}{\partial t}\frac{\delta \ell}{\delta u} + \mathrm{ad}^*_u \frac{\delta \ell}{\delta u} + \frac{\delta \ell}{\delta n} \diamond n + \frac{\delta \ell}{\delta \nu} \diamond \nu = 0$$

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EPMorph

- We therefore obtain the system

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\begin{cases}
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Special case 1: Riemannian metric

- Define an invariant Riemannian metric on $G \times N$, yielding

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- For example $l(u, n, \nu) = |u|_{\mathfrak{g}}^2 + |\nu|_n^2$, for a given norm, $|.|_{\mathfrak{g}}$, on $\mathfrak{g}$ and a pre-existing Riemannian structure on N (Trouvé, Y.).

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- This in turn projects into a Riemannian metric on N.

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- Assume that N is also a group and that $g(n\tilde{n}) = (gn)(g\tilde{n})$ (e.g., N is a vector space and the action of G is linear).

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- Consider the semi-direct product $G \circledS N$ with $(g,n)(\tilde{g},\tilde{n}) = (g\tilde{g},(g\tilde{n})n)$ with a right-invariant metric specified by $\| \ \|_{(\mathrm{id}_G,\mathrm{id}_N)}$ at the identity.

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- Minimize the geodesic energy between (id_G, n_0) and (g_1, n_1) with fixed n_0 and n_1.

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- Minimize the geodesic energy between (id_G, n_0) and (g_1, n_1) with fixed n_0 and n_1.
- Proposition: This is metamorphosis with reduced Lagrangian

$$\ell(u, n, \nu) = \|(u, n^{-1}\nu)\|^2_{(\mathrm{id}_G, \mathrm{id}_N)}.$$

- Note that the evolution equations in this case correspond to the full conservation of momentum on the semi-direct product:

$$\left(\frac{\delta \ell}{\delta u}(t), \frac{\delta \ell}{\delta \nu}(t)\right) = Ad^*_{(g,n)^{-1}}\left(\frac{\delta \ell}{\delta u}(t), \frac{\delta \ell}{\delta \nu}(t)\right)$$

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- When N is a vector space, the Lagrangian does ot depend on n and the equations are

$$\begin{cases} \dfrac{\delta \ell}{\delta u} + \dfrac{\delta \ell}{\delta \nu} \diamond n = 0 \\[2ex] \dfrac{\partial}{\partial t}\dfrac{\delta \ell}{\delta \nu} + u \star \dfrac{\delta \ell}{\delta \nu} = 0 \\[2ex] \dot{n} = \nu + un \end{cases}$$

- From now on, G is a group of diffeomorphisms over some open subset $\Omega \subset \mathbb{R}^d$.
- $\mathfrak{g}$ is equipped with a Hilbert norm $\|u_t\|_{\mathfrak{g}}$ with a continuous inclusion in $C_0^p(\Omega)$ (C^p vector fields vanishing at infinity) with the supremum norm and $p \geq 1$.

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- $\mathfrak{g}$ is equipped with a Hilbert norm $\|u_t\|_\mathfrak{g}$ with a continuous inclusion in $C^p_0(\Omega)$ (C^p vector fields vanishing at infinity) with the supremum norm and $p \geq 1$.
- Under this assumption, the flows associated to time-dependent v.f. that satisfy

$$\int_0^1 \|u_t\|^2_\mathfrak{g} dt < \infty$$

for a subgroup of $\mathrm{Diff}(\Omega)$ (Trouvé; Dupuis et al.).

Notation

- We will write the inner product in $\mathfrak{g}$ under the form $\left\langle u\,,\,v\right\rangle_m g = \left(L_{\mathfrak{g}} u \,|\, v\right)$ where $L_{\mathfrak{g}}$ is the duality operator from $\mathfrak{g}$ to $\mathfrak{g}^*$.
- Its inverse, a kernel operator, is denoted $K_{\mathfrak{g}}$.

Landmarks and Peakons

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- We have $\delta\ell/\delta u = 2L_{\mathfrak{g}} u$, and $(\delta\ell/\delta\nu) = (2/\sigma^2)(\nu_1, \dots, \nu_N)$.

- Let $n = (q_1, \ldots, q_N)$. We have

$$\frac{\delta \ell}{\delta \nu} \diamond n = -2 \sum_{k=1}^{Q} \frac{\nu_k}{\sigma^2} \otimes \delta_{q_k}.$$

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- Notation: if f is a vector field on $\mathbb{R}^d$ and μ a measure on $\mathbb{R}^d$,

$$\left(f \otimes \mu \,|\, w\right) = \int_{\mathbb{R}^d} f(x)^T w(x) d\mu. \quad (1)$$

- Euler equations for landmark metamorphosis (with $p_k = \nu_k/\sigma^2$)

$$\begin{cases} L_{\mathfrak{g}} u = \sum_{k=1}^{Q} p_k \otimes \delta_{q_k}. \\ \\ \frac{dp_k}{dt} + Du(q_k)^T p_k = 0, \quad k = 1, \ldots, Q \\ \\ \dot{q}_k = u(q_k) + \sigma^2 p_k, \quad k = 1, \ldots, Q \end{cases}$$

- The case $\sigma^2 = 0$ gives the singular solutions to EPDiff.

EPMorph/EPDiff Comparison with two 1-D peakons

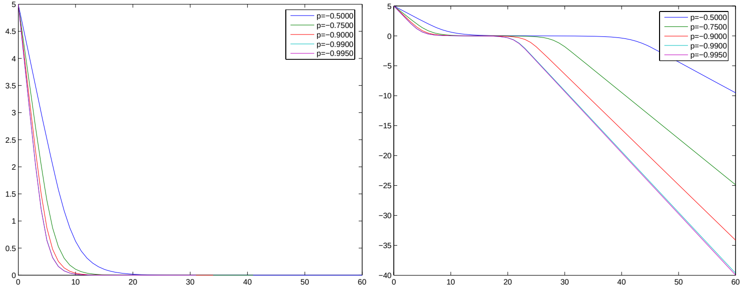

Figure: Head-on collision between two peakons. The plots show the evolution of $q = q_2 - q_1$ over time for several values of $p = p_2 - p_1$, for $\sigma^2 = 0$ (left) and $\sigma^2 = 10^{-4}$ (right)

Rem: Many simulations for peakons with EPdiff, e.g.: Holm and Staley, McLachlan and Marsland...

Head-on collisions

Head-on collision of two peakons; left: EPDIFF; right: metamorphoses

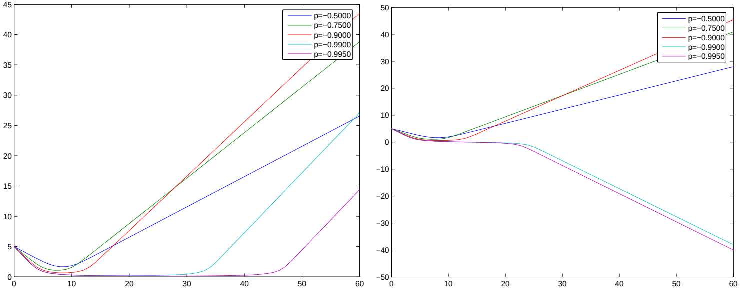

Figure: One peakon overtaking another. The plots show the evolution of $q = q_2 - q_1$ over time for several values of $p = p_2 - p_1$, for $\sigma^2 = 0$ (left) and $\sigma^2 = 0.05$ (right)

Back Collision

Left: EPDIFF; center: metamorphoses, low difference of energy; right: metamorphoses, high difference of energy.

Image Metamorphosis

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- Take
$$\ell(u, \nu) = \|u\|_{\mathfrak{g}}^2 + \frac{1}{\sigma^2}\|\nu\|_{L^2}^2.$$
- Denote $z = \nu/\sigma^2$. The Euler equations are
$$\begin{cases} L_{\mathfrak{g}} u = -z \nabla n \\ \dot{z} + \mathrm{div}(zu) = 0 \\ \dot{n} + \nabla n^T u = \sigma^2 z \end{cases}$$

Remark: H^1_α in 1-D

- Let $m = L_{\mathfrak{g}} u = (1 - \partial_x^2)u$, and $\rho = \sigma z$:

$$\partial_t m + u\partial_x m + 2m\partial_x u = -\rho\partial_x\rho \quad \text{with} \quad \partial_t\rho + \partial_x(\rho u) = 0$$

- This is equivalent to the compatibility for $d\lambda/dt = 0$ of

$$\begin{aligned} \partial_x^2\psi \;&+\; \Big(-\frac{1}{4} + m\lambda + \rho^2\lambda^2\Big)\psi = 0 \\ \partial_t\psi \;&=\; -\Big(\frac{1}{2\lambda} + u\Big)\partial_x\psi + \frac{1}{2}\psi\partial_x u \end{aligned}$$

- This model of image metamorphoses is a semi-direct product model.
- The conserved momentum is $(L_{\mathfrak{g}} u, z)$, yielding the equations $L_{\mathfrak{g}} u_t + z_t \nabla n_t = \mathrm{cst}$ and $z_t = \det(D g_t^{-1}) z_0 \circ g_t^{-1}$.

Examples

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Interpolation between two face images (different persons).

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- The equations are

$$\begin{cases} L_{\mathfrak{g}} u = n \nabla z \\ \dot{z} + \nabla z^T u = 0 \\ \dot{n} + \operatorname{div}(nu) = \sigma^2 z \end{cases}$$

- The conservation of momentum gives $L_{\mathfrak{g}} u + n \nabla z = \mathrm{cst}$ and $z = z_0 \circ g^{-1}$.
- The constant in the first equation vanishes for horizontal geodesics in $G Ⓢ N / G$.

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- Such structures are conveniently represented as measures, e.g.,

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- To apply (Riemannian) metamorphoses to measures, one needs to embed them in a Hilbert space.

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- We assume that $H \subset C^0$ so that for all $x \in \mathbb{R}^d$, $\exists K_x \in H$ such that, for all $f \in H$,

$$\left\langle K_x \, , \, f \right\rangle_H = f(x).$$

- The kernel $K_H(x, y) := K_x(y)$ satisfies the equation $\left\langle K_x \, , \, K_y \right\rangle_H = K_H(x, y)$.

- K_H provides an isometry between N and H via the relation $\eta \mapsto K_H \eta$ with $\left\langle K_H \eta\,,\, f \right\rangle_H = \left(\eta \,|\, f \right)$.
- In particular, the dual inner product on N is given by

$$\left\langle \eta\,,\, \tilde{\eta} \right\rangle_N = \left(\eta \,|\, K_H \tilde{\eta} \right).$$

- Define the action of $G \subset \mathrm{Diff}$ on N by

$$\big(g\eta \,|\, f\big) = \big(\eta \,|\, f \circ g\big),$$

with $g \in G, \eta \in N$ and $f \in H$.

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- For $\eta \in N$ and $w \in \mathfrak{g}$, and for all $f \in H$, $\big(w\eta \,|\, f\big) = \big(\eta \,|\, \nabla f^T w\big)$.
- We use the semi-direct product model and define the Lagrangian

$$\ell(u, \nu) = \frac{1}{2}\|u\|_{\mathfrak{g}}^2 + \frac{1}{2\sigma^2}\|\nu\|_N^2.$$

E-L Equations

- Let $f = \delta\ell/\delta\nu = (1/\sigma^2)K_H\nu$
- The E-L Equations are:

$$\begin{cases} L_{\mathfrak{g}}u = \nabla f \otimes u \\ \dot{f}_t + \nabla f^T u = 0 \\ \dot{n} - u.n = \sigma^2 L_H f \end{cases}$$

Integrated form

- In integrated form, this yields

$$\begin{cases} u = K_{\mathfrak{g}}(\nabla f \otimes n) \\ \\ n_t = g_t n_0 + \sigma^2 g_t \int_0^t g_s^{-1} K_H^{-1}(f_0 \circ g_s^{-1}) ds \end{cases}$$

with $\dot{g}_t = u_t \circ g_t$.

Initial Value Problem

Theorem

Under some conditions on H, f_0, n_0, for all $T > 0$, there exists a unique solution to the previous system over $[0, T]$ with initial conditions n_0 and f_0.

(The conditions essentially imply the loss of one derivative for n_0 and f_0: if $H \sim H^p$, then $f_0 \in H^{p+1}$ and $n_0 \in H^{-p+1}$.)

Boundary Value Problem

- The BVP requires to minimize, with fixed n_0 and n_1

$$E(u,n) := \int_0^1 \|u_t\|_{\mathfrak{g}}^2 dt + \frac{1}{\sigma^2}\int_0^1 \|\dot{n}_t - u_t n_t\|_N^2 dt.$$

Theorem

Under some conditions on H, for any given $n_0, n_1 \in H$, there exists a minimizer for the BVP.

Open issues

- Numerical schemes for measure metamorphoses. How “smooth” are geodesics?
- Extension of VPM methods to EPMorph?