Fluctuation Spectra of Plasmas and Fluids

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Goal: Do statistical mechanics using eigenmodes associated with the continuous spectrum as degrees of freedom to obtain the fluctuation spectra about equilibria.

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Three Parts

1. Vlasov Equation – Grand Unified 2+1 Media Theory

   Plasma Physics, Vortex Dynamics, Stellar Dyns., ...

2. Canonization and Diagonalization of Continuous Spectrum

   Normal Modes of Infinite Hamiltonian System

3. Statistical Mechanics with Continuous Spectrum

   Partition Function Calculation

Part One

Vlasov Equation – Grand Unified 2+1 Media Theory

Vlasov-Poisson System

Phase space density ((1 + 1) + 1 field theory):

$$f : \mathit{\Pi} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\,, \qquad f(x,v,t) \geq 0$$

Conservation of phase space density:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{e}{m}\frac{\partial \phi}{\partial x}\frac{\partial f}{\partial v} = 0$$

Poisson's equation:

$$\frac{\partial^2 \phi}{\partial x^2} = 4\pi\left[e\int_{\mathbb{R}} f(x,v,t)\,dv - \rho_B\right]$$

Energy:

$$H = \frac{m}{2}\int_{\mathit{\Pi}}\int_{\mathbb{R}} v^2 f\,dxdv + \frac{1}{8\pi}\int_{\mathit{\Pi}}\left(\frac{\partial \phi}{\partial x}\right)^2\,dx$$

Grand Unified 2+1 Media Theory (1-field)

Phase space density of something :

$$\zeta : \mathcal{Z} \times \mathbb{R} \to \mathbb{R}, \qquad z = (q,p) \in \mathcal{Z} = \text{symplectic manifold}$$

Conservation of phase space density:

$$\frac{\partial \zeta}{\partial t} + [\mathcal{E}, \zeta] = 0 \qquad [f,g] := \partial_q f \partial_p g - \partial_p f \partial_q g$$

Constraint/elliptic equation:

$$\mathcal{E} = \frac{\delta H}{\delta \zeta} = h_1(z) + \int_{\mathcal{Z}} h_2(z,z')\,\zeta(z')\,d^2z' + \ldots$$

Energy:

$$H[\zeta] = H_1 + H_2 = \int_{\mathcal{Z}} h_1(z)\,\zeta(z)\,d^2z + \frac{1}{2}\int_{\mathcal{Z}}\int_{\mathcal{Z}} \zeta(z)\,h_2(z,z')\,\zeta(z')\,d^2z\,d^2z'$$

Particles and Forces

two particle interactions

attractive: $1 \bullet \longrightarrow \qquad \longleftarrow \bullet 2$

repulsive: $\longleftarrow \bullet 1 \qquad 2 \bullet \longrightarrow$

- Vortex-Vortex (e.g. fluid mechanics)
- Electrostatic (e.g. plasma physics)
- Potential Vorticity (e.g. fluid mechanics)
- Newtonian gravity (e.g. stellar dynamics)
- Vorticity Defects (e.g. fluid mechanics)

many particles & long range interaction $\longrightarrow$
phase space density f governed by Vlasov dynamics

$f(x, v, t)\, dxdv =$ number of particles in $dx\, dv$ at (x, v)

Noncanonical Hamiltonian Structure

Hamiltonian structure of media in Eulerian variables

Kinematic Commonality:

energy, momentum, Casimir conservation; dynamics is measure preserving rearrangement; continuous spectra; $\ldots \longrightarrow$

Noncanonical Poisson Bracket:

$$\{F, G\} = \int_{\mathcal{Z}} \zeta \left[\frac{\delta F}{\delta \zeta}, \frac{\delta G}{\delta \zeta} \right] dq dp$$

Cosymplectic Operator:

$$\mathcal{J} \cdot = - \left(\frac{\partial \zeta}{\partial q} \frac{\partial \ \cdot}{\partial p} - \frac{\partial \ \cdot}{\partial q} \frac{\partial \zeta}{\partial p} \right)$$

Equation of Motion:

$$\frac{\partial \zeta}{\partial t} = \{\zeta, H\} = \mathcal{J} \frac{\delta H}{\delta \zeta} = -[\zeta, \mathcal{E}].$$

Organizing principle. Do one do all!

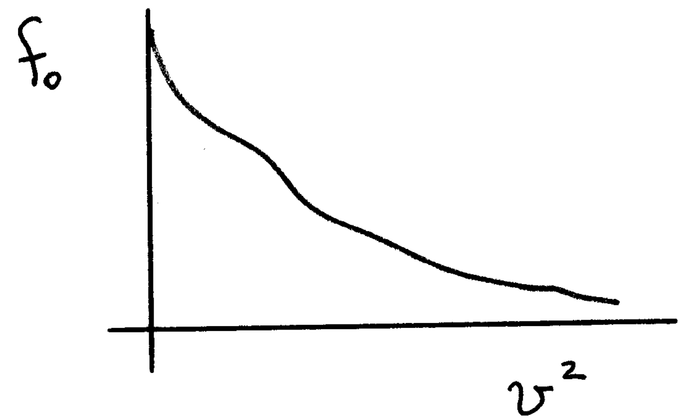
Stable Equilibria (generalized Rayleigh)

Vlasov:

Thermal $f_0 = c\, e^{-mv^2/2k_B T_e}$

Dynamical $f_0(v; T_1, T_2 \ldots)$

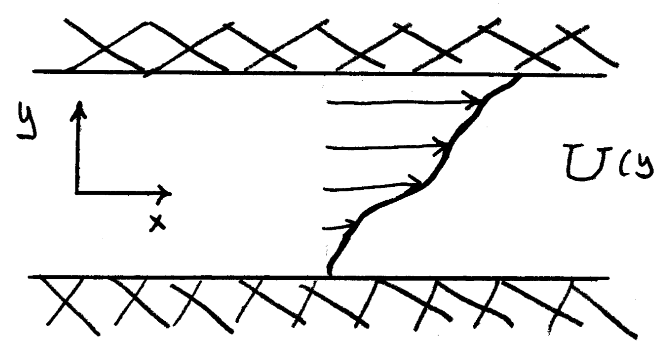
Stable $f_0' < 0.$



Euler:

Dynamical $U(y; a_1, a_2 \ldots) = \psi'$

Stable Shear Flow $U'' \neq 0$



Linear Vlasov-Poisson System

Linearize:

$$f = f_0(v) + \delta f(x, v, t)$$

Linear EOM:

$$\frac{\partial \delta f}{\partial t} + v \frac{\partial \delta f}{\partial x} + \frac{e}{m} \frac{\partial \delta \phi}{\partial x} \frac{\partial f_0}{\partial v} = 0$$

$$\delta \phi_{xx} = 4\pi e \int_{I\!R} \delta f(x, v, t)\, dv$$

Linearized Energy (Kruskal-Oberman):

$$H_L = -\frac{m}{2} \int_{\Pi} \int_{I\!R} \frac{v\,(\delta f)^2}{f_0'}\, dv dx + \frac{1}{8\pi} \int_{\Pi} (\delta \phi_x)^2\, dx$$

Eigenvalue Problem and Continuous Spectrum

Assuming $\delta f = \widehat{f}(v)e^{i\omega t - ikx}$ and $\delta\phi = \widehat{\phi}e^{i\omega t - ikx}$ the LVP becomes

$$i\omega\widehat{f} - ikv\widehat{f} - ik\widehat{\phi}[x;\widehat{f}]f_0'(v) = 0$$

$\Longleftrightarrow$ the eigenvalue problem

$$\boxed{\mathcal{L}\widehat{f} = \omega\widehat{f} \qquad \text{where} \qquad \left(\mathcal{L}\widehat{f}\right)(v) := v\,\widehat{f}(v) - f_0'(v)\int_{I\!R} du\,\widehat{f}(u)}$$

where $\mathcal{L}:\mathcal{B}\to\mathcal{B}$, Banach space $\mathcal{B}$ (e.g. L^2), $\mathcal{L}$ not self-adjoint.

The set of eigenvalues, $\operatorname{spec}\mathcal{L} = \sigma_p \cup \sigma_c \cup \sigma_r$, where

- point: $\omega \in \sigma_p$, if $\mathcal{L} - \omega\mathcal{I}$ is not one-one, $\mathcal{I} =$ identity operator.
- residual: $\omega \in \sigma_r$ if the range of $\mathcal{L} - \omega\mathcal{I}$ is not dense in $\mathcal{B}$.
- continuous: $\omega \in \sigma_c$, if the inverse of $\mathcal{L} - \omega\mathcal{I}$, defined on its range, is unbounded. Can prove $\operatorname{spec}\mathcal{L} = \sigma_c$.

Linear Hamiltonian Theory

Poisson Bracket:

$$\{F, G\}_L = \int f_0 \left[\frac{\delta F}{\delta \delta f}, \frac{\delta G}{\delta \delta f} \right] \, dx \, dv \ ,$$

with Hamiltonian as Kruskal-Oberman energy, H_L, gives the LVP system in the following form:

$$\frac{\partial \delta f}{\partial t} = \{\delta f, H_L\}_L \, ,$$

with variables <u>noncanonical</u> and H_L <u>not diagonal</u>.

Fourier Linear Theory

Assume

$$\delta f = \sum_k f_k(v,t) e^{ikx} \,, \qquad \delta\phi = \sum_k \phi_k(t) e^{ikx}$$

Linearized EOM:

$$\frac{\partial f_k}{\partial t} + ikv f_k + ik\phi_k \frac{e}{m}\frac{\partial f_0}{\partial v} = 0 \,, \qquad k^2 \phi_k = -4\pi e \int_{\mathbb{R}} f_k(v,t)\, dv$$

Three methods:

1. Laplace Transforms (Landau and others 1946)
2. Normal Modes (Van Kampen, ... 1955)
3. Coordinate Change $\Longleftrightarrow$ Integral Transform (PJM, Pfirsch, Shadwick, Balmforth 1992)

Part One Summary

A large class of systems GU2+1MT have common Hamiltonian structure with equilibria with common linear theory that has a continuous spectrum.

Part Two

Canonization and Diagonalization of Continuous Spectrum

Canonization & Diagonalization

Fourier Linear Poisson Bracket:

$$\{F,G\}_L = \sum_{k=1}^{\infty} \frac{ik}{m} \int_{I\!R} f_0' \left(\frac{\delta F}{\delta f_k}\frac{\delta G}{\delta f_{-k}} - \frac{\delta G}{\delta f_k}\frac{\delta F}{\delta f_{-k}} \right) dv$$

Linear Hamiltonian:

$$H_L = -\frac{m}{2} \sum_k \int_{I\!R} \frac{v}{f_0'} |f_k|^2 \, dv \; + \; \frac{1}{8\pi} \sum_k k^2 |\phi_k|^2$$

$$= \sum_{k,k'} \int_{I\!R} \int_{I\!R} f_k(v) \, \mathcal{O}_{k,k'}(v|v') \, f_{k'}(v') \, dv dv'$$

Canonization:

$$q_k(v,t) = f_k(v,t) \, , \qquad p_k(v,t) = \frac{m}{ik f_0'} f_{-k}(v,t) \qquad \Longrightarrow$$

$$\{F,G\}_L = \sum_{k=1}^{\infty} \int_{I\!R} \left(\frac{\delta F}{\delta q_k}\frac{\delta G}{\delta p_k} - \frac{\delta G}{\delta q_k}\frac{\delta F}{\delta p_k} \right) dv$$

Integral Transform

Definintion:

$$\boxed{f(v) = \mathcal{G}[g](v) \;:= \epsilon_R(v)\, g(v) + \epsilon_I(v)\, H[g](v)\,,}$$

where

$$\epsilon_I(v) = -\pi \frac{\omega_p^2}{k^2} \frac{\partial f_0(v)}{\partial v}\,, \qquad\qquad \epsilon_R(v) = 1 + H[\epsilon_I](v)\,,$$

and the Hilbert transform

$$H[g](v) := \frac{P}{\pi} \int_{\mathbb{R}} \frac{g(u)}{u - v}\, du\,,$$

with P denoting Cauchy principal value.

In general Hilbert-like transforms diagonalize continuous spectrum in large class of systems (cf. quantum mechanics, Simon).

Transform Properties

Theorem (G1) *$\mathcal{G}\colon L^p(\mathbb{R}) \to L^p(\mathbb{R})$, $1 < p < \infty$, is a bounded linear operator; i.e.*

$$\|\mathcal{G}[g]\|_p \leq B_p \, \|g\|_p \,,$$

where B_p depends only on p.

Theorem (G2) *If $f_0' \in L^q(\mathbb{R})$, stable, Hölder decay, then $\mathcal{G}[g]$ has a bounded inverse,*

$$\mathcal{G}^{-1}\colon L^p(\mathbb{R}) \to L^p(\mathbb{R}) \,,$$

for $1/p + 1/q < 1$, given by

$$\begin{aligned} g(u) &= \mathcal{G}^{-1}[f](u) \\ &:= \frac{\epsilon_R(u)}{|\epsilon(u)|^2} \, f(u) - \frac{\epsilon_I(u)}{|\epsilon(u)|^2} \, H[f](u) \,. \end{aligned}$$

where $|\epsilon|^2 := \epsilon_R^2 + \epsilon_I^2$.

Transform Identities

Lemma (G3) *If ϵ_I and ϵ_R are as above, then*

(i) *for $f, vf \in L^p(I\!R)$,*

$$\mathcal{G}^{-1}[vf](u) = u\,\mathcal{G}^{-1}[f](u) - \frac{\epsilon_I}{|\epsilon|^2}\frac{1}{\pi}\int_{I\!R} f\,dv\,,$$

(ii) $\mathcal{G}^{-1}[\epsilon_I](u) = \frac{\epsilon_I(u)}{|\epsilon|^2(u)}$

(iii) *and if $f(u,t)$ and $g(v,t)$ are strongly differentiable in t; i.e. the mapping $t \mapsto f(t) = f(t,\cdot) \in L^p(I\!R)$ is differentiable with the usual difference quotient converging in the L^p sense, then*

a) $\mathcal{G}^{-1}\left[\frac{\partial f}{\partial t}\right] = \frac{\partial \widehat{G}[f]}{\partial t} = \frac{\partial g}{\partial t}$,

b) $\mathcal{G}\left[\frac{\partial g}{\partial t}\right] = \frac{\partial \mathcal{G}[g]}{\partial t} = \frac{\partial f}{\partial t}$.

Integral Transformation Solution of LVP

Theorem (S1) *For initial conditions and equilibria as above,*

$$f_k(v,t) = \mathcal{G}\left[\mathcal{G}^{-1}[\mathring{f}_k]e^{-ikut}\right]$$

is a solution of (LVP) in the strong L^p sense [cf. Lemma (G3)].

This solution is arises naturally in the Hamiltonian context. Below it is written in terms of a mixed variable generation functional.

Diagonalization

Mixed Variable Generating Functional:

$$\mathcal{F}[q, P'] = \sum_{k=1}^{\infty} \int_{\mathbb{R}} q_k(v)\, \mathcal{G}[P'_k](v)\, dv$$

Canonical Coordinate changes $(q, p) \longleftrightarrow (Q', P')$:

$$p_k(v) = \frac{\delta \mathcal{F}[q, P']}{\delta q_k(v)} = \mathcal{G}[P_k](v)\,, \qquad Q'_k(u) = \frac{\delta \mathcal{F}[q, P']}{\delta P_k(u)} = \mathcal{G}^{\dagger}[q_k](u)$$

New Hamiltonian:

$$H_L = \sum_{k=1}^{\infty} \int_{\mathbb{R}} i\, \omega_k(u)\, Q'_k(u)\, P'_k(u)\, du = \sum_{k=1}^{\infty} \int_{\mathbb{R}} \omega_k(u)\, (Q_k^2 + P_k^2)/2\, du$$

where $\omega_k(u) = ku$ and $(Q', P') \longleftrightarrow (Q, P)$ is trivial.

Part Two Summary

A large class of noncanonical Hamiltonian systems GU2+1MT with continuous spectra can be diagonalized just like finite degree-of-freedom Hamiltonian systems.

Part Three

Statistical Mechanics with Continuous Spectrum

Some History

Statistical Mechanics Applied to Fluids and Plasmas:

L. Onsager (1949), 2D Euler; Burgers (1929), Euler, T. D. Lee (1952), 3D Euler, MHD; R. H. Kraichnan and D. Montgomery (1980), Salmon, Robert, Sommeria... (1990's) 2D Euler; D. Lynden-Bell (1967), Vlasov-Jeans; Turkington, Majda, Jung et al.... present

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Statistical Mechanics: finite DoF $\longrightarrow$ infinite DoF   thermo limit

Fluid and Plasma PDEs: infinite DoF $\longrightarrow$?

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Why? Turbulence is 'far from absolute equilibrium'....

$\longrightarrow$ Can do it?

Attitude $\longrightarrow$ Do calculation and see what comes out!

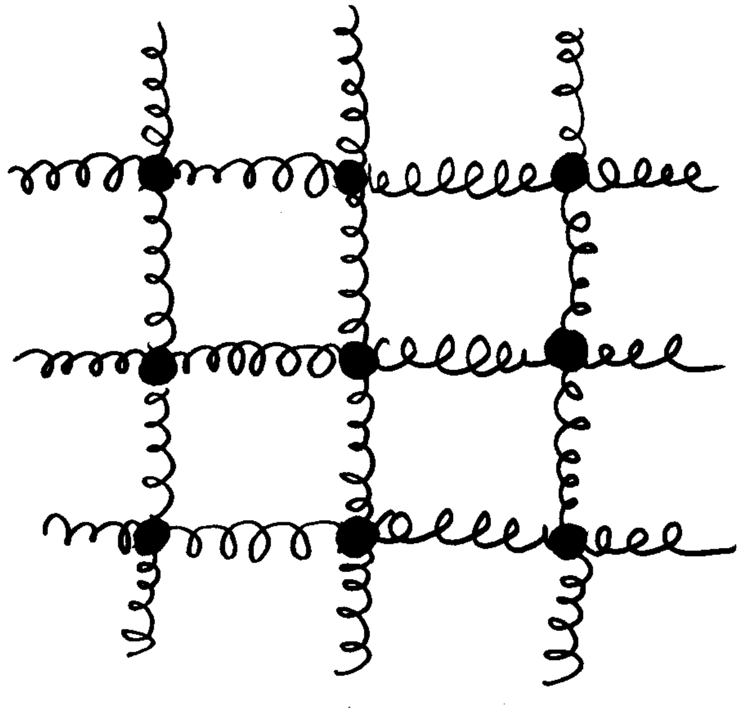
Plasma Fluctuation History

Statistical mechanics of fluctuations (Van Kampen Modes).

Other Conventional Approaches:

- Klimontovich: smooth by $< \delta - \text{functions} >$.
- Liouville Equation: truncate BBGKY hierarchy.
- Dressed Test Particle: W.B. Thompson, Rostoker, ...
- N-Body Stat Mech: partition function calculation.

Partition Function for Solid



Einstein (1907), Debye, . . .

Lattice vibrations of $3N$ SHO's

$$\mathcal{Z} = \sum_{r=0} e^{-\beta E_r}$$

$$\beta := \frac{1}{k_B T_b}$$

E_r = quantized energy levels

Dulong-Petit: classical limit $\Longrightarrow$ $\underline{C_v = 3Nk_B}$

$$\langle E_r \rangle = -\frac{\partial \ln \mathcal{Z}}{\partial \beta} = \frac{1}{\mathcal{Z}} \sum_{r=0} E_r \, e^{-\beta E_r}$$

$k_B T_b$/ DOF

Classical Systems

$$\mathcal{Z} = \int e^{-\beta E} \prod_i^N dq_i dp_i$$

Requirements: • Energy •Volume Measure

Hamiltonian Dynamics: •$E = H(q,p)$ • Liouville's Thm

Diagonalization Procedure: $H = \frac{1}{2} pMp + qGp + \frac{1}{2} qVq$

$$(q,p) \longleftrightarrow (Q,P) \quad \Longrightarrow \quad H = \sum_i^N \omega_i (Q_i^2 + P_i^2)/2 = \sum_i^N i\omega_i {Q'}_i {P'}_i$$

Equipartition: $\frac{1}{2} k_B T_b / (\quad)^2$

Field Theory – Functional Integral

$$\boxed{\mathcal{Z} = \int \; e^{-\beta H[q,p]} \; \mathcal{D}q\mathcal{D}p}$$

Fields: $q(z,t)$ and $p(z,t)$

Evaluate by discretization: $(q,p) \longrightarrow (q_i, p_i)$ and $\int dx \longrightarrow \sum_i$

We do it for Gaussian integrals for which $\exists$ rigor .

(e.g. R. Feynman $\rightarrow$ C. Dewitt-Morette)

Functional Integral

$$\mathcal{Z} = \int \; e^{-\beta H_L[q,p]} \, \mathcal{D}q\mathcal{D}p = \int \; e^{-\beta \sum_k \int \frac{\omega_k}{2}(Q_k^2 + P_k^2)\, du} \; \prod_k \mathcal{D}Q_k \mathcal{D}P_k$$

Gaussian Functional Integral: $u \to u_i\,,\; Q_k' \propto E_k$ and $\langle E_k E_k^* \rangle \Longrightarrow$

$$\boxed{\langle E_k(u) E_{k'}(u')^* \rangle = \delta_{k,k'} \, \frac{16}{V\beta} \frac{\epsilon_I(u)}{u|\epsilon|^2} \delta(u - u')}$$

Units: Volume, V, cf. $\mathcal{E}_D = \frac{V}{16\pi} \frac{\partial(\omega \epsilon_R)}{\partial \omega} |E_k|^2$

Comparison: For Maxwellian agrees with Thompson-Rostoker result, <u>otherwise new</u>. Note: T_e need not equal T_b.

Equipartition

- In previous work it is noted that $\langle E_k E_k^* \rangle$ approaches $k_B T/2$ in the limit $k\lambda_D << 1$, which suggests a failure of the equipartition theorem when this limit is not taken.

- E_k not canonical variable $\Longrightarrow \nexists$ equipartition.

- Equipartition $\Longrightarrow k_B T/2$ for each quadratic term in diagonal Hamiltonian.

- Present context equipartition $\forall\ k \Longleftrightarrow$

$$\langle Q_k(u) Q_{k'}(u') \rangle = \frac{k_B T}{ku}\, \delta_{k,k'}\, \delta(u - u')\,.$$

- Hamiltonian formulation makes this clear.

Fluctuation Spectra

Map Back:

$$E_k(u) \longmapsto f_k(v)$$

Calculational Identities $\Longrightarrow$

$$\boxed{\langle f_k(v) f^*_{k'}(v') \rangle = \delta_{k,k'} \frac{k^2}{\pi^2 e^2 V \beta} \left\{ \frac{\epsilon_I(v)}{v} \delta(v - v') - \frac{1}{\pi} \frac{\epsilon_R(0)}{|\epsilon(0)|^2} \frac{\epsilon_I(v')\epsilon_I(v)}{vv'} \right\}}$$

New Formula (Klimontovich had Maxwellian case.)

Summary

1. Vlasov Equation

   Plasma Physics, Vortex Dynamics, Stellar dyns. ...

2. Diagonalization of Continuous Spectrum

   Normal Modes of Infinite Hamiltonian System

3. Statistical Mechanics with Continuous Spectrum

   Partition Function Calculation reproduced and generalized old results. $\longrightarrow$ Have done calculations for shear flow – can do them for all GU2+1MT's!

Comparison and Comments

- N-particle statistical mechanics with Coulomb interaction: Full dynamics, approximate Z vs. linear dynamics, exact Z. Hamiltonian diagonalization $\longrightarrow$ static form factor.

- Liouville, Klimontovich, dressed test particle: Same $< EE >$ expression only. Klimontovich $< ff >$ but Maxwellian only.

- Vlasov vs. Klimontovich: Vlasov smooth, Klim. not. Same eq. different i.c.. Sum over states? Landau incomplete.

- Bath temperature vs equilibrium temp: $T_b = 1/(k_B \beta)$ a Z property; T_e is equlibrium (steady state) property.

- Thompson and Rostoker nonequilibrium $< EE >$'s: No calculation, expansion in charge; dimensionless parameter = ?

- Method of general utility: fluid shear flow, rossby-drift waves; etc. Noncanonical Poisson brackets. Organizing principle.

Shear Flow Fluctuation Spectra

Euler/QG:

$$\langle \omega_k(y) \omega_k^*(y') \rangle = \delta_{k,k'} \left\{ f(y)\delta(y-y') + g_k(y,y') \right\}$$

where f, g are determined by the equilibrium velocity profile.

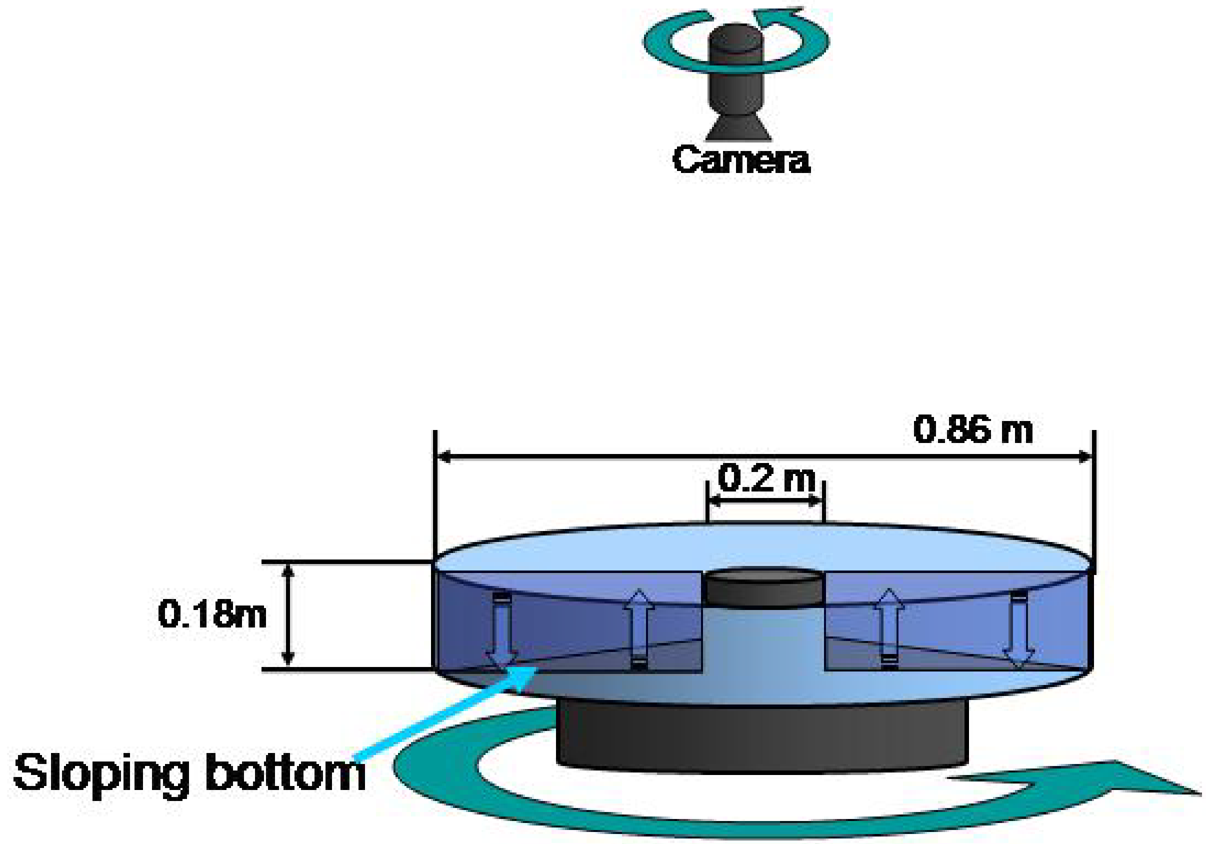
Nontrivial calculation is in S. Jung Ph.D. thesis.

Fluid Experiment

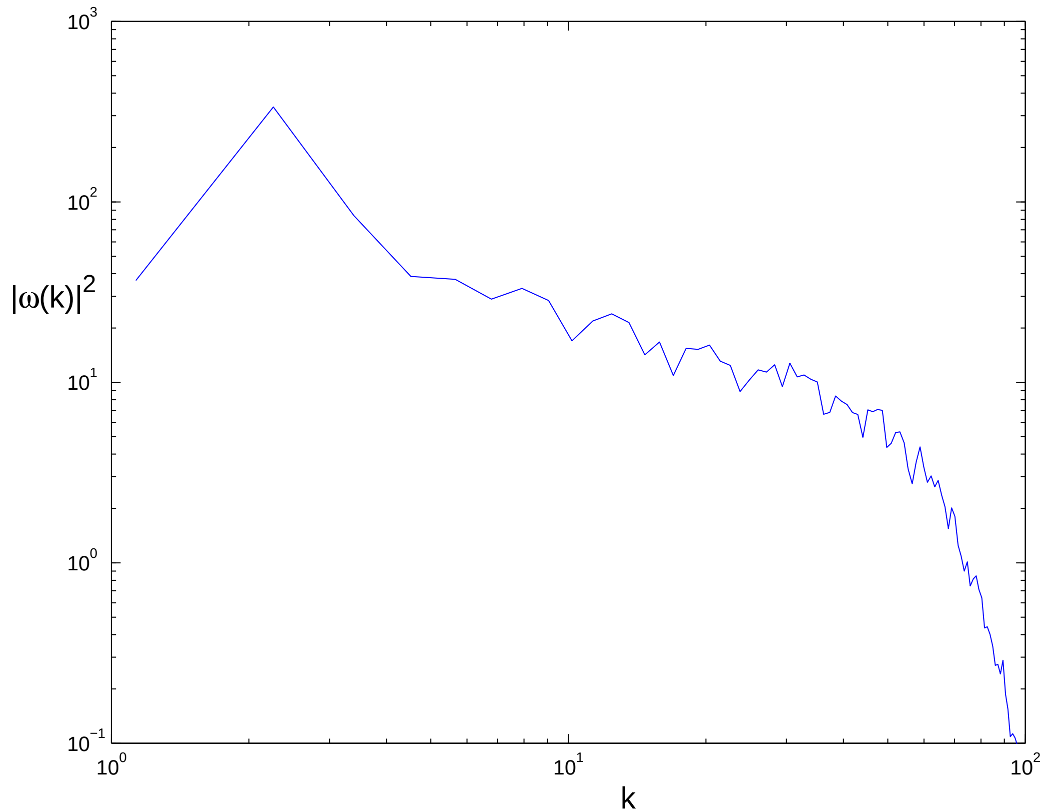
Swinney CNLD Lab

S. Jung

damped and driven



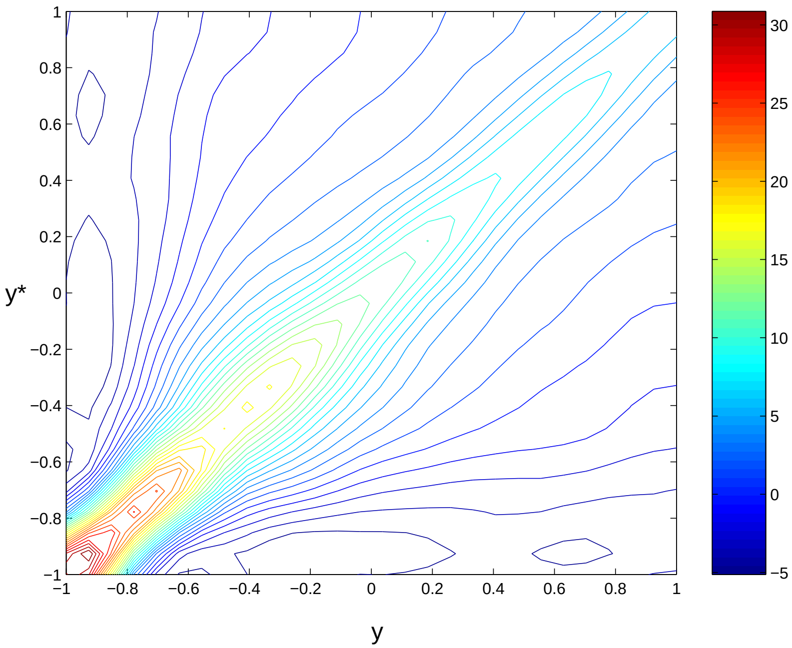
Vorticity k-Spectrum



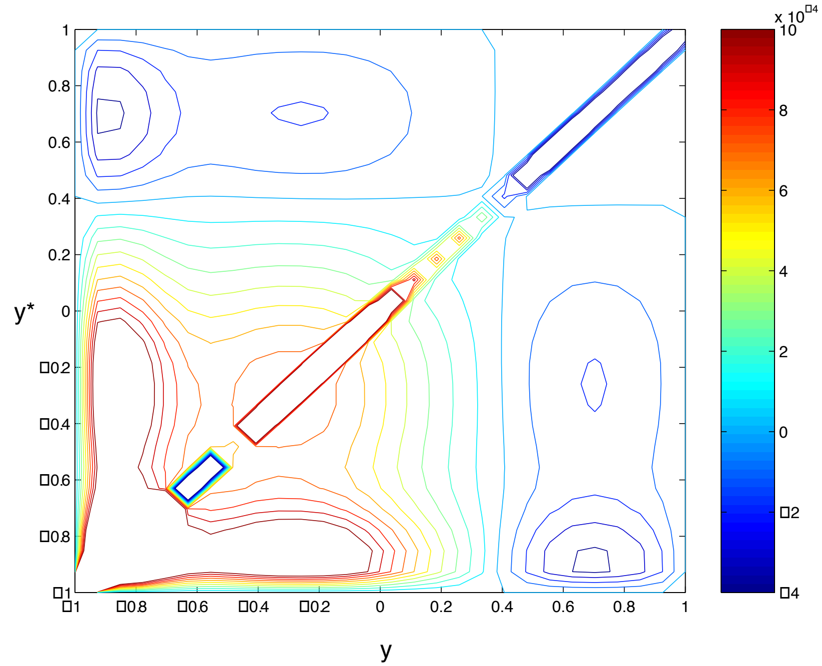
$$\langle \omega_k(y)\omega_k^*(y')\rangle \longrightarrow \langle |\omega_k|^2\rangle \sim k^{\alpha}\,, \qquad \alpha \sim \frac{1}{50}\,, \qquad \alpha \to 0 \quad \Longrightarrow$$

$$\boxed{\langle |v_k|^2\rangle \sim \frac{1}{k^2}}$$

Fluctuation Spectra



Experiment



Theory

Properties of Vlasov Systems

- most important equation in plasma physics
- nonlinear pde
- hyperbolic
- elliptic
- characteristics $\longrightarrow$ nonlinear ode
- Hamiltonian system (symplectic geometry)

$\Longrightarrow$ rich and challenging area of study

Global Existence and Uniqueness

R. Kurth (1952) and J. Batt (1963,1977)

Theorem (local) *Every initial datum* $\overset{\circ}{f} \in C^1_c(\mathbb{R}^6)$, $\overset{\circ}{f} \geq 0$, *launches a unique classical solution* f *on some time interval* $[0, T[$ *with* $f(0) = \overset{\circ}{f}$. *For all* $t \in [0, T[$ *the function* $f(t)$ *is compactly supported and non-negative. If* $T > 0$ *is chosen maximal and if*

$$\sup\{|v| \mid (x, v) \in \operatorname{supp} f(t),\ 0 \leq t < T\} < \infty$$

or

$$\sup\left\{\rho(t, x) \mid 0 \leq t < T,\ x \in \mathbb{R}^3\right\} < \infty,$$

then the solution is global, i.e., $T = \infty$.

K. Pfaffelmoser, P.-L. Lions and B. Perthame (1989)

Theorem (global) *Any non-negative initial datum* $\overset{\circ}{f} \in C^1_c(\mathbb{R}^6)$ *launches a global classical solution of the Vlasov-Poisson system.*