The LANS-α model of ocean dynamics: fluid and numerical stability

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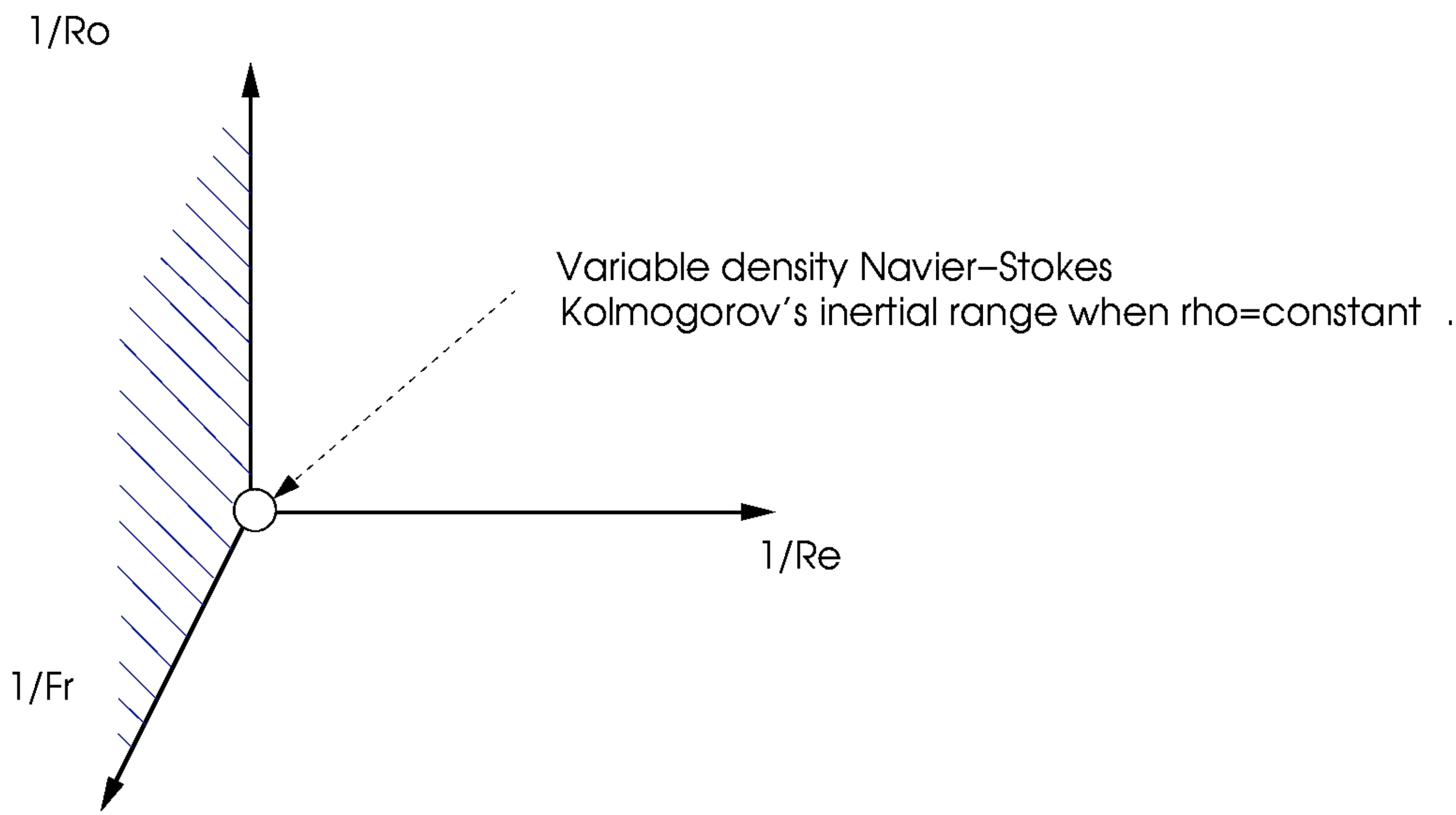
D. D. Holm and B. A. Wingate,,"Baroclinic Instabilities of the Two-Layer Quasigeostrophic Alpha Model", *Journal of Physical Oceanography*, **35**, 2005

B. A. Wingate, "The Maximum allowable time step for the shallow water alpha model and its relation to time-implicit differencing", *Monthly Weather Review*, **132**, 2004

Outline

1. Some key ideas in ocean Modeling
   a) How ocean dynamics is different from 3D Navier-Stokes
   b) Current theory - two layer turbulence and baroclinic instability
   c) Scott and Wang 2005 - satellite measurements to support the role of baroclinic instability
2. The two layer baroclinic instability
3. The numerical stability of the rotating shallow water equations

The differences between Rotating and Stratified flow and 3D Incompressible Navier Stokes



In the ocean we solve *reduced* forms of the Boussinesq equations.

$$\frac{D}{Dt}\mathbf{u} + \frac{1}{Ro}\widehat{\mathbf{z}} \times \mathbf{u} + \nabla\, p + \frac{1}{Fr}\tilde{\theta}\,\widehat{\mathbf{z}} = \frac{1}{Re}\Delta\mathbf{u}$$

$$\frac{D}{Dt}\tilde{\theta} - \frac{1}{Fr}\,w = \frac{1}{Pr Re}\Delta\tilde{\theta} \quad \text{with} \quad \nabla\cdot\mathbf{u} = 0$$

$$\frac{D}{Dt}\,q = \frac{1}{Re}\left(-\frac{1}{Fr}\Delta\omega_3 + (\nabla\tilde{\theta})\cdot\Delta\omega\right) + \frac{1}{Pr Re}\left(\frac{1}{Ro}\Delta\frac{\partial\tilde{\theta}}{\partial z} + \omega\cdot\Delta(\nabla\tilde{\theta})\right)$$

$$\boxed{q = \omega\cdot\nabla\tilde{\theta} + \frac{1}{Ro}\frac{\partial\tilde{\theta}}{\partial z} - \frac{1}{Fr}\omega_3}$$

For the Boussinesq equations, energy is independent of rotation and stratification. Potential enstrophy is not.

$$\frac{D}{Dt}\langle|\mathbf{u}|^2+\tilde{\theta}^2\rangle = \frac{1}{Re}\langle\nabla^2|\mathbf{u}|^2\rangle + \frac{1}{RePr}\langle\nabla^2\tilde{\theta}^2\rangle$$

$$\begin{aligned}\frac{D}{Dt}\langle|Q|\rangle = \quad & \frac{1}{Re}\left(\frac{1}{Fr}\langle q\ \nabla^2\omega_3\rangle - \langle q\ \nabla\tilde{\theta}\cdot\nabla^2\omega\rangle\right) - \\ & \frac{1}{RePr}\left(\frac{1}{Ro}\langle q\ \nabla^2\frac{\partial\tilde{\theta}}{\partial z}\rangle + \langle q\ \omega\cdot\nabla^2(\nabla\tilde{\theta})\rangle\right)\end{aligned}$$

$$\text{where} \qquad q = \omega\cdot\nabla\tilde{\theta} + \frac{1}{Ro}\frac{\partial\tilde{\theta}}{\partial z} - \frac{1}{Fr}\omega_3$$

$$\text{and} \qquad Q = 1/2q^2$$

The non-dimensional two-point correlation equation for q:

Ensemble averaging and assuming statistical homogeneity gives,

$$\begin{aligned}\frac{\partial \langle q\, q'\rangle}{\partial t} - \partial_{r_i}\langle q\, q'(u_i - u_i')\rangle &= Re^{-1}\partial_{r_i}\langle q\rho'\partial^2_{j'}\omega_i' - q'\rho\partial^2_j\omega_i\rangle \\ &+ (RePr)^{-1}\partial_{r_i}\langle q\omega'_{a_i}\partial^2_{j'}\tilde{\rho}' - q'\omega_{a_i}\partial^2_j\tilde{\rho}\rangle\end{aligned}$$

Where $q = \boldsymbol{\omega}\cdot\nabla\tilde{\rho} + Ro^{-1}\dfrac{\partial\tilde{\rho}}{\partial z} - Fr^{-1}\omega_3$. The primed quantities denote the variable writing a location $\boldsymbol{x}'$ while the unprimed variables indicate the variable is measured at location $\boldsymbol{x}$. The spatial increment is $\boldsymbol{r} = \boldsymbol{x}' - \boldsymbol{x}$.

- This equation is the physical space equivalent of the Herring *et al.* spectral space equation for potential enstrophy except that the above applies to all of parameter space.
- As Herring *et. al* pointed out, it is not clear there will be a separation of scales.

Limiting cases

Case	Ro	Fr	q	viscous-diffusion terms
i	$\frac{1}{\epsilon}\frac{N}{f}$	$\frac{1}{\epsilon}$	$\omega \cdot \nabla\tilde{\rho}$	$Re^{-1}\partial_{r_i}\langle(q\tilde{\rho}'\partial^2_{j'}\omega'_i - q'\tilde{\rho}\partial^2_j\omega_i)\rangle$
				$+(RePr)^{-1}\partial_{r_i}\langle(q\omega'_i\partial^2_{j'}\tilde{\rho}' - q'\omega_i\partial^2_j\tilde{\rho})\rangle$
ii	$\epsilon\frac{N}{f}$	ϵ	$\omega_3 - \frac{f}{N}\frac{\partial\tilde{\rho}}{\partial z}$	$Re^{-1}\partial^2_{r_j}\langle 2(\omega_3\omega'_3) - \frac{f}{N}(\frac{\partial\tilde{\rho}}{\partial z}\omega'_3 + \frac{\partial\tilde{\rho}'}{\partial z'}\omega_3)\rangle$
				$+(RePr)^{-1}\partial^2_{r_j}\langle\frac{f^2}{N^2}\frac{\partial\tilde{\rho}}{\partial z}\frac{\partial\tilde{\rho}'}{\partial z'} - \frac{f}{N}(\omega_3\frac{\partial\tilde{\rho}'}{\partial z'} + \omega'_3\frac{\partial\tilde{\rho}}{\partial z})\rangle$
iii	ϵ	$\mathcal{O}(1)$	$\frac{\partial\tilde{\rho}}{\partial z}$	$2(RePr)^{-1}\partial^2_{r_j}\langle qq'\rangle$
iv	$\mathcal{O}(1)$	ϵ	ω_3	$2Re^{-1}\partial^2_{r_j}\langle qq'\rangle$
v	$\frac{1}{\epsilon}$	$\mathcal{O}(1)$	$Fr^{-1}\omega_3 + \omega_i\partial_i\tilde{\rho}$	$Re^{-1}\partial_{r_i}\langle(q\rho'\partial^2_{j'}\omega'_i - q'\rho\partial^2_j\omega_i)\rangle$
				$+(RePr)^{-1}\partial_{r_i}\langle(q\omega'_i\partial^2_{j'}\tilde{\rho}' - q'\omega_i\partial^2_j\tilde{\rho})\rangle$
vi	$\mathcal{O}(1)$	$\frac{1}{\epsilon}$	$Ro^{-1}\frac{\partial\tilde{\rho}}{\partial z} + \omega_i\partial_i\tilde{\rho}$	$Re^{-1}\partial_{r_i}\langle(q\tilde{\rho}'\partial^2_{j'}\omega'_i - q'\tilde{\rho}\partial^2_j\omega_i)\rangle$
				$+(RePr)^{-1}\partial_{r_i}\langle(q\omega'_{a_i}\partial^2_{j'}\tilde{\rho}' - q'\omega_{a_i}\partial^2_j\tilde{\rho})\rangle$

TABLE 1. The form of q and the viscous-diffusion terms of (2.10) in various cases.

The role of potential enstrophy?

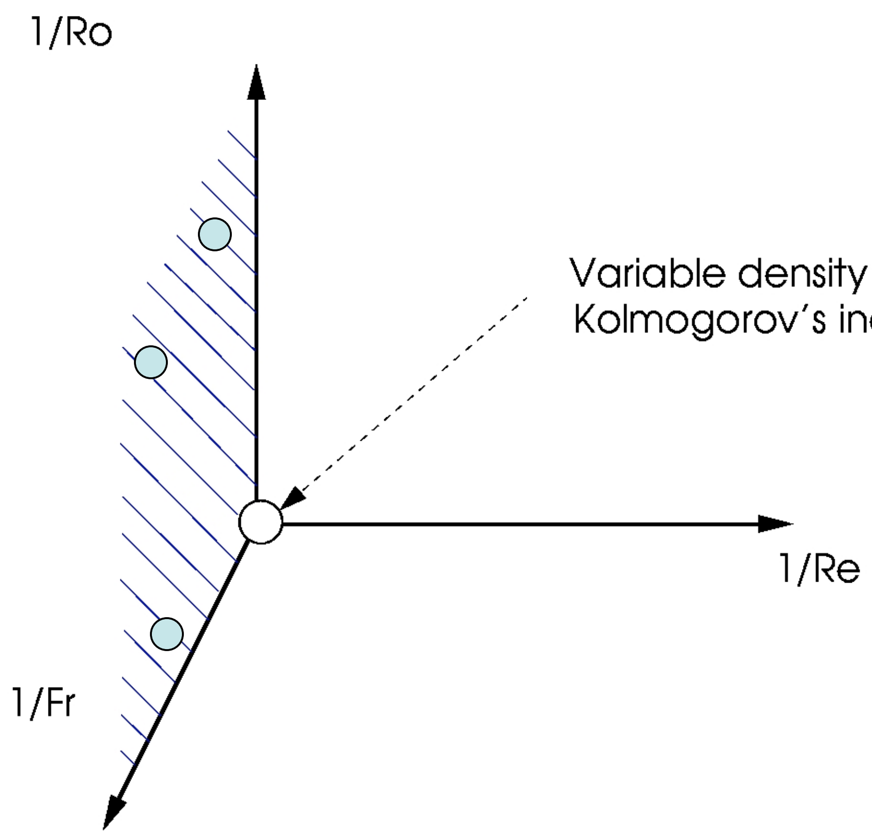
- By examining all of parameter space in physical space we derived an equation for the two-point, second-order correlation function of q.
- We examined six interesting limits in Ro and Fr. There is the possibility of an inertial range for 3 of these cases (ii-iv). For two of these case (i and ii) we found a "2/3 law" analogous to the 4/5 law of Kolmogorov.
- Currently verifying our new results with DNS.

$$\langle (\delta U(x,r))^3 \rangle = -\frac{4}{5}\epsilon r \quad \longrightarrow \quad E(k) = C\epsilon^{2/3}k^{-5/3}$$

$$\langle q(x)q(x+r)\big(u(x) - u(x+r)\big) \cdot \hat{r} \rangle = -\frac{2}{3}\epsilon_Q r \quad \longrightarrow \quad Q(k) = ??$$

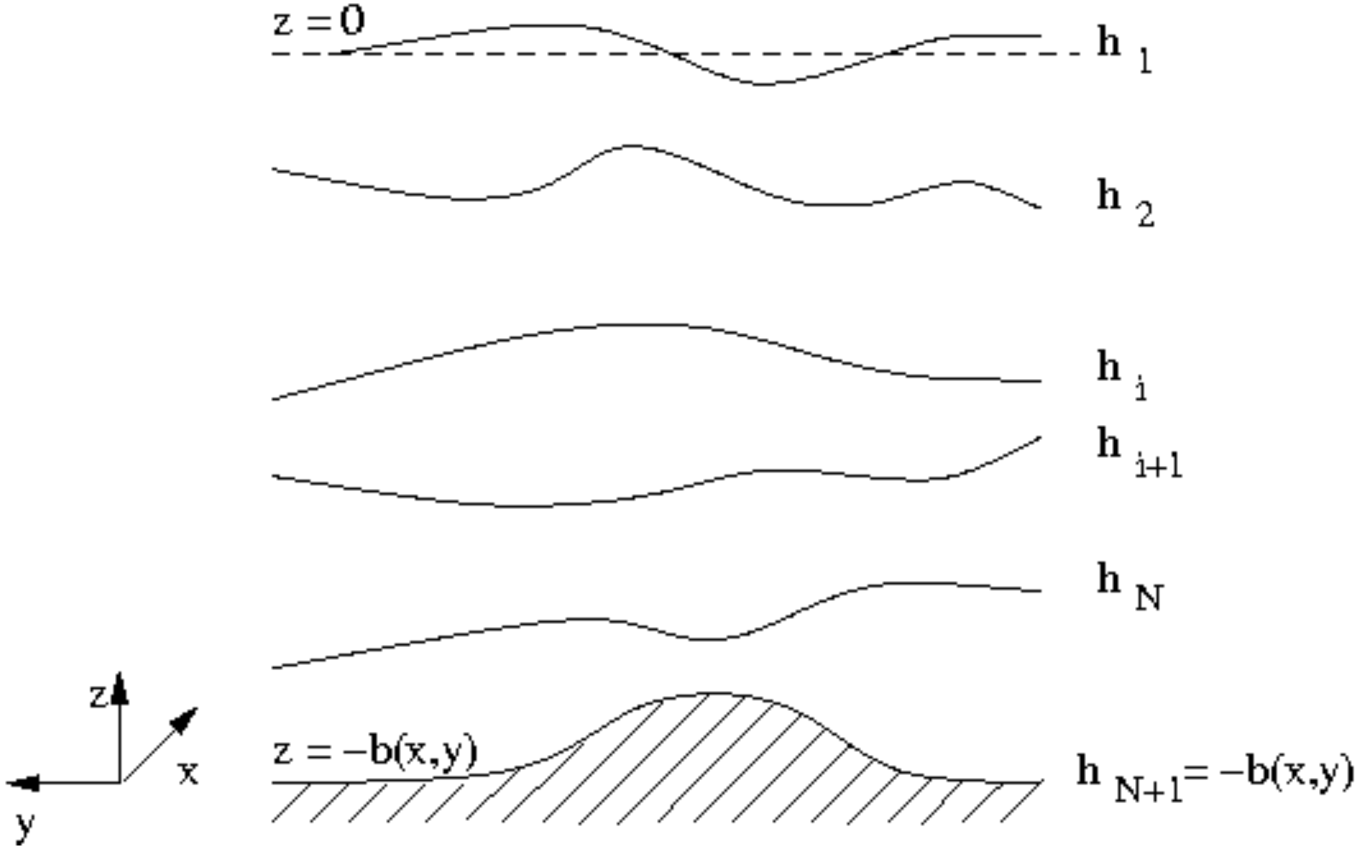
Kurien, S, Smith, L, Wingate, B. A. "On the two point correlation of potential vorticity in rotating and stratified flow", *Journal of Fluid Mechanics*, **255**, 2005

The differences between Rotating and Stratified flow and 3D Incompressible Navier Stokes



Wingate, B. A., Holmes, M, Kurien, S.K., Taylor, M "On potential enstrophy in 3 limits of the Boussinesq Equations", 2007

Layer Model QG-α



Layer Model QG-α

$$\frac{\partial q_i}{\partial t} + \mathbf{u}_i \cdot \nabla \; q_i = 0 \quad \text{with} \quad \mathbf{u}_i = \hat{\mathbf{z}} \times \nabla \psi_i$$

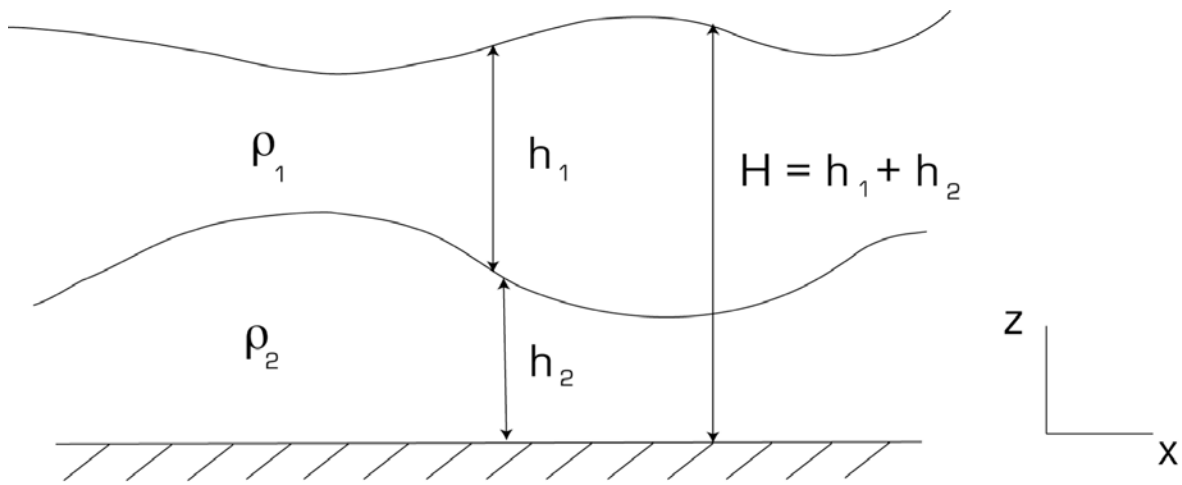
$$q_i = \nabla^2 (1 - \alpha^2 \nabla^2) \; \psi_i + F_i \sum_{j=1}^{N} T_{ij} \psi_j + f$$

$$\text{with} \quad F_i = \frac{f_o^2}{g' H_i^2}$$

$$\text{and} \qquad T = \begin{bmatrix} -1 & 1 & 0 & \ldots & \ldots \\ 1 & -2 & 1 & \ldots & \ldots \\ 0 & 1 & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots - 2 & 1 & \\ \ldots & \ldots & \ldots & 1 & -1 \end{bmatrix}$$

Some ideas about geostrophic turbulence - Salmon

Studied numerical solutions of two-layer Quasi-geostrophic Turbulence and used spectral analysis.



Salmon R, 1980, "Baroclinic instability and geostrophic turbulence", *Geophysical and Astrophysical Fluid Dynamics*, **15**, 167-211

Salmon R, 1989, *Lectures on Geophysical Fluid Dynamics,* Oxford University Press, 378 pp.

Some ideas about geostrophic turbulence - Salmon

For the ridged approximation in the QG limit Salmon finds there are two Rossby wave normal modes,

$$\psi \equiv \frac{F_2\psi_1 + F_1\psi_2}{F_1 + F_2}$$
$$\tau \equiv \frac{1}{2}\left(\psi_1 - \psi_2\right)$$

That obey,

$$\frac{\partial}{\partial t}\nabla^2\psi + \beta\psi_x = 0$$
$$\frac{\partial}{\partial t}(\nabla^2\tau - k_{int}^2\tau) + \beta\tau_x = 0$$

$$k_{int}^2 = F_1 + F_2$$
$$F_i = \frac{f_o^2}{g' H_i}$$

Some ideas about geostrophic turbulence - Salmon

Setting For the ridged approximation in the QG limit Salmon finds there are two Rossby wave normal modes,

$$\frac{\partial}{\partial t}\nabla^2\psi + J(\psi,\nabla^2\psi) + J(\tau,\nabla^2\tau) = 0$$
$$\frac{\partial}{\partial t}\nabla^2\tau + J(\psi,\nabla^2\tau) + J(\tau,\nabla^2\psi) = k_R^2\left(\frac{\partial\tau}{\partial t} + J(\psi,\tau)\right)$$

Some ideas about geostrophic turbulence - Salmon

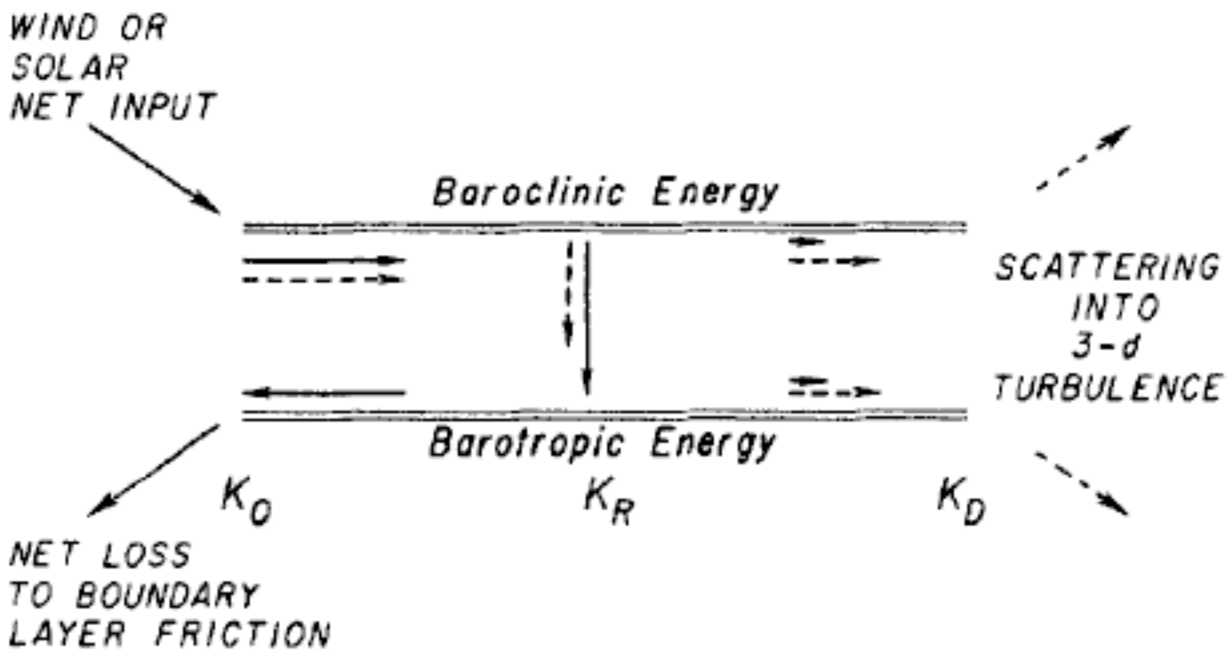
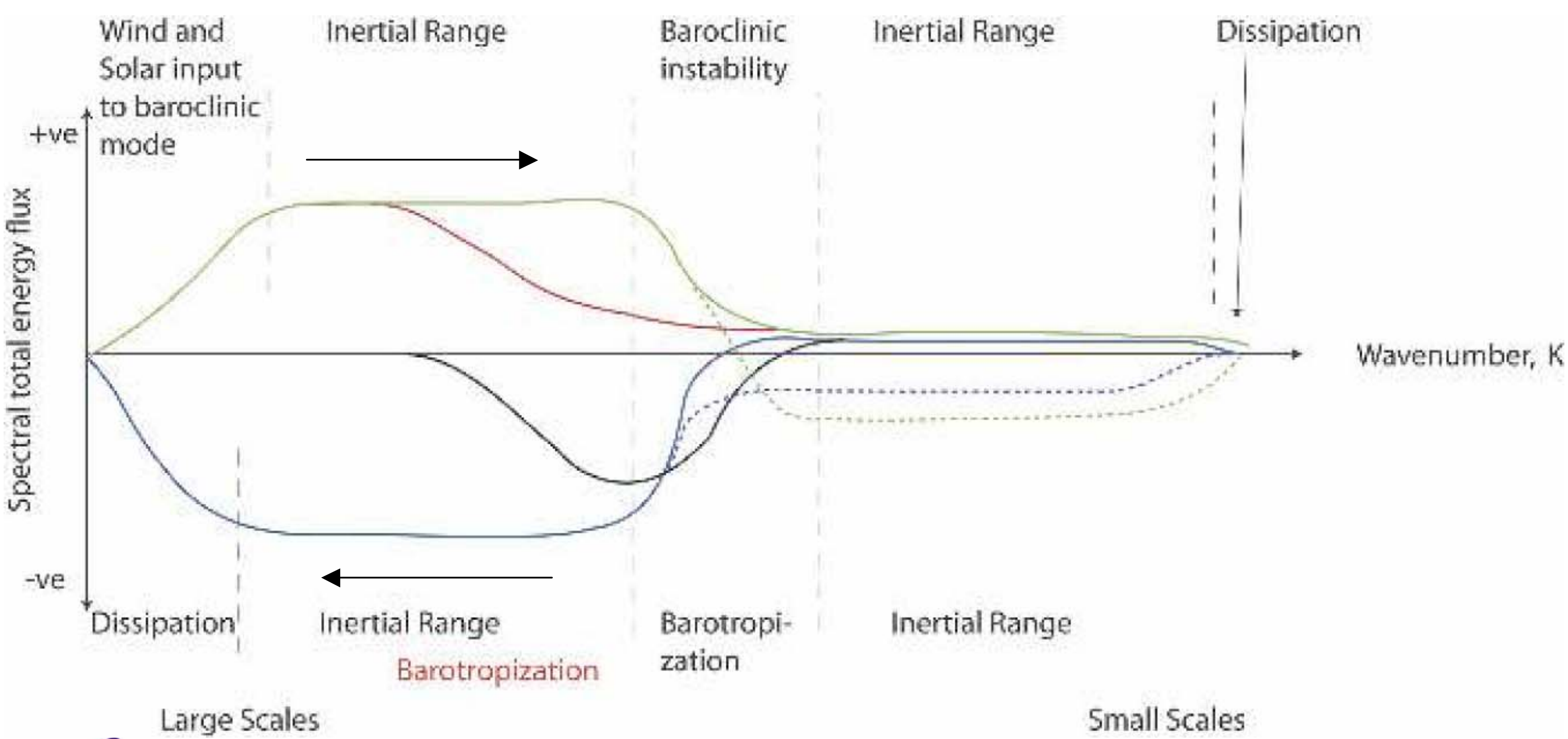


FIGURE 1 Energy (solid arrows) and sum potential enstrophy (dashed arrows) flow in wavenumber space. For an explanation, see the text.

Some ideas about geostrophic turbulence - Fu and Flierl 1982

- One of the main differences between the ocean and atmosphere, aside from horizontal boundaries, is that the stratification is surface intensified in the ocean.
- With realistic ocean profiles, F&F analyzed the first barotropic and the first 3 baroclinic modes of the Quasi-Geostrophic Equations (evolution of potential vorticity).
- The main difference between the uniform stratification case of Salmon is that the higher baroclinic modes transfer energy into the first baroclinic mode
- This energy then cascades toward the deformation radius, *then* transfers to the barotropic mode (as in Rhines 1977).
- All the barotropic energy then cascades to lower wave number (inverse cascade).
- Wave numbers higher than the deformation radius are different.

Scott & Wang 2005 - satellite measurements



Scott & Wang 2005 - spectral energy flux

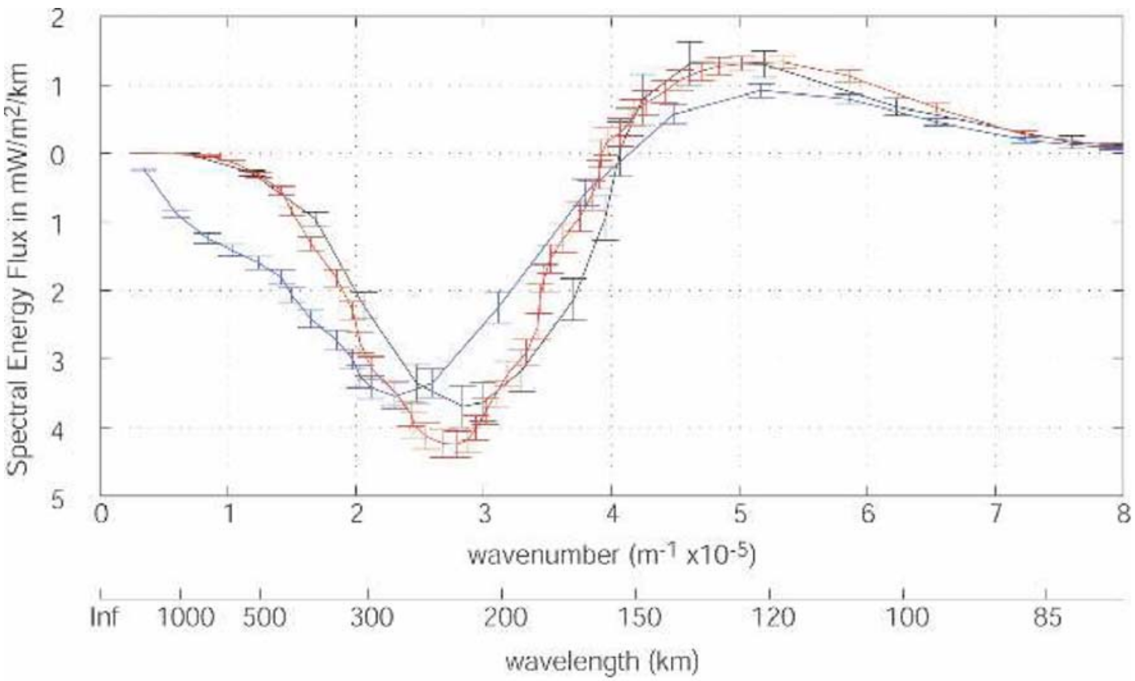


FIG. 2. Time mean, spectral kinetic energy flux $\Pi(K)$ vs total wavenumber K in a homogeneous ACC region (rectangles centered at 57 °S, 120°W): black curve using SSH on a 32 × 32 grid, red curve using SSH on a 64 × 64 grid, blue curve using velocity on a 64 × 64 grid. Positive slope reveals a source of energy. The larger negative lobe reveal s a net inverse cascade to lower wavenumber. Error bars represent standard error.

Scott & Wang 2005 - spectral energy flux - various longitudes.

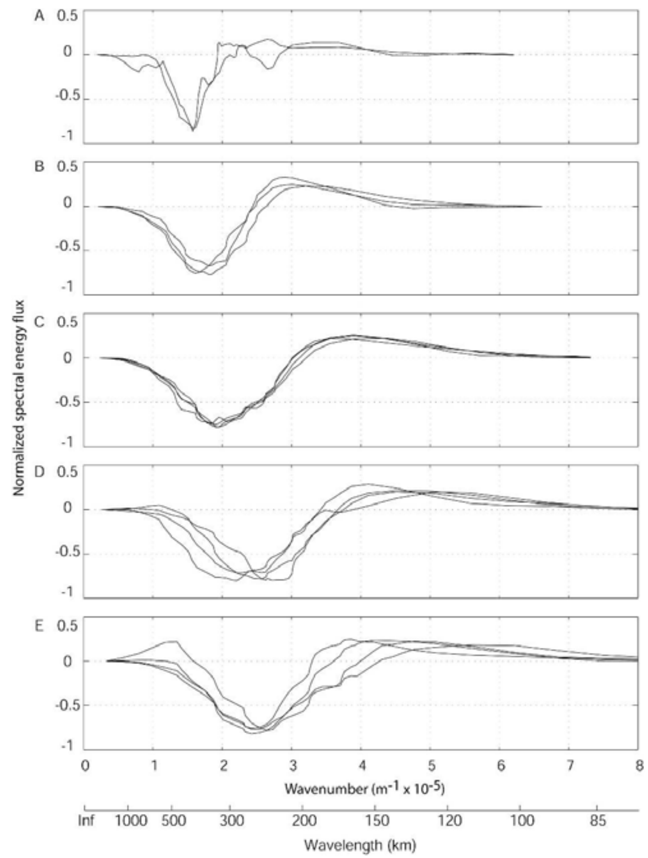
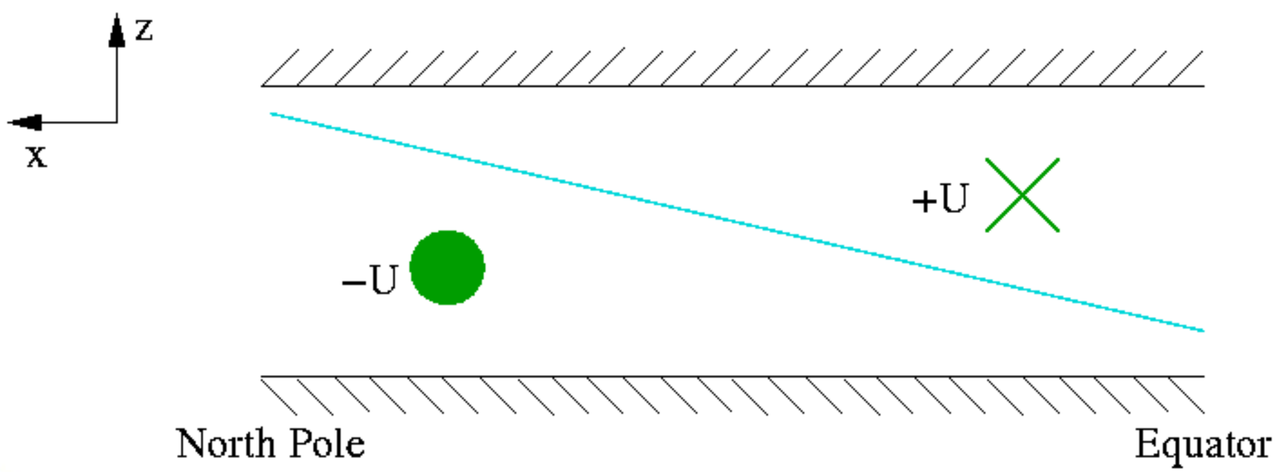


FIG. 4. Time-mean, spectral kinetic energy flux normalized by peak-to-peak amplitude $\overline{\Pi}'(K)$ vs total wavenumber K for various longitudes in the South Pacific. Latitude bands centered at (a) 15°, (b) 25°, (c) 35°, (d) 45°, and (e) 55°S. Curves were smoothed with a five-point binomial filter prior to normalization. Each curve represents an average over 64 × 64 grid points.

LANS-alpha preserves the fundamental fluid stability mechanism of baroclinic instability.

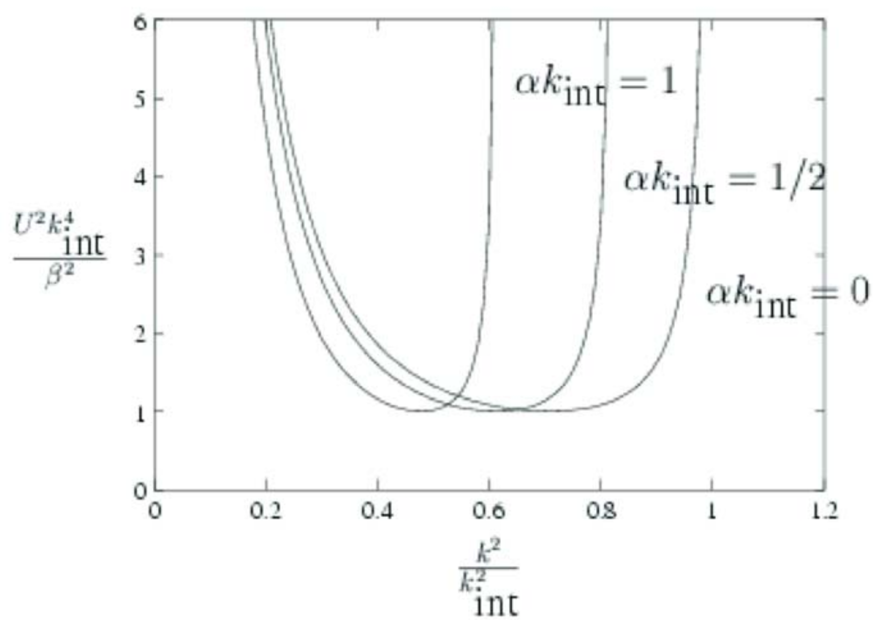
o Coarse resolution climate models do not resolve the Rossby deformation radius, the key length scale for resolving baroclinic instability. This instability converts available potential energy to kinetic energy and is responsible for creating eddies (oceanic weather). Even though our length scales are too large to resolve this phenomenon, any model we use to incorporate the effects of the small scales on the large needs to predict this important phenomenon.

o The QG-α model preserves the fundamental mechanism of baroclinic instability – gradients of the potential vorticity (PV). This leads to very different behavior between a LANS- α model and a model which dissipates PV.



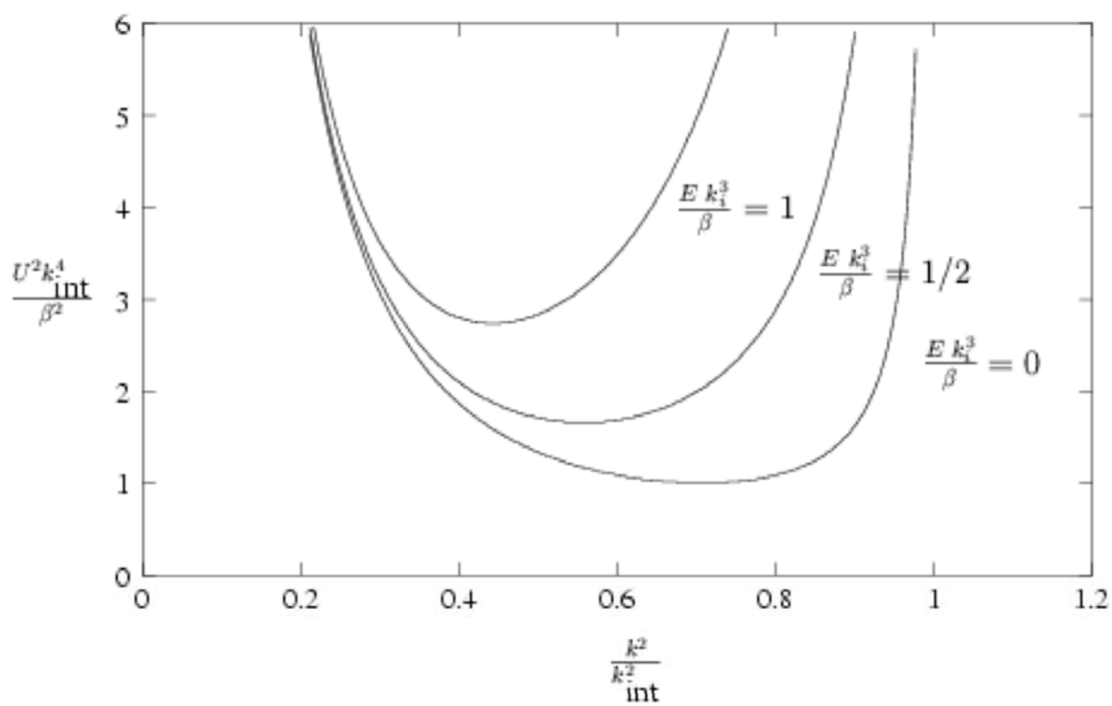
The LANS-alpha model preserves the fundamental mechanisms of baroclinic stability

- Baroclinic Instability is the fundamental process by which eddies (weather) are created in the ocean and atmosphere.
- The alpha model lowers the critical wave number, reduces the bandwidth of instability and preserves the value of forcing at onset in the baroclinic case.
- It also preserves the fundamental dependence of baroclinic instability on



Baroclinic Instability – dissipation of Potential Vorticity

- Dissipation of PV does not preserve the fundamental fluid structures or theorems. It uses damping to stabilize the solution.
- Dissipation of PV also moves the critical wave number to lower values, but requires higher physical forcing for the instability to occur. This is different than the LANS-alpha model who requires the same amount of forcing for instability to occur.



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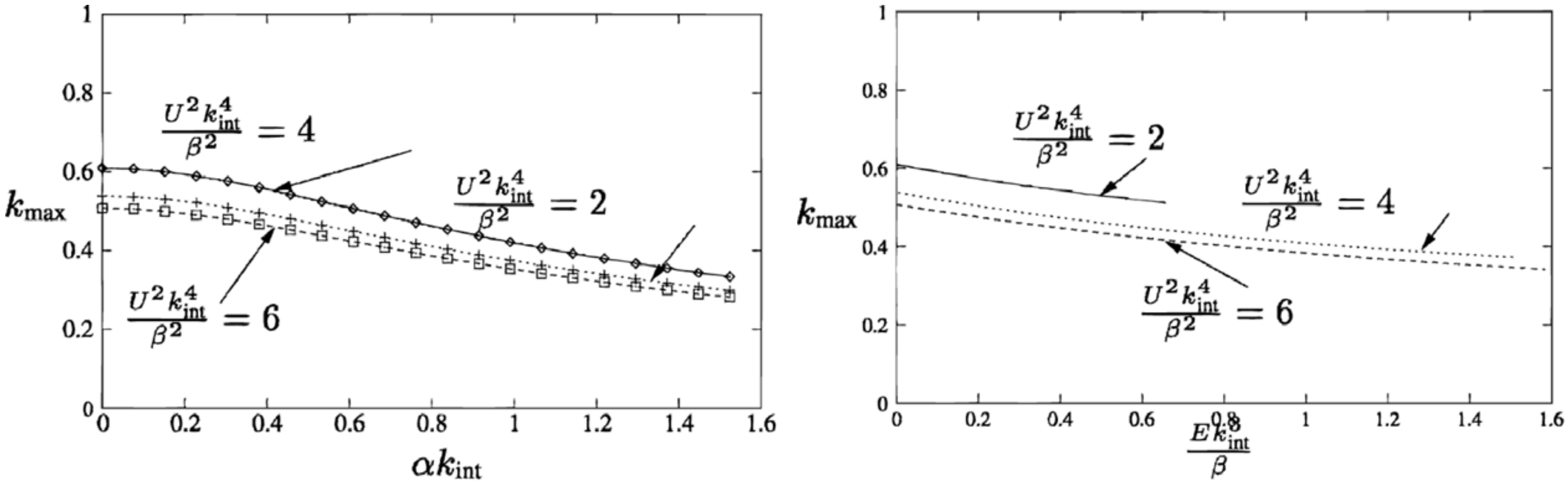
Wave number of the maximum growth rate



Growth rates of the baroclinic instability of the alpha model on the left and an eddy viscosity model on the right.

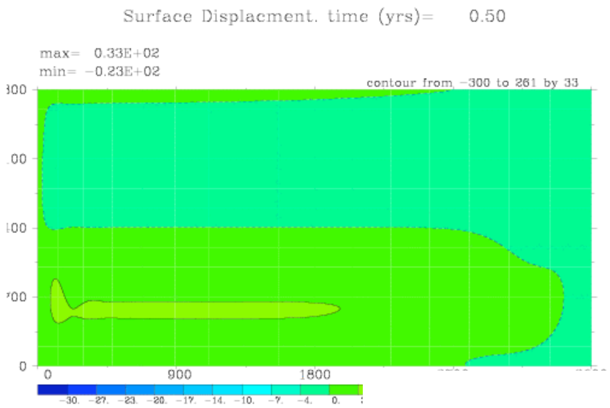
Baroclinic Instability – Nonlinear Stability

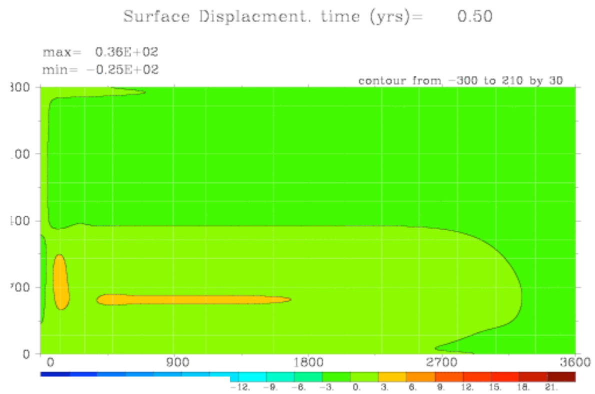
- Howard-Miles Theorem and formal stability analysis all hold, making the critical measure of stability or instability the gradient of Potential Vorticity. A necessary condition for instability is that the potential vorticity must be positive in some region and negative in another.
- Formal stability of the two-layer QG-α system says that a sufficient condition for stability is that grad-PV- α does not change sign.
- Thus the sufficient condition for stability is converse of the necessary condition for instability and both express their results in terms of the gradient of the potential vorticity.
- Lagrangian averaging preserves the potential vorticity, and therefore, the fundamental mechanism of baroclinic instability.
- These theorems do not exist for models which rely on dissipation of potential vorticity to account for the effect of the small scales on the large.

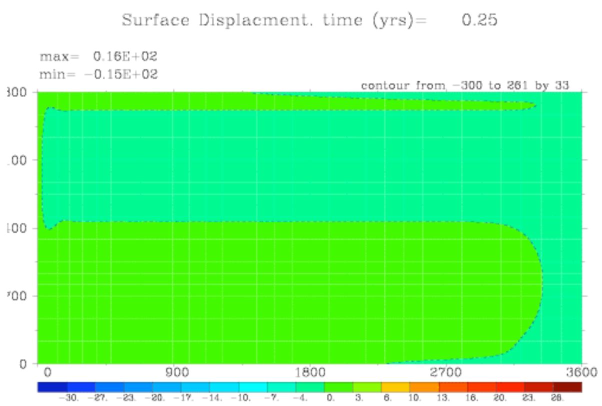
Summary of classical two-layer baroclinic instability

- The alpha model (and Leray too), moves baroclinic instability to a lower wave number, making the conversion of available potential to kinetic energy resolvable on a coarser mesh.
- Eddy viscosity has the same effect but requires a higher forcing to see the instability appear.
- The alpha model preserves the fundamental mechanism of baroclinic instability – that the potential vorticity must change sign somewhere in the domain. Leray??

Numerical stability - the maximum allowable time step and the rotating shallow water equations







Numerical stability - the maximum allowable time step

$$\frac{\partial}{\partial t}\,\mathbf{v} + \mathbf{u}\cdot\nabla\,\mathbf{v} + \nabla\,\mathbf{u}^T\cdot\,\,\mathbf{v} - \mathbf{f}\times\mathbf{u} + \nabla p = \nu\nabla(H\nabla\mathbf{v})$$

$$\frac{\partial\,h}{\partial t} + \nabla\cdot(H\mathbf{u}) = 0$$

where $\quad p = gh - \frac{1}{2}|\mathbf{u}|^2 - \frac{\alpha^2}{2}|\nabla\mathbf{u}|^2$

and $\quad H\mathbf{v} = (H - \alpha^2\nabla(H\nabla))\;\mathbf{u}$

Conserves total energy :

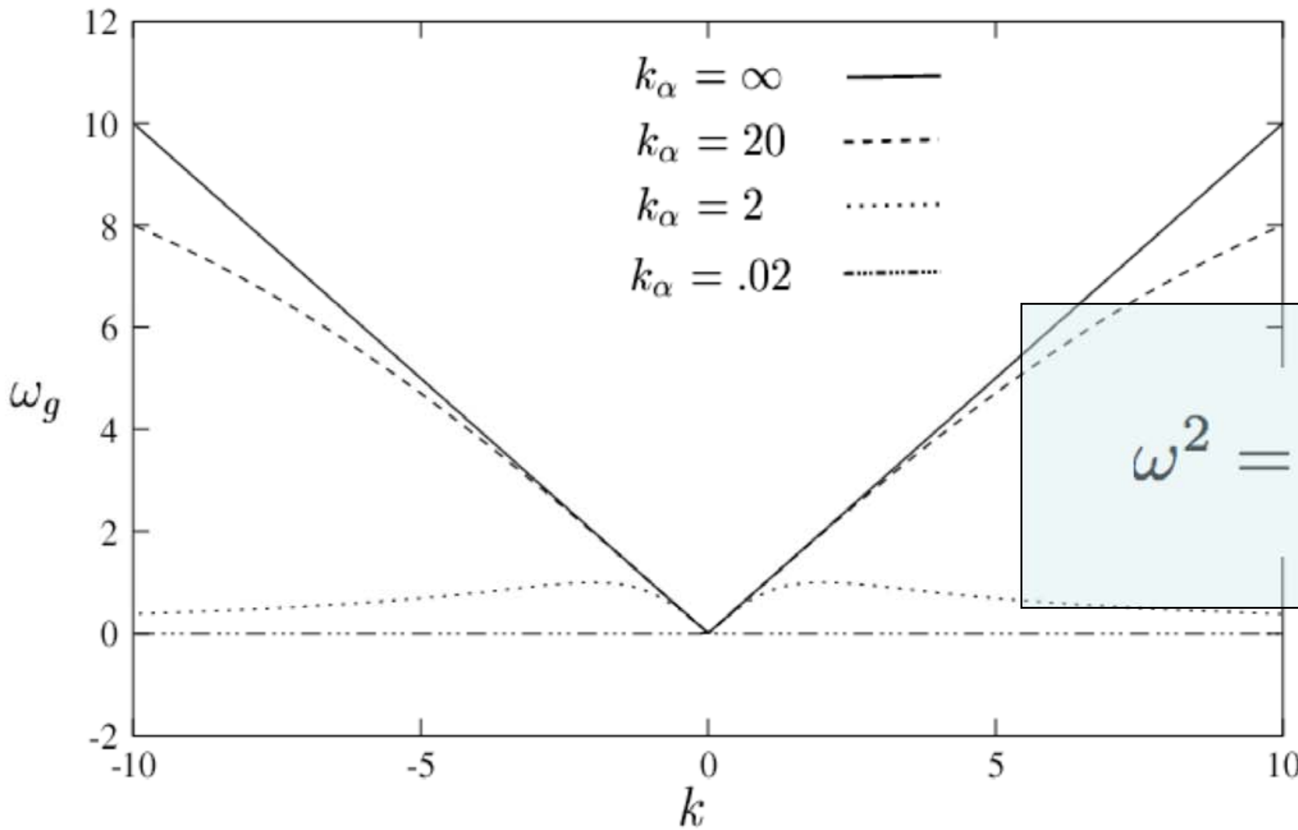
$$\boxed{\frac{1}{2}\int d^2x\;H(|\mathbf{u}^2 + \alpha^2|\nabla|^2) + gh^2 = 0}$$

And potential vorticity :

$$\boxed{\frac{d}{dt}\left(\frac{Q}{H}\right) = 0 \quad \text{where} \quad Q = \widehat{\mathbf{z}}\cdot\mathrm{curl}(\mathbf{v}+\mathbf{R})}$$

What does the LANS-alpha model do to the barotropic modes? It slows down the fastest gravity waves

Wingate, B. A. "The maximum allowable time step for the shallow water alpha model and its relation to time implicit differencing", *Mon. Wea. Rev, ,2004*



$$\omega^2 = \frac{k_h^2}{F^2(1+\alpha^2 k_h^2)}$$

An explicit formulation of the LANS-alpha model for the barotropic mode has the following maximum allowable time step

$$\frac{d\mathbf{U}}{dt} = L\mathbf{U}, \qquad \omega_g \Delta t \leq C.$$

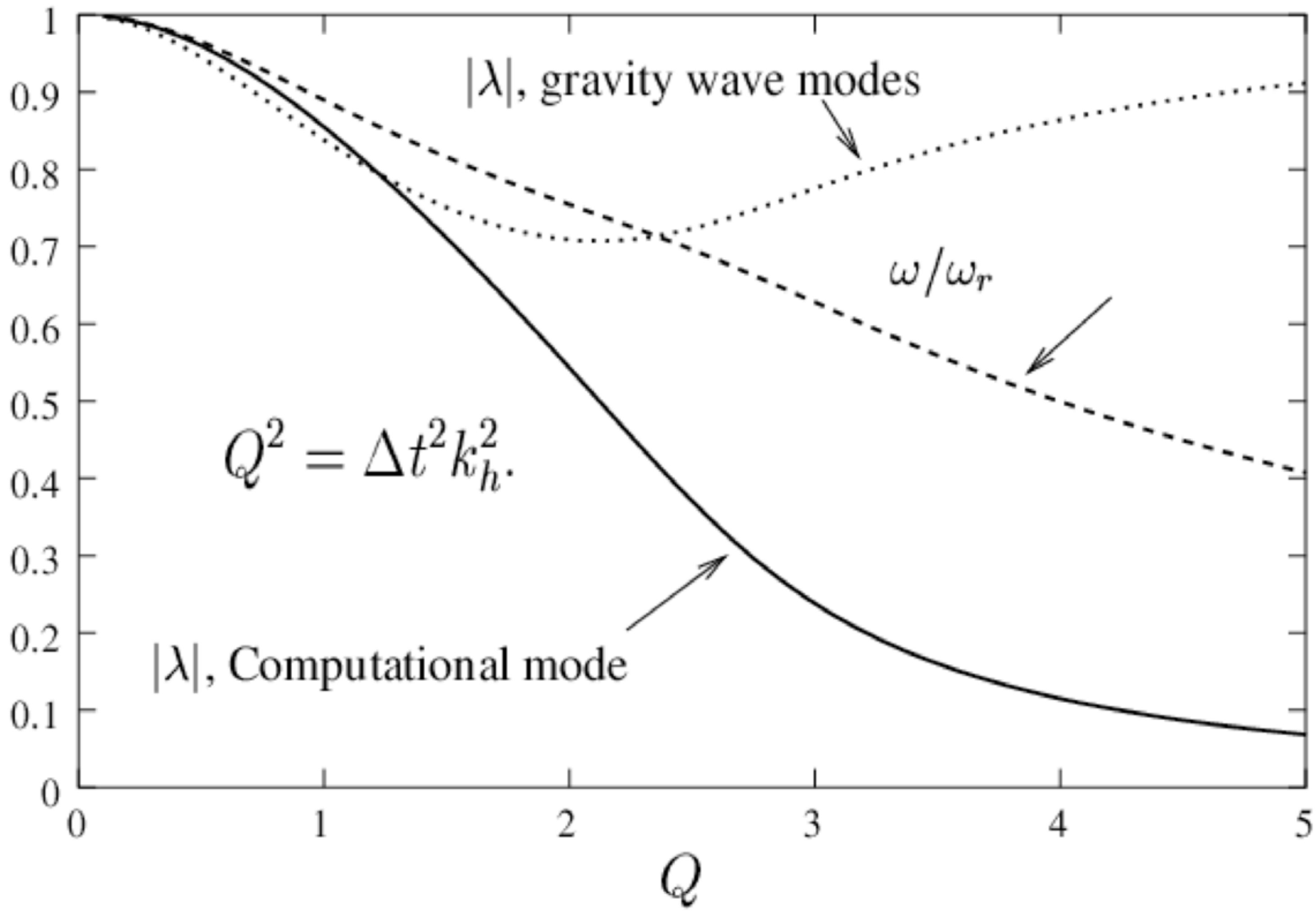
$$\begin{aligned}\Delta t &\leq \frac{C}{\omega_g}, \\ &\leq \frac{CF\sqrt{(1+2\alpha^2N^2)}}{\sqrt{2N^2}}.\end{aligned}$$

$N \to \infty,$

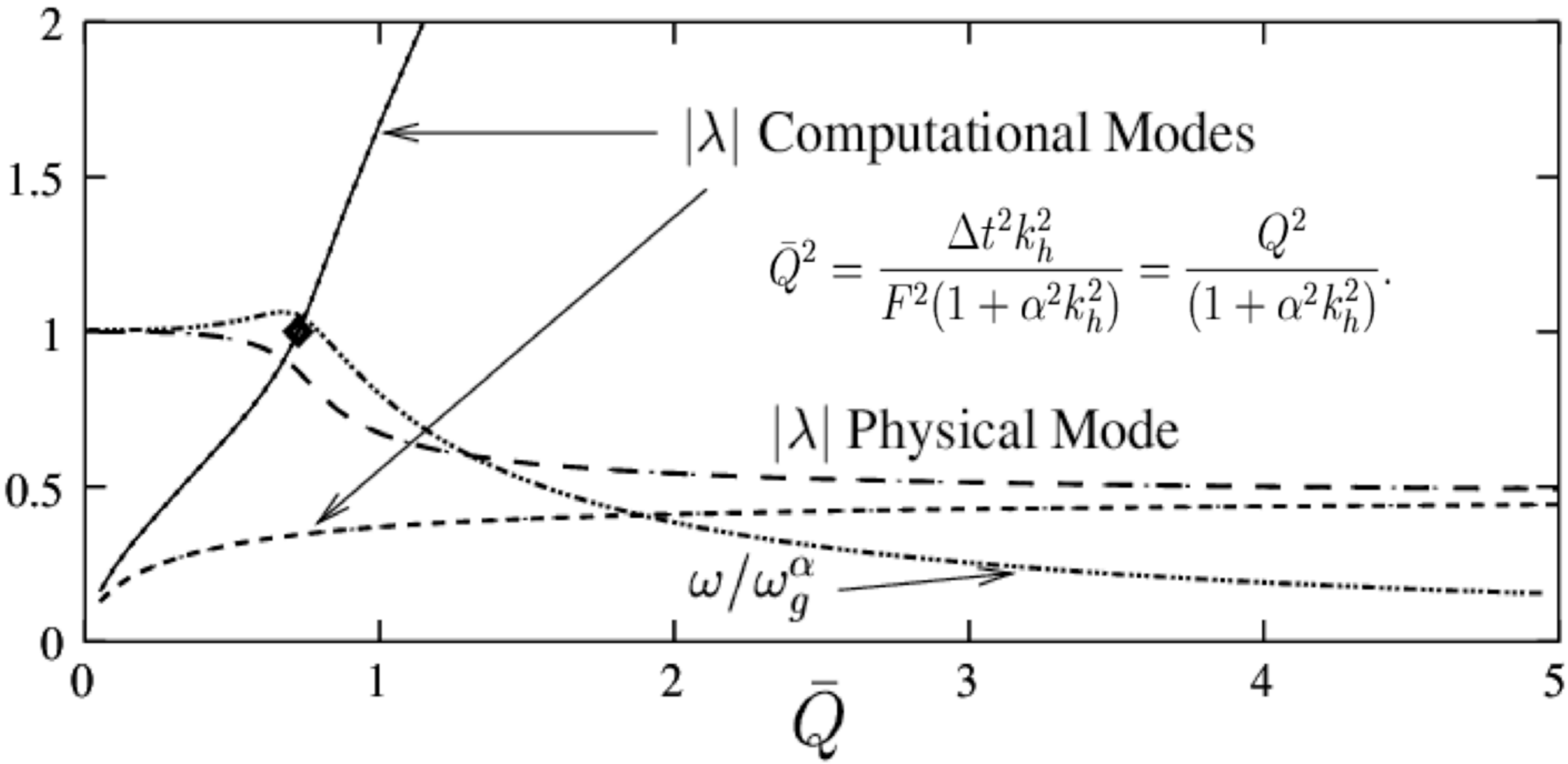
$$\alpha = 0 \quad \text{gives} \quad \Delta t \leq \frac{CF}{\sqrt{2}N}$$

$$\alpha \neq 0 \quad \text{gives} \quad \Delta t \leq CF\alpha.$$

The implicit method of DS94 slows down the fastest gravity waves and reduces the amplitude (damping)



An explicit in time formulation of the LANS-alpha model also reduces the phase speed of the fastest gravity waves, redefining the CFL criteria from Q to Qbar.



Barotropic equations: Do we need to do the splitting in the alpha model? This is a research topic.

- Implicit in time methods also lowers the frequency of the high wave numbers. Comparing the equations on the right and left we see the major difference between the LANS-alpha PDE and the implicit modified equation is damping.
- By examining the modified equation for linear gravity waves we can see a connection between what the implicit method for the barotropic mode is doing relative to what the alpha model will do.
- Research Topics: Does the alpha model already incorporate ideas that the implicit free surface method is addressing? Does this mean we can avoid splitting the equations into the barotropic and baroclinic equations?

Gravity wave pde forLANS-alpha model

$$\partial_t \ (1-\alpha^2\nabla^2)\bar{\delta} + \frac{1}{F^2}\nabla^2 h = 0,$$
$$\partial_t h + \bar{\delta} = 0,$$

Modified Equation of an Implicit method

$$\begin{aligned}\left(1-\frac{\Delta t^2}{4F^2}\nabla^2\right)\partial_t\delta + \frac{1}{F^2}\nabla^2 h &= \frac{\Delta t}{2F^2}\nabla^2\delta \\ \partial_t h + \delta &= 0.\end{aligned}$$

$$\alpha^2 = \frac{\Delta t^2}{4F^2}$$

What's Next?

- Mark Petersen next. Finally, the implementation of PE-α and PE-Leray into POP both for a baroclinic instability test problem and with first results in the North Atlantic.

- What about the splitting? Do we need to solve the barotropic mode separately with the POP-α? Or can we remove the splitting, which will take away one of the bottlenecks in the parallel performance of POP?

- Boundary Conditions.
  1. Irregular boundaries, C^0 most places.
  2. Variable alpha such that $\alpha \to 0$ at the boundaries?
  3. Use matching?