THE EULER-WEIL-PETERSSON EQUATIONS

Progress Report

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THE UNIVERSAL TEICHMÜLLER SPACE

Notation and Elementary Facts

$\widehat{\mathbb{C}}$ Riemann sphere, $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$, $\mathbb{D}^* := \{z \in \widehat{\mathbb{C}} \mid |z| > 1\}$

$L^1(\mathbb{D}^*) := \left\{ \phi : \mathbb{D}^* \to \mathbb{C} \;\middle|\; \|\phi\|_1 := \int_{\mathbb{D}^*} |\phi(z)| d^2 z < +\infty \right\}$ separable

$\left[L^1(\mathbb{D}^*)\right]^* \cong L^\infty(\mathbb{D}^*) := \{\phi : \mathbb{D}^* \to \mathbb{C} \mid \|\mu\|_\infty := \sup_{z \in \mathbb{D}^*} |\mu(z)| < +\infty\}$,
non-separable, isometry: $\forall L \in \left[L^1(\mathbb{D}^*)\right]^*, \exists! \mu \in L^\infty(\mathbb{D}^*)$ such that

$$L(\phi) = \int_{\mathbb{D}^*} \mu(z)\phi(z) d^2 z,$$

In Teichmüller theory, $\mu \in L^\infty(\mathbb{D}^*)$ **Beltrami differential on $\mathbb{D}^*$**.

$A_1(\mathbb{D}^*) = \left\{ \phi \in L^1(\mathbb{D}^*) \;\middle|\; \phi \text{ holomorphic} \right\} \subset L^1(\mathbb{D}^*)$ closed subspace

$[A_1(\mathbb{D}^*)]^* \cong L^\infty(\mathbb{D}^*)/\mathcal{N}(\mathbb{D}^*)$ as Banach spaces, where

$$\mathcal{N}(\mathbb{D}^*) := \left\{ \mu \in L^\infty(\mathbb{D}^*) \;\middle|\; \int_{\mathbb{D}^*} \mu(z)\phi(z) d^2z = 0,\, \forall \phi \in A_1(\mathbb{D}^*) \right\}$$

is the space of **infinitesimally trivial Beltrami differentials**.

Canonical splitting: $L^\infty(\mathbb{D}^*) = \mathcal{N}(\mathbb{D}^*) \oplus \Omega^{-1,1}(\mathbb{D}^*)$, where

$$\Omega^{-1,1}(\mathbb{D}^*) := \left\{ \mu \in L^\infty(\mathbb{D}^*) \;\middle|\; \mu(z) = (1 - |z|^2)^2 \overline{\phi(z)}, \right.$$
$$\left. \phi \text{ a holomorphic map on } \mathbb{D}^* \right\}$$

is the closed non separable Banach subspace in $L^\infty(\mathbb{D}^*)$ consisting of **harmonic Beltrami differentials on $\mathbb{D}^*$**. This decomposition identifies the Banach spaces $L^\infty(\mathbb{D}^*)/\mathcal{N}(\mathbb{D}^*)$ and $\Omega^{-1,1}(\mathbb{D}^*)$.

The duality pairing restricted to the closed subspace

$$\Omega_0^{-1,1}(\mathbb{D}^*) := \left\{ \mu \in \Omega^{-1,1}(\mathbb{D}^*) \;\middle|\; \lim_{|z| \to 1_+} \mu(z) = 0 \right\}$$

identifes $A_1(\mathbb{D}^*)$ with the dual space of $\Omega_0^{-1,1}(\mathbb{D}^*)$.

Summarizing, we have

$$\left[\Omega_0^{-1,1}(\mathbb{D}^*)\right]^* \cong A_1(\mathbb{D}^*), \qquad [A_1(\mathbb{D}^*)]^* \cong \Omega^{-1,1}(\mathbb{D}^*),$$
$$\left[\Omega_0^{-1,1}(\mathbb{D}^*)\right]^{**} \cong \Omega^{-1,1}(\mathbb{D}^*).$$

Define the non-separable complex Banach space

$$A_\infty(\mathbb{D}^*) := \left\{\phi \text{ holomorphic in } \mathbb{D}^* \;\middle|\; \sup_{z\in\mathbb{D}^*} |\phi(z)(1-|z|^2)^2| < \infty\right\}$$

and the closed subspace

$$A_\infty^0(\mathbb{D}^*) := \left\{\phi \in A_\infty(\mathbb{D}^*) \;\middle|\; \lim_{|z|\to 1_+} (1-|z|^2)^2\phi(z) = 0\right\}.$$

Then the harmonic Beltrami differentials are written in terms of $A_\infty(\mathbb{D}^*)$ as

$$\Omega^{-1,1}(\mathbb{D}^*) := \left\{\mu(z) = (1-|z|^2)^2\overline{\phi(z)} \;\middle|\; \phi \in A_\infty(\mathbb{D}^*)\right\}$$
$$\Omega_0^{-1,1}(\mathbb{D}^*) := \left\{\mu(z) = (1-|z|^2)^2\overline{\phi(z)} \;\middle|\; \phi \in A_\infty^0(\mathbb{D}^*)\right\}.$$

All these results remain valid when $\mathbb{D}^*$ is replaced by $\mathbb{D}$.

Quasiconformal Maps on the Disc

S^1 counterclockwise oriented unit circle. This orientation is the boundary orientation of the closed unit disc cl($\mathbb{D}$) which is oriented by giving a positively oriented basis of $\mathbb{R}^2$.

B_1 the unit open ball in $L^\infty(\mathbb{D})$.

- Let $\phi : \mathbb{D} \to \mathbb{D}$ be an orientation preserving homeomorphism that has all directional derivatives (in the sense of distributions) in $L^1_{\mathrm{loc}}(\mathbb{D})$. The ϕ is said to be **quasiconformal** if there is $\mu \in B_1$ such that

$$\partial_{\bar{z}}\phi = \mu \partial_z \phi.$$

This is called the **Beltrami equation** with coefficient μ.

- Any quasiconformal map extends to an orientation preserving homeomorphism of the closed disc cl($\mathbb{D}$).

The Universal Teichmüller Space and its Banach Manifold Structure

$\widehat{\mathbb{C}}$ Riemann sphere. Denote by B_1^* the unit open ball in $L^\infty(\mathbb{D}^*)$.

- **Model A**. Extend every $\mu \in B_1^*$ to $\mathbb{D}$ by the reflection

$$\mu(z) = \overline{\mu\left(\frac{1}{\bar{z}}\right)} \frac{z^2}{\bar{z}^2}, \; z \in \mathbb{D}.$$

Get a new map, also denoted by $\mu \in L^\infty(\mathbb{C})$. Let $\omega_\mu : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the unique solution of the Beltrami equation

$$\partial_{\bar{z}} \omega_\mu = \mu \partial_z \omega_\mu$$

which fixes $\pm 1, -i$. This ω_μ is obtained by applying the existence and uniqueness theorem of Ahlfors-Bers; it is a homeomorphism of $\widehat{\mathbb{C}}$ and

$$\omega_\mu(z) = \overline{\omega_\mu\left(\frac{1}{\bar{z}}\right)}$$

due to the reflection symmetry of μ. Hence, $S^1, \mathbb{D}$ and $\mathbb{D}^*$ are invariant under ω_μ.

• **Model B.** Extend every $\mu \in B_1^*$ to be zero outside $\mathbb{D}^*$. We denote by $\omega^\mu : \mathbb{C} \to \mathbb{C}$ the unique solution of the Beltrami equation

$$\partial_{\bar{z}} \omega^\mu = \mu \partial_z \omega^\mu,$$

satisfying the conditions $f(0) = 0, \partial_z f(0) = 1$, and $\partial_z^2 f(0) = 0$, where f is the holomorphic mapping $f := \omega^\mu|_{\mathbb{D}}$. This ω^μ is also obtained by existence and uniqueness theorem of Ahlfors-Bers; it is a homeomorphism of $\widehat{\mathbb{C}}$.

If $\mu, \nu \in B_1^*$ we have $\omega_\mu|_{S^1} = \omega_\nu|_{S^1} \Longleftrightarrow \omega^\mu|_{\mathbb{D}} = \omega^\nu|_{\mathbb{D}}$.

(1) We define the following equivalence relation on B_1^*:

$$\mu \sim \nu \Longleftrightarrow \omega_\mu|_{S^1} = \omega_\nu|_{S^1} \Longleftrightarrow \omega^\mu|_{\mathbb{D}} = \omega^\nu|_{\mathbb{D}}.$$

(2) The **universal Teichmüller** space is the quotient space:

$$T(1) := B_1^* / \sim .$$

(1) $T(1)$ has a unique structure of a complex Banach manifold such that the projection map

$$\pi : B_1^* \to T(1)$$

is a holomorphic submersion.
(2) The kernel of the tangent map $T_0\pi : L^\infty(\mathbb{D}^*) \to T_0T(1)$ is

$$\ker(T_0\pi) = \mathcal{N}(\mathbb{D}^*),$$

so the decomposition $L^\infty(\mathbb{D}^*) = \mathcal{N}(\mathbb{D}^*) \oplus \Omega^{-1,1}(\mathbb{D}^*)$ identifies the holomorphic tangent space $T_0T(1) = L^\infty(\mathbb{D}^*)/\mathcal{N}(\mathbb{D}^*)$ with the Banach space $\Omega^{-1,1}(\mathbb{D}^*)$.

The universal Teichmüller space $T(1)$ endowed with its complex Banach manifold structure is denoted by $T(1)^B$.

The manifold $T(1)^B$ is called universal Teichmüller space since it contains as complex submanifolds all the Teichmüller spaces of Fuchsian groups.

The Bers Embedding

The Bers embedding

$$\beta : T(1)^B \to A_\infty(\mathbb{D}), \qquad \beta([\mu]) := S\left(\omega^\mu|_{\mathbb{D}}\right),$$

is a biholomorphic mapping from $T(1)^B$ onto an open subset of $A_\infty(\mathbb{D})$; S denotes the Schwarzian derivative of a conformal map f,

$$S(f) = \frac{\partial_z^3 f}{\partial_z f} - \frac{3}{2}\left(\frac{\partial_z^2 f}{\partial_z f}\right)^2.$$

In particular, the tangent map $T_{[0]}\beta$ induces a isomorphism $T_{[0]}\beta : \Omega^{-1,1}(\mathbb{D}^*) \to A_\infty(\mathbb{D})$ of complex Banach spaces, given by

$$T_{[0]}\beta(\nu)(z) = -\frac{6}{\pi}\int_{\mathbb{D}^*} \frac{\nu(\zeta)}{(\zeta - z)^4} d^2\zeta$$

with inverse

$$T_{[0]}\beta^{-1}(\phi)(z) = -\frac{1}{2}(1 - |z|^2)^2 \phi\left(\frac{1}{\bar{z}}\right)\frac{1}{\bar{z}^4}$$

Quasisymmetric Homeomorphisms of the Circle

An orientation preserving homeomorphism η of S^1 is **quasisymmetric** if there is $M > 0$ such that for every x and every $|t| \leq \pi/2$

$$\frac{1}{M} \leq \frac{\eta(x+t) - \eta(x)}{\eta(x) - \eta(x-t)} \leq M.$$

Here we identify the homeomorphisms of the circle with the strictly increasing homeomorphisms of the real line satisfying the condition $\eta(x + 2\pi) = \eta(x) + 2\pi$. The set of all quasisymmetric homeomorphisms of the circle is denoted by QS(S^1); it is a group under the composition of maps. The link with the quasiconformal mappings on the disc is given by the

Beurling-Ahlfors Extension Theorem [1956]: An orientation preserving homeomorphism of the circle admits a quasiconformal extension to the disc if and only if it is quasisymmetric.

This extension is not unique.

It follows that $\omega_\mu|_{S^1}$ is a quasisymmetric homeomorphism of S^1, where ω_μ is a solution of the Beltrami equation for $\mu \in B_1^*$. Hence

$$\Phi : T(1) \longrightarrow \mathsf{QS}(S^1)_{\mathsf{fix}}, \qquad [\mu] \longmapsto \omega_\mu|_{S^1}, \qquad \text{is bijective,}$$

where $\mathsf{QS}(S^1)_{\mathsf{fix}} := \{\eta \in \mathsf{QS}(S^1) \mid \eta(\pm 1) = \pm 1, \eta(-i) = -i\}$

This bijection endows the group $\mathsf{QS}(S^1)_{\mathsf{fix}}$ with the structure of a complex Banach manifold by pushing forward this structure from $T(1)^B$. The resulting Banach manifold is denoted by $\mathsf{QS}(S^1)^B_{\mathsf{fix}}$. This bijection also endows the set $T(1)$ with a group structure by pulling back the group structure of $\mathsf{QS}(S^1)_{\mathsf{fix}}$. The multiplication is

$$[\nu]\cdot[\mu] = \left[\frac{\mu + (\nu \circ \omega_\mu) r_\mu}{1 + \bar{\mu}(\nu \circ \omega_\mu) r_\mu}\right], \quad \text{where} \quad r_\mu := \frac{\overline{\partial_z \omega_\mu}}{\partial_z \omega_\mu}.$$

Relative to the Banach manifold structure, the right translations $R_{[\mu]}$ are biholomorphic mappings for all $[\mu] \in T(1)$. Indeed, we have

$$R_{[\mu]} : [\nu] \in T(1)^B \longmapsto \left[\frac{\mu + (\nu \circ \omega_\mu) r_\mu}{1 + \bar{\mu}(\nu \circ \omega_\mu) r_\mu}\right] \in T(1)^B,$$

which depends holomorphically on the variable $[\nu]$. The left translations are not continuous, in general, therefore $T(1)^B$ is *not* a topological group.

Note that $\mathrm{QS}(S^1)_{\mathrm{fix}}$ can be identified with the quotient spaces $\mathrm{QS}(S^1)/\mathrm{PSU}(1,1)$ (or $\mathrm{PSU}(1,1)\backslash \mathrm{QS}(S^1)$). Indeed, given $\eta \in \mathrm{QS}(S^1)$, there exists only one $\gamma \in \mathrm{PSU}(1,1)$ such that $\eta \circ \gamma$ (or $\gamma \circ \eta$) fixes the points ± 1 and $-i$. Note that the projections

$$\mathrm{QS}(S^1) \to \mathrm{QS}(S^1)/\mathrm{PSU}(1,1) \quad \text{and} \quad \mathrm{QS}(S^1) \to \mathrm{PSU}(1,1)\backslash \mathrm{QS}(S^1)$$

are not group homomorphisms, when the quotient space is endowed with the group structure of $\mathrm{QS}(S^1)_{\mathrm{fix}}$.

$T_e\,\mathrm{QS}(S^1)^B_{\mathrm{fix}}$ is given by the vector fields u on the circle belonging to the **Zygmund space**, which is defined to be

$$Z(S^1) = \Big\{ u \in C^0(S^1, \mathbb{R}) \,\Big|\, \text{there is a } C \text{ such that} \\ |u(x+t) + u(x-t) - 2u(x)| \le C|t| \text{ for all } x, t \in S^1 \Big\},$$

and verifying the condition $u(\pm 1) = u(-i) = 0$. In particular $Z(S^1)$ contains the vector fields whose flows are quasisymmetric homeomorphisms (Reimann). Here the continuous vector fields u on the circle are identified with continuous 2π-periodic functions on the real line.

The isomorphism between $T_e\,\mathrm{QS}(S^1)^B_{\mathrm{fix}}$ and the model Banach space $A_\infty(\mathbb{D})$ of $T(1)^B$ is given by taking the tangent map at e to the map

$$\mathrm{QS}(S^1)^B_{\mathrm{fix}} \xrightarrow{\Phi^{-1}} T(1)^B \xrightarrow{\beta} A_\infty(\mathbb{D}),$$

where β denotes the Bers embedding. This isomorphism is

$$\sum_{n \in \mathbb{Z}} u_n e^{inx} \in T_e\,\mathrm{QS}(S^1)^B_{\mathrm{fix}} \quad \stackrel{\sim}{\longmapsto} \quad i \sum_{n \ge 2} (n^3 - n) u_n z^{n-2} \in A_\infty(\mathbb{D}).$$

Known: for all $s < 1$ we have the inclusion $T_e \mathrm{QS}(S^1)_{\mathrm{fix}} \subset H^s(S^1)$,

$$H^s(S^1) := \left\{ u(x) = \sum_{n\in\mathbb{Z}} u_n e^{inx} \,\middle|\, u_{-n} = \overline{u_n} \text{ and } \sum_{n\in\mathbb{Z}} |n|^{2s}|u_n|^2 < \infty \right\},$$

is the space of Sobolev class H^s real valued maps on S^1.

Using the Banach manifold structure of $\mathrm{QS}(S^1)^B_{\mathrm{fix}}$, it is possible to endow the whole group $\mathrm{QS}(S^1)$ with a real Banach manifold structure, by assuming that the bijection

$$\Psi : \mathrm{QS}(S^1) \longrightarrow \mathrm{PSU}(1,1) \times \mathrm{QS}(S^1)^B_{\mathrm{fix}},$$

defined by the condition

$$\Psi(\eta) = (\widehat{\eta}, \eta_0) \Longleftrightarrow \eta = \widehat{\eta} \circ \eta_0,$$

is a diffeomorphism. The group $\mathrm{QS}(S^1)$ endowed with this Banach manifold structure is denoted by $\mathrm{QS}(S^1)^B$.

The choice of an other subgroup fixing three points does not change the Banach manifold structure on $\mathrm{QS}(S^1)$, because of:

Let $\mathsf{QS}(S^1)_1 = \{\eta \in \mathsf{QS}(S^1) \mid \eta$ fixes three points$\}$. Then $\mathsf{QS}(S^1)_1$ can be endowed with a Banach manifold structure in the same way as $\mathsf{QS}(S^1)_{\mathsf{fix}}$. The bijection

$$\mathsf{PSU}(1,1) \times \mathsf{QS}(S^1)^B_{\mathsf{fix}} \to \mathsf{PSU}(1,1) \times \mathsf{QS}(S^1)^B_1$$

$$(\gamma_0, \eta_0) \mapsto (\gamma_1, \eta_1), \quad \text{such that} \quad \gamma_0 \circ \eta_0 = \gamma_1 \circ \eta_1,$$

is a smooth diffeomorphism. (We do not know if this is new.)

Hence: the identifications of $\mathsf{QS}(S^1)$ with $\mathsf{PSU}(1,1) \times \mathsf{QS}(S^1)^B_{\mathsf{fix}}$ or $\mathsf{PSU}(1,1) \times \mathsf{QS}(S^1)^B_1$ gives the same Banach manifold structure.

Properties of $\mathsf{QS}(S^1)^B$**:** The tangent space at the identity to the real Banach manifold $\mathsf{QS}(S^1)^B$ is the Zygmund space $Z(S^1)$. The group $\mathsf{QS}(S^1)^B$ is not a topological group but the right translations are smooth; it contains the subgroup $\mathsf{QS}(S^1)^B_{\mathsf{fix}}$ as a closed submanifold of codimension 3. (We do not know if this is new.)

Symmetric Homeomorphisms of the Circle

An orientation preserving homeomorphism η of S^1 is **symmetric** if there is a continuous function $\epsilon(t)$, with $\epsilon(t) \stackrel{t\to 0}{\longrightarrow} 0$, such that for every x and every $|t| \leq \pi/2$ we have

$$\frac{1}{1+\epsilon(t)} \leq \frac{|\,\eta(x+t)-\eta(x)|}{|\,\eta(x)-\eta(x-t)|} \leq 1+\epsilon(t).$$

The set of all symmetric homeomorphisms $\mathsf{S}(S^1)$ contains the group $\mathsf{Diff}_+^{C^1}(S^1)$ of all orientation preserving C^1 diffeomorphisms of the circle and is a subgroup of $\mathsf{QS}(S^1)$.

Define $\mathsf{S}(S^1)_{\mathsf{fix}} := \{\eta \in \mathsf{S}(S^1) \mid \eta(\pm 1) = \pm 1, \eta(-i) = -i\}$ which is a subgroup of $\mathsf{S}(S^1)$. It is known that the embedding

$$\beta \circ \Phi^{-1} : \mathsf{QS}(S^1)^B_{\mathsf{fix}} \longrightarrow A_\infty(\mathbb{D})$$

restricts to an injective map $\mathsf{S}(S^1)_{\mathsf{fix}} \longrightarrow A^0_\infty(\mathbb{D})$, making the group $\mathsf{S}(S^1)_{\mathsf{fix}}$ into a smooth Banach submanifold $\mathsf{S}(S^1)^B_{\mathsf{fix}}$ of $\mathsf{QS}(S^1)^B_{\mathsf{fix}}$.

• $T_e\,\mathsf{S}(S^1)^B_{\mathsf{fix}}$ consists of vector fields u on the circle belonging to the subspace $Z_{\mathsf{sym}}(S^1)_0 := \{u \in Z_{\mathsf{sym}}(S^1) \mid u(\pm 1) = u(-i) = 0\}$, where

$$Z_{\mathsf{sym}}(S^1) = \left\{ u \in Z(S^1) \,\middle|\, |u(x+t) + u(x-t) - 2u(x)| \le \epsilon(t)|t|,\ \forall x, t \in S^1 \right\}$$

for $\epsilon(t)$ is independent of x and $\epsilon(t) \overset{t \to 0}{\longrightarrow} 0$.

• The group $\mathsf{S}(S^1)_{\mathsf{fix}}$ is the closure in $\mathsf{QS}(S^1)^B_{\mathsf{fix}}$ of the subgroup $\mathsf{Diff}_+(S^1)_{\mathsf{fix}}$ consisting of orientation preserving smooth diffeomorphisms of the circle fixing the points ± 1 and $-i$.

$\mathsf{S}(S^1)$ becomes a real Banach manifold declaring that the bijection

$$\Psi : \mathsf{S}(S^1) \longrightarrow \mathsf{PSU}(1,1) \times \mathsf{S}(S^1)^B_{\mathsf{fix}},$$

$$\Psi(\eta) = (\widehat{\eta}, \eta_0) \Longleftrightarrow \eta = \widehat{\eta} \circ \eta_0,$$

is a diffeomorphism. The group $\mathsf{S}(S^1)$ endowed with this Banach manifold structure is denoted by $\mathsf{S}(S^1)^B$. As before, we have

$\mathsf{S}(S^1)^B$ is a closed real Banach submanifold of $\mathsf{QS}(S^1)^B$ and a topological group with smooth right translations. It contains the subgroup $\mathsf{S}(S^1)^B_{\mathsf{fix}}$ as a closed submanifold of codimension 3 and is the closure in $\mathsf{QS}(S^1)^B$ of the subgroup $\mathsf{Diff}_+(S^1)$. $T_e\mathsf{S}(S^1)^B = Z_{\mathsf{sym}}(S^1)$, a closed spubspace of $Z(S^1)$.

Summary

$$\mathsf{Diff}_+(S^1) \subset \mathsf{Diff}^s_+(S^1) \subset \mathsf{Diff}^{C^1}_+(S^1) \subset \mathsf{S}(S^1) \subset \mathsf{QS}(S^1),$$

for all $s > 3/2$. The differential properties are the following:

- $\mathsf{Diff}_+(S^1)$ is a C^∞ Fréchet Lie group.

- $\mathsf{Diff}^s_+(S^1)$ denotes the group of all orientation preserving Sobolev class H^s diffeomorphisms with the H^s Hilbert manifold structure (which is possible for $s > 3/2$).

- $\mathsf{Diff}^{C^1}_+(S^1)$ is endowed with the C^1 Banach manifold structure.

• All these manifold structures are real and not complex.

• $\mathrm{Diff}^s_+(S^1)$, $\mathrm{Diff}^{C^1}_+(S^1)$, $\mathrm{S}(S^1)$ are topological groups with smooth right translations.

• $\mathrm{QS}(S^1)$ has smooth right translations; *not* a topological group.

• All inclusions are smooth. The three first inclusions have dense range. The last inclusion is a weak submanifold inclusion.

• Curious fact: The tangent space to $\mathrm{QS}^1(S^1)^B$ is the bidual space of the tangent space to its weak submanifold $\mathrm{S}(S^1)^B$.

Same differential properties for the subgroups fixing ± 1 and $-i$:

$$\mathrm{Diff}_+(S^1)_0 \subset \mathrm{Diff}^s_+(S^1)_0 \subset \mathrm{Diff}^{C^1}_+(S^1)_0 \subset \mathrm{S}(S^1)_0 \subset \mathrm{QS}(S^1)_0,$$

for $s > 3/2$. They are, in addition, complex manifolds.

The tangent spaces at e to these subgroups are obtained by requiring that $u(\pm 1) = u(-i) = 0$ for the elements of the tangent vectors at e to the corresponding large groups. They will be denoted by:

$$\mathfrak{g}^{\infty} \subset \mathfrak{g}^{s} \subset \mathfrak{g}^{C^1} \subset \mathfrak{g}^{S} \subset \mathfrak{g}^{QS}.$$

Another realization: impose $u_{-1} = u_0 = u_1 = 0$ in the Fourier coefficients. This corresponds to think of these subgroups as quotients of the corresponding groups by the Möbius group PSU$(1,1)$; so the vector fields are taken modulo $\mathfrak{psu}(1,1)$. Notation:

$$\mathfrak{h}^{\infty} \subset \mathfrak{h}^{s} \subset \mathfrak{h}^{C^1} \subset \mathfrak{h}^{S} \subset \mathfrak{h}^{QS}.$$

Isomorphism between the two interpretations: restrict isomorphism

$$T : u \in \left\{ u \in C^0(S^1, \mathbb{R}) \mid u(\pm 1) = u(-i) = 0 \right\} \longmapsto [u] \in C^0(S^1, \mathbb{R})/\mathfrak{psu}(1,1)$$

to the $\mathfrak{g}$'s. Since $\mathfrak{psu}(1,1) = \left\{ \bar{c}e^{-ix} + b + ce^{ix} \mid b \in \mathbb{R}, c \in \mathbb{C} \right\}$, we have

$$C^0(S^1, \mathbb{R})/\mathfrak{psu}(1,1) = \left\{ \sum_{n \in \mathbb{Z}} u_n e^{inx} \in C^0(S^1, \mathbb{R}) \,\middle|\, \overline{u}_{-n} = u_n,\, u_1 = u_0 = 0 \right\}.$$

Complex Structure in Fourier Representation

Recall that the complex structure of $T(1)^B$ is determined by requiring that the projection $B_1^* \to T(1)^B$ be a holomorphic submersion. Therefore the complex structure on $T(1)^B$ is multiplication by i.

The complex structure induced on $\mathrm{QS}(S^1)_0^B$ is right-invariant and its value $\mathsf{J} : \mathfrak{h}^{QS} \to \mathfrak{h}^{QS}$ at the identity is

$$\mathsf{J}\left(\sum_{n \neq -1,0,1} u_n e^{inx} \right) = i \sum_{n \neq -1,0,1} \mathrm{sgn}(n) u_n e^{inx}$$

which is the expression of the Hilbert transform on the circle

$$\mathsf{J}(u)(x) = \frac{1}{2\pi} \int_{S^1} u(s) \cot\left(\frac{s-x}{2} \right) ds.$$

in Fourier representation. In particular,

$$\mathsf{J}(\sin(nx)) = \cos(nx) \quad \text{and} \quad \mathsf{J}(\cos(nx)) = -\sin(nx).$$

The Weil-Petersson Metric

The **Weil-Petersson metric** on $T(1)^B$ is the right-invariant Hermitian metric whose value at the identity is given by

$$\langle \mu, \nu \rangle := \int_{\mathbb{D}^*} \mu(z)\overline{\nu(z)} \frac{4}{(1-|z|^2)^2} d^2z.$$

Nag and Verjovsky [1990] introduced it as a direct generalization of the Weil-Petersson metric on the finite dimensional Teichmüller spaces. It does not make sense for all $\mu, \nu \in \Omega^{-1,1}(\mathbb{D}^*)$ since it converges only for $\mu, \nu \in H^{-1,1}(\mathbb{D}^*) \subset \Omega^{-1,1}(\mathbb{D}^*)$, where

$$\begin{aligned} H^{-1,1}(\mathbb{D}^*) :&= \left\{ \mu \in \Omega^{-1,1}(\mathbb{D}^*) \;\middle|\; \int_{\mathbb{D}^*} |\mu(z)|^2 \frac{1}{(1-|z|^2)^2} d^2z < \infty \right\} \\ &= \left\{ \mu(z) = (1-|z|^2)^2 \overline{\phi(z)} \;\middle|\; \phi \in A_2(\mathbb{D}^*) \right\} \end{aligned}$$

and

$$A_2(\mathbb{D}^*) := \left\{ \phi \text{ holomorphic in } \mathbb{D}^* \;\middle|\; \int_{\mathbb{D}^*} |\phi(z)|^2 (1-|z|^2)^2 d^2z < \infty \right\}.$$

Using the identification

$$T_e \, \mathrm{QS}(S^1)^B_{\mathsf{fix}} = \mathfrak{h}^{QS} \stackrel{T_{[0]}\Phi}{\longleftarrow} T_{[0]}T(1)^B \stackrel{T_{[0]}\beta}{\longrightarrow} A_\infty(\mathbb{D}),$$

the metric on $\mathfrak{h}^{QS}$ has the expression

$$h_e(u,v) = \frac{\pi}{2}\sum_{n=2}^{\infty} n(n^2-1)u_n\overline{v_n}$$

and one can see that it converges only for $u, v \in \mathfrak{h}^{3/2}$, the subspace of $H^{3/2}$ real vector fields on the circle with $u_0 = u_1 = 0$ which is strictly included in $\mathfrak{h}^{QS}$. Therefore, $T_{[0]}\Phi\left(H^{-1,1}(\mathbb{D})\right) = \mathfrak{h}^{3/2}$.

On the other hand, the metric on $A_\infty(\mathbb{D})$ is given by

$$h_e(\phi,\psi) = \frac{1}{4}\int_{\mathbb{D}} \phi(z)\overline{\psi(z)}(1-|z|^2)^2 d^2z,$$

which converges only if $\phi, \psi \in A_2(\mathbb{D})$, a strict subspace of $A_\infty(\mathbb{D})$. Therefore $T_{[0]}\beta\left(H^{-1,1}(\mathbb{D})\right) = A_2(\mathbb{D})$.

The corresponding Weil-Petersson Riemannian metric on $\mathrm{QS}(S^1)^B_{\mathrm{fix}}$ (considered as a real manifold), is given by

$$g_e(u,v) = \frac{\pi}{2}\,\mathrm{Re}\left(\sum_{n=2}^{\infty} n(n^2-1)u_n\overline{v}_n\right) = \frac{\pi}{4}\sum_{n\neq -1,0,1} |n|(n^2-1)u_n\overline{v}_n.$$

The imaginary part of the Hermitian metric is the symplectic form

$$\omega_e(u,v) = \frac{\pi}{2}\,\mathrm{Im}\left(\sum_{n=2}^{\infty} n(n^2-1)u_n\overline{v}_n\right) = -\frac{i\pi}{4}\sum_{n\neq -1,0,1} n(n^2-1)u_n\overline{v}_n.$$

As it was the case for the Weil-Petersson Hermitian metric, g and ω are only defined on the subspace $\mathfrak{h}^{3/2}$ of $T_e\,\mathrm{QS}(S^1)^B_{\mathrm{fix}} = \mathfrak{h}^{QS}$.

In order to solve the convergence problem, Takhtajan and Teo [2003] introduce a new complex Hilbert manifold structure on $T(1)$, such that the natural inner product is given by the Weil-Petersson Hermitian metric.

TAKHTAJAN-TEO THEORY

The Complex Hilbert Manifold Structure on $T(1)$

GOAL: define a Hilbert manifold structure on $T(1)$ with model the Hilbert space $A_2(\mathbb{D})$.

Use continuity of $A_2(\mathbb{D}) \hookrightarrow A_\infty(\mathbb{D})$ and $H^{-1,1}(\mathbb{D}^*) \hookrightarrow \Omega^{-1,1}(\mathbb{D}^*)$.

$A_2(\mathbb{D})$-Hilbert manifold structure on $A_\infty(\mathbb{D})$: Coordinate chart at $\phi \in A_\infty(\mathbb{D})$ is $\phi + A_2(\mathbb{D})$. The resulting Hilbert manifold $A_\infty(\mathbb{D})$ modeled on $A_2(\mathbb{D})$ is not connected and is the union of uncountably many components $\phi + A_2(\mathbb{D})$.

$A_2(\mathbb{D})$-Hilbert manifold structure on $T(1)$: restrict the Banach manifold charts from $A_\infty(\mathbb{D})$ to $A_2(\mathbb{D})$. The resulting Hilbert manifold $T(1)$ is also not connected, with uncountably many components.

The sets $T(1)$ and $A_\infty(\mathbb{D})$ endowed with the $A_2(\mathbb{D})$-Hilbert manifold structure are denoted by $T(1)^H$ and $A_\infty(\mathbb{D})^H$.

As in the Banach case, the bijection $\Phi : T(1) \to \mathrm{QS}(S^1)_{\mathrm{fix}}$ endows the group $\mathrm{QS}(S^1)_{\mathrm{fix}}$ with the structure of a complex Hilbert manifold denoted by $\mathrm{QS}(S^1)^H_{\mathrm{fix}}$.

The Bers embedding

$$\beta : T(1)^H \to A_\infty(\mathbb{D})^H, \qquad \beta([\mu]) := S\left(\omega^\mu|_{\mathbb{D}}\right),$$

is a biholomorphic mapping from $T(1)^H$ onto an open subset of $A_\infty(\mathbb{D})^H$. In particular, the tangent map $T_{[0]}\beta$ induces an isomorphism $H^{-1,1}(\mathbb{D}^*) \cong A_2(\mathbb{D})$. The connected components of $T(1)^H$ are the inverse images of the connected components of $\beta\left(T(1)^H\right)$.

The connected component of $[0] \in T(1)^H$ is denoted by $T(1)^H_\circ$. The manifolds $T(1)^H$ and $T(1)^H_\circ$ have the following good properties:

- The Weil-Petersson metric is *strong* on $T(1)^H$, since it is the natural Hermitian inner product on the tangent spaces. As a consequence, g and ω are also strong with respect to the Hilbert manifold structure.

- $(T(1)^H, \mathsf{J}, \omega)$ is a strong Kähler-Einstein Hilbert manifold with negative constant Ricci curvature and negative sectional and holomorphic sectional curvatures.

- The right translations on $T(1)^H$ are biholomorphic mappings.

- The connected component $T(1)^H_\circ$ is a topological group.

- The connected component $T(1)^H_\circ$ is the closure in $T(1)^H$ of the subgroup $\Phi^{-1}\left(\mathsf{Diff}_+(S^1)_{\mathsf{fix}}\right)$.

The closure of the subgroup $\mathsf{Diff}_+(S^1)_{\mathsf{fix}}$ in the Hilbert manifold $\mathsf{QS}(S^1)^H_{\mathsf{fix}}$ is $\Phi\left(T(1)^H_\circ\right)$. Recall that the closure of the subgroup $\mathsf{Diff}_+(S^1)_{\mathsf{fix}}$ in the Banach manifold $\mathsf{QS}(S^1)^B_{\mathsf{fix}}$ is $\mathsf{S}(S^1)_{\mathsf{fix}}$. These completions do not coincide. Indeed, for $[\mu] \in T(1)$ we have the following characterization in terms of the Bers embedding β:

$$[\mu] \in T(1)^H_\circ \Longleftrightarrow \beta([\mu]) \in A_2(\mathbb{D}) \qquad \text{and}$$
$$[\mu] \in \Phi^{-1}\left(\mathsf{S}(S^1)\right) \Longleftrightarrow \beta([\mu]) \in A^0_\infty(\mathbb{D}).$$

Since $A_2(\mathbb{D}) \subsetneq A^0_\infty(\mathbb{D})$ we get $\Phi\left(T(1)^H_\circ\right) \subsetneq \mathsf{S}(S^1)_{\mathsf{fix}}$.

Endow $\mathsf{QS}(S^1)$ with a real Hilbert manifold structure, by declaring the bijection

$$\Psi : \mathsf{QS}(S^1) \longrightarrow \mathsf{PSU}(1,1) \times \mathsf{QS}(S^1)^H_{\mathsf{fix}} \qquad \text{given by}$$
$$\Psi(\eta) = (\widehat{\eta}, \eta_0) \Longleftrightarrow \eta = \widehat{\eta} \circ \eta_0,$$

to be a diffeomorphism. $\mathsf{QS}(S^1)$ endowed with this Hilbert manifold structure is denoted by $\mathsf{QS}(S^1)^H$. $\mathsf{QS}(S^1)^H_\circ$ connected component of e.

$T_e\,\mathrm{QS}(S^1)^H = H^{3/2}(S^1,\mathbb{R})$. The manifold $\mathrm{QS}(S^1)^H$ has smooth right translations and contains the subgroup $\mathrm{QS}(S^1)^H_{\mathrm{fix}}$ as a closed submanifold of codimension 3.

$$
\begin{array}{ccccccc}
A_\infty(\mathbb{D}) & \xleftarrow{\;T_{[0]}\beta\;} & \Omega^{-1,1}(\mathbb{D}^*) & \xrightarrow{\;T_{[0]}\Phi\;} & Z(S^1)_0 & \longrightarrow & Z(S^1) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
A^0_\infty(\mathbb{D}) & \xleftarrow{\;T_{[0]}\beta\;} & \Omega_0^{-1,1}(\mathbb{D}^*) & \xrightarrow{\;T_{[0]}\Phi\;} & Z_{\mathrm{sym}}(S^1)_0 & \longrightarrow & Z_{\mathrm{sym}}(S^1) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
A_2(\mathbb{D}) & \xleftarrow{\;T_{[0]}\beta\;} & H^{-1,1}(\mathbb{D}^*) & \xrightarrow{\;T_{[0]}\Phi\;} & H^{3/2}(S^1,\mathbb{R})_0 & \longrightarrow & H^{3/2}(S^1,\mathbb{R})
\end{array}
$$

$T_{[0]}\beta$ and $T_{[0]}\Phi$ complex Banach space isomorphisms. Other arrows: continuous inclusions. Last column of horizontal arrows have images are codimension three subspaces. First row of vertical arrows have closed ranges that are not complemented. Ranges of the second row of vertical arrows are not closed.

$$
\begin{array}{ccccccc}
A_\infty(\mathbb{D}) \supset \beta(T(1)^B) & \xleftarrow{\ \beta\ } & T(1)^B & \xrightarrow{\ \Phi\ } & \mathsf{QS}(S^1)^B_{\text{fix}} & \longrightarrow & \mathsf{QS}(S^1)^B \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
A^0_\infty(\mathbb{D}) \supset \beta(T_{\text{sym}}(1)^B) & \xleftarrow{\ \beta\ } & T_{\text{sym}}(1)^B & \xrightarrow{\ \Phi\ } & \mathsf{S}(S^1)^B_{\text{fix}} & \longrightarrow & \mathsf{S}(S^1)^B \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
A_2(\mathbb{D}) \supset \beta(T(1)^H_\circ) & \xleftarrow{\ \beta\ } & T(1)^H_\circ & \xrightarrow{\ \Phi\ } & \mathsf{QS}(S^1)^H_{\text{fix},\circ} & \longrightarrow & \mathsf{QS}(S^1)^H_\circ
\end{array}
$$

β and Φ are diffeomorphisms relative to the indicated complex Banach and complex Hilbert manifold structures. The three images of β are open in the indicated Banach spaces. The last column is formed by real Banach manifolds, the first three columns are complex Banach manifolds. The right horizontal column of four arrows are codimension three embeddings with closed range. All spaces in this diagram are connected. The top row of vertical arrows are weak embeddings, that is, injective immersions whose derivatives have closed (but not necessarily complemented) ranges and have the induced subspace topologies. In addition, the ranges of these four inclusions are closed in the topological spaces of the first row. The bottom row of vertical arrows are smooth inclusions whose inverses from their ranges are discontinuous; the first three arrows are holomorphic maps.

$T_{\mathsf{sym}}(1)^B := \Phi^{-1}\left(\mathsf{S}(S^1)^B_{\mathsf{fix}}\right)$. Since this space will play no role in what follows we shall not define it intrinsically; $T(1)^B/T_{\mathsf{sym}}(1)^B$ is called the **asymptotic universal Teichmüller space**.

$\mathfrak{T} := \Phi(T(1)^H_\circ) = \mathrm{QS}(S^1)^H_{\mathrm{fix},\circ}$ is the object of study.

1. $\eta \in \mathfrak{T}$ symmetric homeomorphisms $\eta : S^1 \to S^1$ that fix ± 1, $-i$

2. $\mathfrak{T}$ complex Hilbert manifold and connected topological group

2. Right translations on $\mathfrak{T}$ are biholomorphic maps $R_\eta : \mathfrak{T} \to \mathfrak{T}$

3. The group $\mathrm{Diff}_+(S^1)_{\mathrm{fix}}$ of smooth orientation preserving diffeomorphisms of S^1 that fix the three points $\pm 1, -i$ is dense in $\mathfrak{T}$

4. $T_e\mathfrak{T} = H^{3/2}(S^1, \mathbb{R})_0 = \mathfrak{g}^{3/2}$.

5. Weil-Petersson metric on $\mathfrak{T}$ is smooth, right invariant, strong. $\mathfrak{T}$ is Kähler-Einstein with negative constant Ricci curvature and negative sectional and holomorphic sectional curvatures.

$T_e\mathfrak{T} = H^{3/2}(S^1, \mathbb{R})_0 = \mathfrak{g}^{3/2}$ is a Hilbert space of real vector fields, identified with real valued functions, that also has a complex structure. Since the derivative of right translation by $\eta \in \mathfrak{T}$ from $T_e\mathfrak{T} \to T_\eta\mathfrak{T}$ is a Hilbert space isomorphism, the topology on the other tangent space is equivalent to the $H^{3/2}$ topology as well. Thus, the topology induced on the tangent spaces of $\mathfrak{T}$ by the Weil-Petersson Riemannian metric is also the $H^{3/2}$ topology (metric is strong).

Moreover, again by strongness and smoothness of the Weil-Petersson Riemannian metric, its exponential maps form coordinate charts. Since these charts map into the space of $H^{3/2}$ functions, it follows that $\mathfrak{T}$ is a $H^{3/2}$ Hilbert manifold.

Note that the relation $\eta(e^{ix}) = e^{i\bar{\eta}(x)}$ identifies the elements $\eta \in \mathfrak{T}$ with the strictly increasing maps $\bar{\eta}$ of $\mathbb{R}$ such that $\bar{\eta}(x + 2\pi) = \bar{\eta}(x) + 2\pi$ and $\bar{\eta}, \bar{\eta}^{-1} \in H^{3/2}_{\text{loc}}(\mathbb{R}, \mathbb{R})$.

We do not know if this is $\text{Diff}^{3/2}_{+,\text{fix}}(S^1)$.

“Lie algebra structure” on $T_e\mathfrak{T} = H^{3/2}(S^1, \mathbb{R})_0 = \mathfrak{g}^{3/2}$.

$$[f, g](\theta) = g(\theta)f'(\theta) - g'(\theta)f(\theta)$$

This makes sense for $f, g \in \mathfrak{g}^{3/2}$, producing a vector field in $H^{1/2}$. Pointwise multiplication in H^r:

Palais [1968]: If $t > (n/2)$ and $r \geq -t$, pointwise multiplication extends from

$$C^\infty(M, \mathbb{R}) \times C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$$

to a continuous bilinear map

$$H^t(M, \mathbb{R}) \times H^r(M, \mathbb{R}) \to H^{\min\{r,t\}}(M, \mathbb{R}).$$

In particular, for $|r| \leq t$, $H^r(M, \mathbb{R})$ is an $H^t(M, \mathbb{R})$-module.

THE BOTT-VIRASORO GROUP

The Group and its Lie Algebra

The **Bott-Virasoro group** BVir(S^1) is, up to isomorphism, the unique nontrivial central extension of the diffeomorphism group of the circle by $\mathbb{R}$. As a set

$$\mathrm{BVir}(S^1) = \mathrm{Diff}_+(S^1) \times \mathbb{R}$$

where $\mathrm{Diff}_+(S^1)$ is the group of orientation preserving smooth diffeomorphisms of the circle $S^1 := \{e^{ix} \mid x \in \mathbb{R}\} \equiv \mathbb{R}/2\pi\mathbb{Z}$. Thus, $\xi \in \mathrm{Diff}_+(S^1)$ can be thought of as a strictly increasing diffeomorphism of $\mathbb{R}$ satisfying $\xi(x + 2\pi) = \xi(x) + 2\pi$ for all $x \in \mathbb{R}$. Group multiplication on BVir(S^1) is defined by

$$(\xi, \alpha)(\eta, \beta) = (\xi \circ \eta, \alpha + \beta + B(\xi, \eta)),$$

where $B : \mathrm{Diff}_+(S^1) \times \mathrm{Diff}_+(S^1) \to \mathbb{R}$ is the **Bott-Thurston two-cocycle** defined by

$$B(\xi, \eta) = \int_{S^1} \log \partial_x(\xi \circ \eta) \mathbf{d} \log \partial_x \eta.$$

$\partial_x \xi$ is the derivative of ξ on $\mathbb{R}$.

The Lie algebra $\mathfrak{v}(S^1) := \mathfrak{X}(S^1) \times \mathbb{R}$ of BVir(S^1), called the **Virasoro algebra**, has the bracket

$$[(v,a),(w,b)] = ([v,w], C(v,w)),$$

where $[v,w] := (\partial_x v)w - (\partial_x w)v$ is the negative Jacobi-Lie bracket of the vector fields $v, w \in \mathfrak{X}(S^1)$, identified with 2π-periodic real valued functions on $\mathbb{R}$, and $C : \mathfrak{X}(S^1) \times \mathfrak{X}(S^1) \to \mathbb{R}$ is the **Gelfand-Fuchs two-cocycle** defined by

$$C(v,w) = 2\int_{S^1} (\partial_x v)(\partial_x^2 w)dx.$$

Identify $\mathfrak{v}(S^1)^*$ with $\mathfrak{v}(S^1)$ using the weak L^2 pairing

$$\langle (v,a),(w,b) \rangle := \int_{S^1} v(x)w(x)dx + ab.$$

Then the coadjoint action has the expression

$$\mathrm{Ad}^*_{(\xi,\alpha)}(u,c) = ((\partial_x \xi)^2 (u \circ \xi) + 2cS(\xi), c),$$

where S is the **Schwarzian derivative**

$$S(\xi) = \frac{\partial_x^3 \xi}{\partial_x \xi} - \frac{3}{2}\left[\frac{\partial_x^2 \xi}{\partial_x \xi}\right]^2.$$

Coadjoint Orbits

$\mathcal{O}$ a BVir(S^1) coadjoint orbit. Orbit symplectic form at $(u,c) \in \mathcal{O}$:

$$\omega_{\mathcal{O}}(u,c)(\mathrm{ad}^*_{(v,a)}(u,c), \mathrm{ad}^*_{(w,b)}(u,c))$$
$$= \int_{S^1} u((\partial_x v)w - (\partial_x w)v)dx + 2c \int_{S^1} (\partial_x^3 v) w dx,$$

where ad^* denotes the infinitesimal coadjoint action given by

$$\mathrm{ad}^*_{(v,a)}(u,c) = (u\partial_x v + v\partial_x u + 2c\partial_x^3 v, 0).$$

$\omega_{\mathcal{O}}$ is invariant under the coadjoint action. Note that

$$(\dot{u}, \dot{c}) = -\,\mathrm{ad}^*_{(u,c)}(u,c) \Longleftrightarrow \left\{ \begin{array}{l} \dot{u} + 3uu' + 2cu''' = 0 \\ \dot{c} = 0 \end{array} \right\} \Longleftrightarrow \mathrm{KdV}\,.$$

So KdV is the spatial representation of the L^2 geodesic spray.

If $(u,c) \in \mathcal{O}$, then $\varphi_{(u,c)} : \mathrm{BVir}(S^1)/\,\mathrm{BVir}(S^1)_{(u,c)} \to \mathcal{O}$ given by $\varphi_{(u,c)}\left(\mathrm{BVir}(S^1)_{(u,c)}(\xi,a)\right) = \mathrm{Ad}^*_{(\xi,a)^{-1}}(u,c)$ is a Fréchet manifold diffeomorphism, where $\mathrm{BVir}(S^1)_{(u,c)}$ is the coadjoint isotropy group of (u,c); the quotient is taken with respect to left multiplication.

Note that $\mathrm{BVir}(S^1)_{(u,c)}$ is of the form

$$\mathrm{BVir}(S^1)_{(u,c)} = \mathrm{Diff}_+(S^1)_{(u,c)} \times \mathbb{R},$$

therefore we have $\mathrm{BVir}(S^1)/\,\mathrm{BVir}(S^1)_{(u,c)} = \mathrm{Diff}_+(S^1)/\,\mathrm{Diff}_+(S^1)_{(u,c)}$. The tangent space at $[e]$ can be identified with the quotient vector space $\mathfrak{X}(S^1)/\mathfrak{X}(S^1)_{(u,c)}$, where

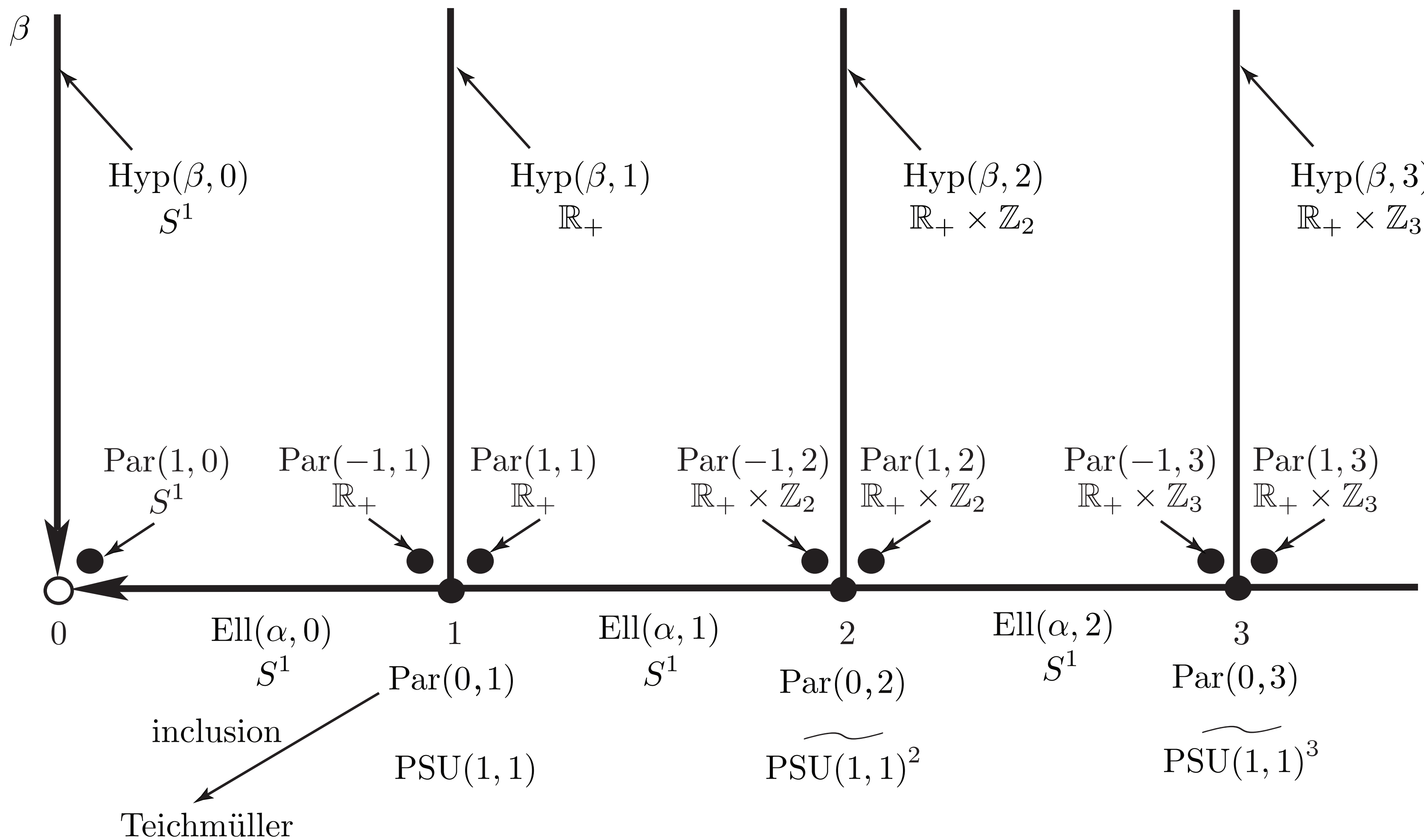
$$\mathfrak{X}(S^1)_{(u,c)} = \{v \in \mathfrak{X}(S^1) \mid 2u\partial_x v + v\partial_x u + 2c\partial_x^3 v = 0\}$$

is the Lie algebra of $\mathrm{Diff}_+(S^1)_{(u,c)}$. At $[e] \in \mathrm{Diff}_+(S^1)/\,\mathrm{Diff}_+(S^1)_{(u,c)}$ the symplectic form $\omega_{(u,c)} := \varphi^*_{(u,c)}\omega_{\mathcal{O}}$ is given by

$$\omega_{(u,c)}([e])([v],[w]) = \int_{S^1} u((\partial_x v)w - (\partial_x w)v)dx + 2c\int_{S^1}(\partial_x^3 v)w dx,$$

where $[v],[w] \in \mathfrak{X}(S^1)/\mathfrak{X}(S^1)_{(u,c)}$. In addition, $r_\eta^*\omega_{(u,c)} = \omega_{(u,c)}$, where $r_\eta : \mathrm{Diff}_+(S^1)/\,\mathrm{Diff}_+(S^1)_{(u,c)} \to \mathrm{Diff}_+(S^1)/\,\mathrm{Diff}_+(S^1)_{(u,c)}$ is given by $r_\eta([\xi]) := [\xi \circ \eta]$, for any $\xi, \eta \in \mathrm{Diff}_+(S^1)$.

Coadjoint Orbit Classification



Points on the "comb" and the "floating points" represent the space of coadjoint orbits for nonzero charge c (Balog, Fehér, Palla [1998]).

The vertical lines, each of which is parametrized by $\beta > 0$, and labeled by an integer $n = 0, 1, 2, \ldots$, represent the hyperbolic orbits. The non-integer points on the horizontal axis represent the elliptic orbits, while the integer points represent those parabolic orbits $\mathrm{Par}(\varepsilon, n)$ with $\varepsilon = 0$. The parabolic orbits $\mathrm{Par}(\varepsilon, n)$ with $\varepsilon = \pm 1$ and $n = 0, 1, 2, \ldots$ are represented by the "floating points". The open circle at $n = 0, \beta = 0$ is an empty point with no corresponding orbit. The groups under each orbit are the coadjoint isotropy subgroups.

$\mathrm{Par}(0, 1)$ is special and is related to the universal Teichmüller space. Consider the particular momentum $(1/4, 1/4) \in \mathfrak{v}(S^1)^* \cong \mathfrak{v}(S^1) = \mathfrak{X}(S^1) \times \mathbb{R}$. The $(1/4, 1/4)$-coadjoint isotropy group is $\mathrm{PSU}(1,1) \times \mathbb{R}$ and the corresponding orbit is diffeomorphic to $\mathrm{Diff}_+(S^1)/\,\mathrm{PSU}(1,1)$. If $[u], [v] \in T_{[e]}(\mathrm{Diff}_+(S^1)/\,\mathrm{PSU}(1,1)) = \mathfrak{X}(S^1)/\mathfrak{psu}(1,1)$,

$$\omega_{(1/4,1/4)}([e])([u],[v]) = \left\langle \left(\frac{1}{4}, \frac{1}{4}\right), \left[\left(u, \frac{1}{4}\right), \left(v, \frac{1}{4}\right)\right]\right\rangle$$
$$= \frac{1}{4}\int_{S^1}(u'v - uv') + \frac{1}{2}\int_{S^1} u'v'' = \frac{1}{2}\int_{S^1}(u' + u''')v.$$

u and v are taken modulo

$$\begin{aligned}\mathfrak{psu}(1,1) &= \{a + b\sin(x) + c\cos(x) \mid a,b,c \in \mathbb{R}\} \\ &= \{\overline{u_1}e^{-ix} + u_0 + u_1 e^{ix} \mid u_0 \in \mathbb{R}, u_1 \in \mathbb{C}\}.\end{aligned}$$

In terms of Fourier series, for

$$u = \sum_{n\in\mathbb{Z}} u_n e^{inx} \quad \text{and} \quad v = \sum_{n\in\mathbb{Z}} v_n e^{inx}$$

we have

$$\omega_{(1/4,1/4)}([e])([u],[v]) = -i\pi \sum_{n\neq -1,0,1} (n^3 - n)u_n \overline{v_n}$$

which is $\mathrm{Diff}_+(S^1)$ right invariant.

Identify $\mathrm{Diff}_+(S^1)/\,\mathrm{PSU}(1,1)$ with the subgroup $\mathrm{Diff}_+(S^1)_0$; so for $\eta \in \mathrm{Diff}_+(S^1)_0$ r_η is right translation by η on $\mathrm{Diff}_+(S^1)_0$. In addition, *the orbit symplectic structure equals four times the imaginary part of the Weil-Petersson Hermitian metric:* $\omega_{(1/4,1/4)} = 4\omega$.

Conclusion: $\Phi(T_0(1)^H)$ is the the completion of the Teichmüller orbit on which the orbit symplectic form $\omega_{(1/4,1/4)}$ is strong.

THE EULER-WEIL-PETERSSON EQUATION

g^H the Weil-Petersson Riemannian metric on $T(1)^H$; $(T(1)^H, g^H)$ is a strong Riemannian Hilbert manifold. g^B the Weil-Petersson Riemannian metric on $T(1)^B$; g^B is not everywhere defined.

Weil-Petersson Geodesics on $T(1)^H$

Right invariance of the metric and the fact that it is strong implies:

The Riemannian manifold $(T(1)^H, g^H)$ is geodesically complete. The geodesic spray of the metric g^H is smooth and there is an associated Levi-Civita connection. The curvature and Ricci tensors are bounded operators. Moreover, since the sectional curvature is negative, there are no conjugate points.

Weil-Petersson Geodesics on $T(1)^B$

One cannot say very much. The identity map $j : T(1)^H \to T(1)^B$ is holomorphic but the inverse j^{-1} is not even continuous, since left multiplication is not continuous on $T(1)^B$.

The tangent map $T_{[\mu]}j : T_{[\mu]}T(1)^H \to T_{[\mu]}T(1)^B$ is not an isomorphism; more precisely, $T_{[\mu]}j\left(T_{[\mu]}T(1)^H\right)$ is striclty included in $T_{[\mu]}T(1)^B$. At the identity we have $T_{[0]}j\left(T_0T(1)^H\right) = H^{-1,1}(\mathbb{D}^*)$, a space strictly included in $T_{[0]}T(1)^B = \Omega^{-1,1}(\mathbb{D}^*)$.

Recall that the Weil-Petersson metric g^B on $T(1)^B$ is not everywhere defined, but we have the relation $g^H = j^*g^B$ on $T(1)^H$. Given a geodesic $\gamma \subset T(1)^H$ with respect to the metric g^H, we can consider the curve $j \circ \gamma \subset T(1)^B$. This curve is smooth with respect to the Banach manifold structure. Its derivative $\dot{\gamma}$ belongs to the distribution

$$\bigcup_{[\mu]\in T(1)^B} T_0R_{[\mu]}(H^{-1,1}(\mathbb{D}^*)) = T_{[\mu]}j\left(T_{[\mu]}T(1)^H\right) \subset TT(1)^B.$$

The Euler-Weil-Petersson Equation

By Euler-Poincaré reduction, if $\gamma(t)$ a geodesic of the Weil-Petersson metric on $\Phi(T(1)^H_\circ)$, the curve

$$u(t) := \dot{\gamma}(t) \circ \gamma(t)^{-1} \in T_e(\Phi(T(1)^H_\circ)) = \mathfrak{g}^{3/2}$$

should formally be a solution of the Euler-Poincaré equation

$$\frac{d}{dt}\frac{\delta l}{\delta u} = -\operatorname{ad}^*_u \frac{\delta l}{\delta u},$$

where $l : \mathfrak{g}^{3/2} \to \mathbb{R}$, $l(u) = g_e(u,u)/2$ is the Weil-Petersson Lagrangian; this is the **Euler-Weil-Petersson (EWP) equation**. For the moment we proceed formally and later discuss the rigorous interpretation of the equation.

The Lagrangian of the EWP Equation: The solution of the Euler-Poincaré equation formally does not depend on the choice of the duality pairing.

The L^2 strong pairing on $H^0(S^1) = L^2(S^1)$, $\langle u, v\rangle = \int_{S^1} uv$ for $u, v \in L^2(S^1)$, extends to a strong pairing between $H^s(S^1)$ and $H^{-s}(S^1)$ for any $s \in \mathbb{R}$. Therefore the dual space to the closed subspace $\mathfrak{g}^s = \{u \in H^s(S^1) \mid u(\pm 1) = u(-i) = 0\} \subset H^s(S^1), s > 1/2$ is $H^{-s}(S^1)/N$, where $N = \{v \in H^{-s}(S^1) \mid \langle v, u\rangle = 0$, for all $u \in \mathfrak{g}^s\}$.

With respect the L^2 pairing, the Weil-Petersson Lagrangian reads

$$l(u) = \frac{1}{8}\langle Q_{\text{op}}(u), v\rangle,$$

where $Q_{\text{op}} : H^s(S^1) \to H^{s-3}(S^1)$, $s \in \mathbb{R}$, is the symmetric operator given by

$$Q_{\text{op}}\left(\sum_{n\in\mathbb{Z}} u_n e^{inx}\right) = \sum_{n\in\mathbb{Z}} |n|(n^2-1)u_n e^{inx} = \sum_{n\neq -1,0,1} |n|(n^2-1)u_n e^{inx}.$$

Does this formal expression make sense?

Properties of the Operator Q_{op}: $Q_{\mathrm{op}} = \mathsf{J} \circ (\partial^3 + \partial)$ where J is the Hilbert-transform on S^1. Thus, while Q_{op} is not literally a third order elliptic differential operator, it has similar properties. Namely, since $\mathsf{J} : H^s \to H^s$ is an isomorphism for all s and since $\partial^3 + \partial$ is literally a third order elliptic differential operator, we conclude

(1) $Q_{\mathrm{op}} : H^s \to H^{s-3}$,
(2) $Q_{\mathrm{op}}(u) \in H^s \Rightarrow u \in H^{s+3}$,
(3) $\ker(Q_{\mathrm{op}}) = \mathfrak{psu}(1,1)$, and
(4) $\mathrm{Im}(Q_{\mathrm{op}}) = \left\{ \sum_{n \neq -1,0,1} v_n e^{inx} \,\middle|\, v_{-n} = \overline{v}_n \right\} \cap H^{s-3}(S^1)$.

Study the kernel and image of $Q_{\mathrm{op}}|_{\mathfrak{g}^s}$ for $s > 1/2$. Since $\ker(Q_{\mathrm{op}}) = \mathfrak{psu}(1,1)$ it follows that $\ker Q_{\mathrm{op}}|_{\mathfrak{g}^s} = \ker Q_{\mathrm{op}} \cap \mathfrak{g}^s = \{0\}$ because elements of $\mathfrak{psu}(1,1)$ that vanish at three points are identically zero. Thus, $Q_{\mathrm{op}}|_{\mathfrak{g}^s}$ is injective.

One proves that $\mathrm{Im}(Q_{\mathrm{op}}|_{\mathfrak{g}^s}) = \mathrm{Im}(Q_{\mathrm{op}})$. Hence $Q_{\mathrm{op}} : \mathfrak{g}^{3/2} \to \mathrm{Im}(Q_{\mathrm{op}}) \subset H^{-3/2}(S^1)$ is an isomorphism.

Conclusion: l is well-defined on $\mathfrak{g}^{3/2}$ since $u \in H^{3/2}(S^1)$ and $Q_{\mathrm{op}}(u) \in H^{-3/2}(S^1)$.

Formal Derivation of the EWP Equation: With respect to the L^2 pairing, the infinitesimal coadjoint action is

$$\mathrm{ad}_u^* m = 2mu' + m'u;$$

more precisely we should write $\mathrm{ad}_u^*[m] = [2mu' + m'u]$, where [ ] denotes the equivalence class modulo N. One can check that $[2mu' + m'u]$ does not depend on the choice of $m \in [m]$.

The functional derivative of l is

$$\frac{\delta l}{\delta u} = 2Q_{\mathrm{op}}(u).$$

Thus, *the Euler-Weil-Petersson equation reads*

$$\dot{m} + 2mu' + m'u = 0, \quad m = Q_{op}(u) \in H^{-3/2}(S^1).$$

Fundamental Problem: This equation as well as the formula for the coadjoint action make no sense as written if $u \in \mathfrak{g}^{3/2}$!

We comment on this difficulty now.

Palais: If $t > (\dim M)/2$ and $r \geq -t$, pointwise multiplication extends from $C^\infty(M, \mathbb{R}) \times C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ to a continuous bilinear map

$$H^t(M, \mathbb{R}) \times H^r(M, \mathbb{R}) \to H^{\min\{r,t\}}(M, \mathbb{R}).$$

In particular, for $|r| \leq t$, $H^r(M, \mathbb{R})$ is an $H^t(M, \mathbb{R})$-module.

(1) The theorem does not apply to our equation.
(2) Try weak form

$$\left\langle \frac{d}{dt} m, \varphi \right\rangle = \langle m, [u, \varphi] \rangle, \quad \forall \varphi \in C^\infty(S^1), \quad m = Q_{\mathrm{op}}(u).$$

This is also not well-defined since on the right hand side there is a L^2 pairing between $m \in H^{-3/2}$ and $[u, \varphi] \in H^{1/2}$.
(3) This difficulty does not occur for the Camassa-Holm (or Euler, or Euler-alpha) equation. CH: $\dot{m} + 2mu' + m'u = 0$, that is $\dot{u} + Q_{\mathrm{op}}^{-1}(2mu' + m'u) = 0$, where here, $m = Q_{\mathrm{op}}(u) = (1 - \alpha^2 \partial^2)u$. Since $u \in H^s, s > 3/2$, we have $m \in H^{s-2}$ and so, by Palais, $2mu' + m'u \in H^{s-3}$. Therefore $Q_{\mathrm{op}}^{-1}(2mu' + m'u) \in H^{s-1}$. We also know that $u \in C^0(I, H^s) \cap C^1(I, H^{s-1})$. Thus, it is meaningful to write the Camassa-Holm equation in Euler-Poincaré form.

The Geometric Form of the EWP Equations: The preceding difficulties disappear if one writes the equations directly in terms of u *without* introducing the dual space. In doing so, we will heavily exploit the fact that the spray of the WP metric is smooth.

$\gamma(x,t)$ a WP geodesic, for $x \in S^1$. Thus, as a function of t, and thought of as a curve in $H^{3/2}$, it is smooth because the spray of the WP metric is smooth. So, $\dot{\gamma}$ and $\ddot{\gamma}$ are well defined. According to the fact that there is a smooth WP spray, we can write $\ddot{\gamma} + \Gamma(\gamma)(\dot{\gamma}, \dot{\gamma}) = 0$ for a well defined operator Γ that is quadratic in $\dot{\gamma}$.

By definition of u, we have

$$u(\gamma(x,t),t) = \dot{\gamma}(x,t).$$

This makes sense and defines $u \in \mathfrak{g}^{3/2}$ because, by Takhtajan-Teo theory, $T(1)^H_\circ$ is a group, right multiplication is smooth, and the tangent space at the identity is $\mathfrak{g}^{3/2}$. Now differentiate in t:

$$\dot{u}(y,t) + u'(y,t)u(y,t) = \ddot{\gamma}(x,t)$$

where $y = \gamma(x,t)$. The term $u'(y,t)u(y,t)$ is well defined in $H^{1/2}$ by Palais. From $\ddot{\gamma}(x,t) + \Gamma(\gamma)(\dot{\gamma},\dot{\gamma}) = 0$, we get the **geometric form of the EWP equation**

$$\dot{u} + uu' + \Gamma(u,u) = 0$$

which is well defined because the geodesic spray is smooth.

$$\Gamma(u,u) = -uu' + Q_{\mathrm{op}}^{-1}\left(2Q_{\mathrm{op}}(u)u' + (Q_{\mathrm{op}}(u))'\,u\right)$$

Geodesic spray on $T\mathfrak{T}$:

$$\mathcal{Z}(u_\eta) = S \circ u_\eta - \mathrm{Ver}_{u_\eta}(\overline{F}(u_\eta)),$$

$S \in \mathfrak{X}(TS^1)$ is the geodesic spray of the natural metric on S^1,
$F(u) = \Gamma(u,u) = Q_{\mathrm{op}}^{-1}(Q_{\mathrm{op}}(u)'u - Q_{\mathrm{op}}(u'u) + 2Q_{\mathrm{op}}(u)u'),$
$\overline{F}(u_\eta) = F(u_\eta \circ \eta^{-1}) \circ \eta.$

Still to do: If $u_t = \dot{\gamma}_t \circ \gamma_t^{-1}$ show $\gamma \in C^0(I, H^{3/2}) \cap C^1(I, H^{1/2})$. Looks promising.