SEM: Stress-Energy-Momentum Tensors

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Joint work with others, mentioned as we proceed

CALTECH
Control & Dynamical Systems

Goals of the Lecture

- ☑ Explain a little about classical field theories, multisymplectic geometry, multimomentum maps, etc.
- ☑ Discuss the corresponding discrete theory and associated variational integrators that lead to things like AVIs (Asynchronous Variational Integrators) for field theories such as nonlinear elasticity and electromagnetism.
- ☑ Define SEM (Stress-Energy-Momentum) tensors using multimomentum maps --- that is in terms of the flux of momentum and energy across surfaces.
- ☑ Introduce the Kuchar trick to make any field theory diffeomorphism invariant
- ☑ Establish the Hilbert-Belinfante-Rosenfeld formula:

$$T^{\mu\nu} = 2\frac{\partial L}{\partial G_{\mu\nu}}$$

Some Classical Field Theory

- ☑ Fields, jet bundles, dual jet bundles
- ☑ Multisymplectic forms
- ☑ Multimomentum maps
- ☑ Euler--Lagrange equations
- ☑ Field theoretic analog of “Flow is symplectic”

Fields

- $\pi_{XY} : Y \to X$ a *covariant configuration bundle*. Coordinates on X are x^μ, $\mu = 0, 1, \ldots, n$ (and so $\dim X = n+1$) and coordinates on Y denoted y^A.
- ***Fields*** are sections $\phi : X \to Y$.
- Each such field determines a section $j^1\phi$ of the first jet bundle $J^1Y \to X$, an *affine bundle*; in coordinates, $j^1\phi$ is given by

$$x^\mu \mapsto \left(x^\mu, \phi^A(x^\mu), \partial_\nu \phi^A(x^\mu)\right).$$

- **Electromagnetism**: X is spacetime and Y is the bundle of one-forms. Sections $A : X \to Y$ are electromagnetic potentials.

Dual Jet Bundle

- Field-theoretic analogue of the cotangent bundle.
- ***Dual jet bundle*** $J^1Y^\star$ is the *vector* bundle over Y whose fiber at $y \in Y_x$ is the set of *affine* maps from J^1_yY to $\Lambda^{n+1}_x X$, where $\Lambda^{n+1}X$ denotes the bundle of $(n+1)$-forms on X.
- Coordinates on $J^1Y^\star$ are $(x^\mu, p, p_A{}^\mu)$, which correspond to the affine map given in coordinates by

$$v^A{}_\mu \mapsto \left(p + p_A{}^\mu v^A{}_\mu\right) d^{n+1}x$$

where

$$d^{n+1}x = dx^0 \wedge dx^1 \wedge \cdots \wedge dx^n.$$

Canonical Forms

- Analogous to the canonical one- and two-forms on a cotangent bundle, there are canonical forms on $J^1Y^\star$.
- An $n+1$ form Θ and an $n+2$ form Ω:

$$\Theta = p_A{}^\mu dy^A \wedge d^n x_\mu + p\, d^{n+1}x,$$

and

$$\Omega = -\mathbf{d}\Theta = dy^A \wedge dp_A{}^\mu \wedge d^n x_\mu - dp \wedge d^{n+1}x.$$

Lagrangian

- Lagrangian density:

$$\mathcal{L} : J^1Y \longrightarrow \Lambda^{n+1}X,$$

In coordinates, write

$$\mathcal{L} = L\left(x^\mu, y^A, v^A{}_\mu\right) d^{n+1}x.$$

- *Covariant Legendre transformation*: a map

$$\mathbb{F}\mathcal{L} : J^1Y \longrightarrow J^1Y^\star$$

$$p_A{}^\mu = \frac{\partial L}{\partial v^A{}_\mu}, \quad p = L - \frac{\partial L}{\partial v^A{}_\mu} v^A{}_\mu$$

- Note that the multimomenta $p_A{}^\mu$ are "conjugate" to the space as well as the time derivatives.

Euler–Lagrange Equations

- Variational principle:

$$\delta \int \mathcal{L}(j^1\phi) = 0$$

- Equivalent to the Euler–Lagrange equations:

$$\frac{\partial L}{\partial y^A}(j^1\phi) - \frac{\partial}{\partial x^\mu}\left(\frac{\partial L}{\partial v^A{}_\mu}(j^1\phi)\right) = 0$$

- Analog of symplecticity in Lag/Ham mechanics: ***Multisymplecticity***: a consequence of the variational principle. *The integral of the contraction of Ω with two first variation solutions of EL over the boundary of a spacetime region is zero.*

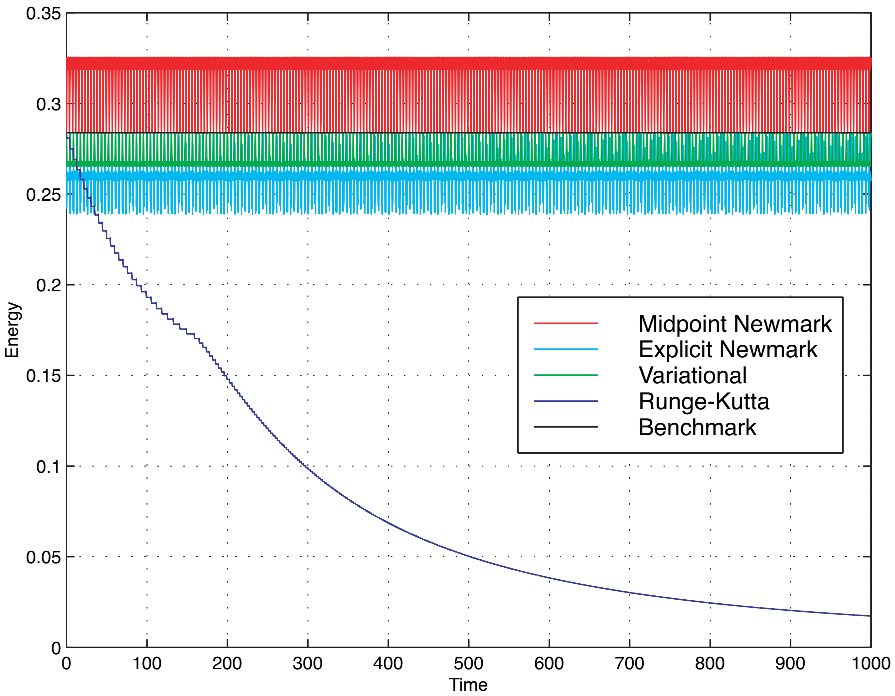
People: Patrick, Shkoller, JM

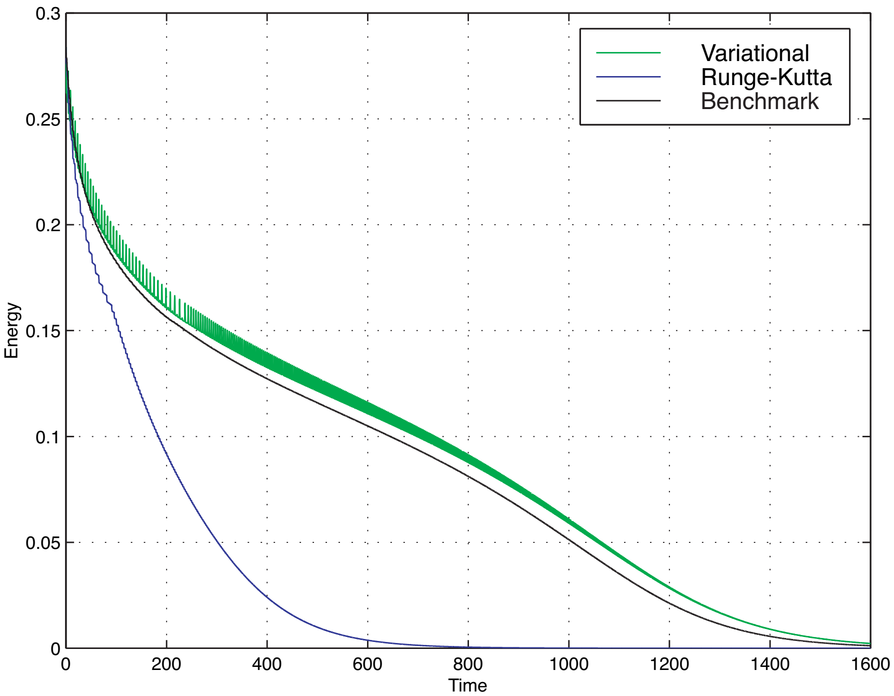
Discrete version of this theory

- ☑ Leads to a field-theoretic version of variational integrators
- ☑ Integrators are similar to spacetime finite element methods
- ☑ Multisymplectic, good energy behavior
- ☑ Lead to AVI methods
- ☑ Applied to nonlinear wave equations (sine-Gordon), nonlinear elasticity, etc.

Variational Integrators

- ☑ Based on discrete versions of the variational principles of mechanics.
- ☑ Respects the structure of mechanics (symplectic, energy, momentum).
- ☑ Gets the energy budget right even for forced or dissipative systems

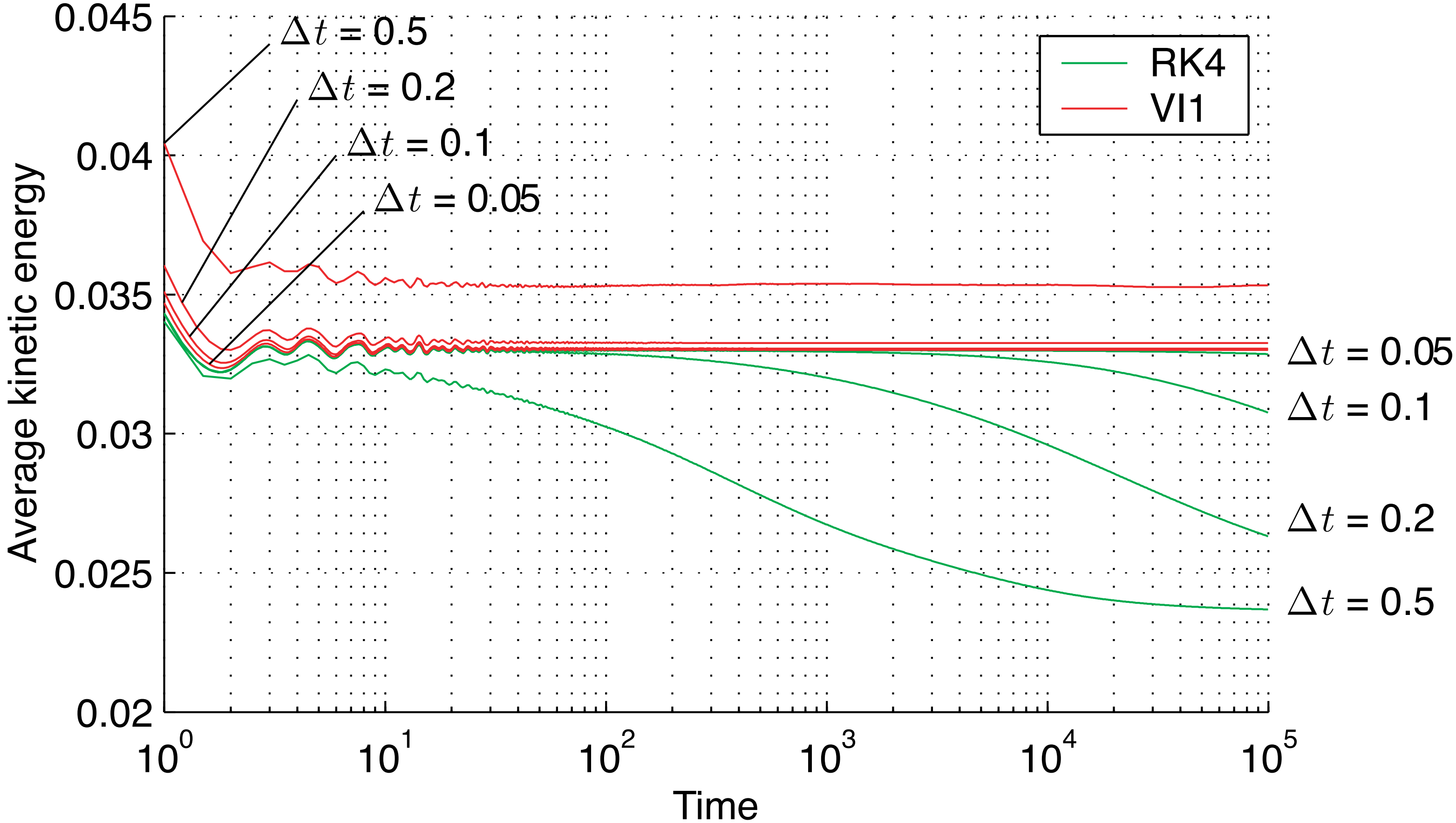




People: Kane, West, Ortiz and JM

Statistical Properties

In addition, these algorithms get the statistics right—in the sense of uncertainty propagation in time (and, eventually in space, through a network). No theorem yet, but it is surely related to symplecticity.



People: Lew, West, Ortiz and JM

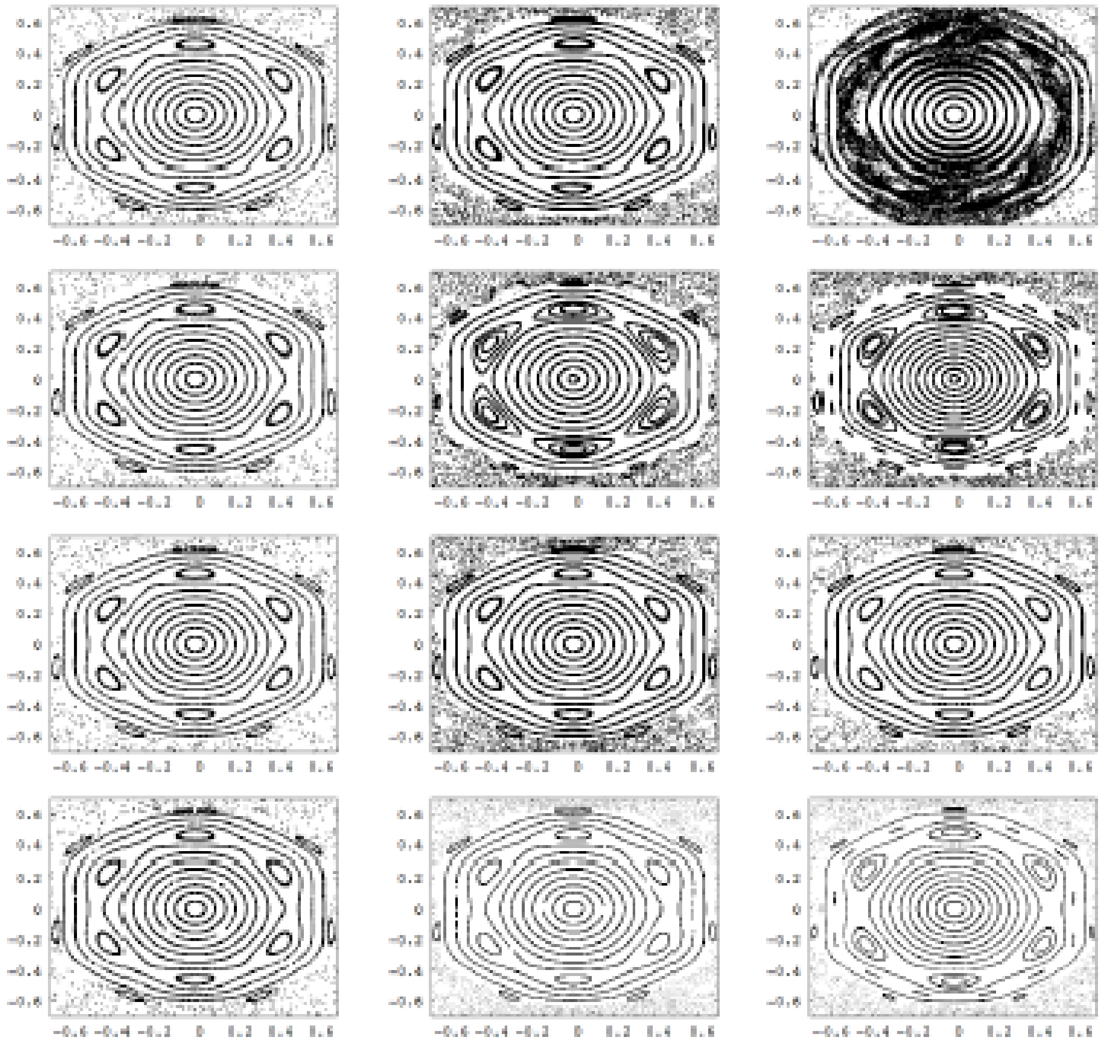
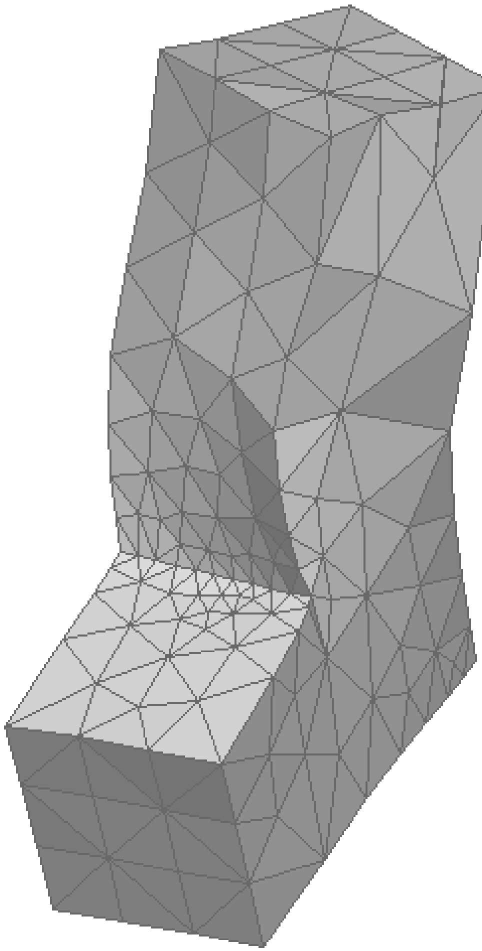


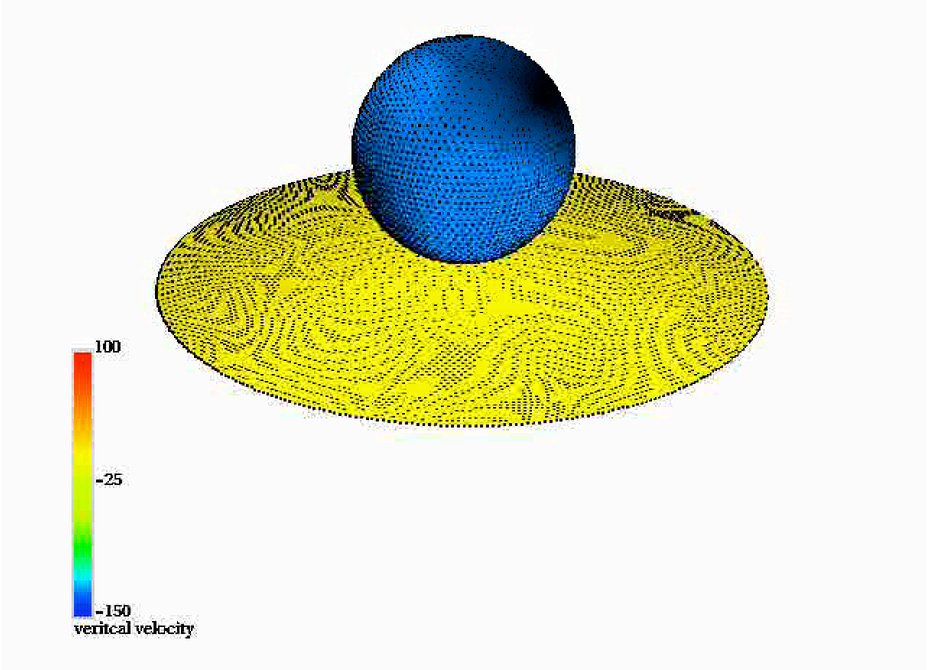
Figure 6.2: **Poincaré sections vs. Time-Step.** From top Poincaré sections computed using RK4, CAY, TLN, and FLV. From left the time-step used is $h = 0.0125, 0.025, 0.05$ and the time-interval of integration is $[0, 10^6]$. These Poincaré sections are for a underwater vehicle with the following values of the integrals of motion $\mathbf{\Pi} \cdot \mathbf{p} = 0$, $\mathbf{p} \cdot \mathbf{p} = 5.2^2$ and $\mathcal{H} = 4.0$. The section is obtained by plotting points Π_x, Π_z for which $p_z = 0$. RK4 is the only method that does not perform well in this experiment.

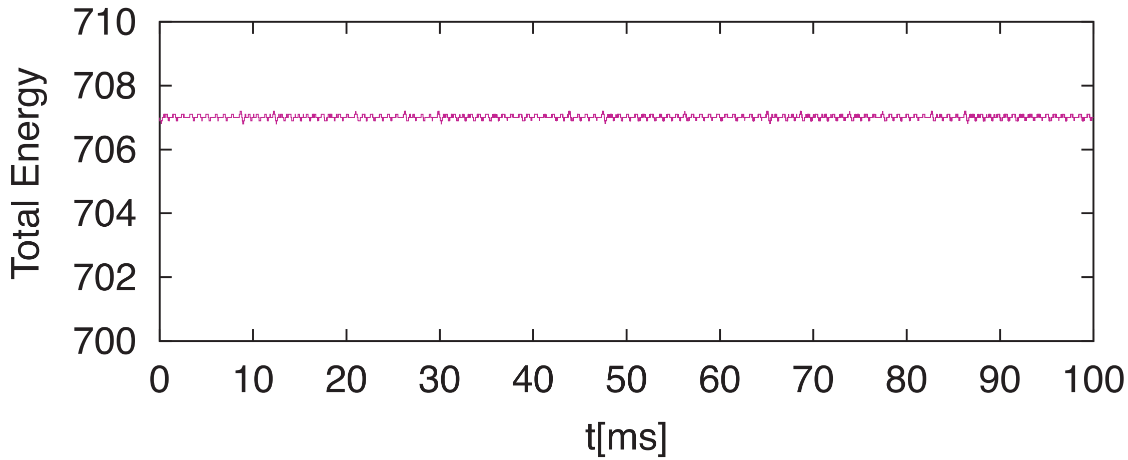
Variational integrators compute chaotic invariant sets much more robustly than even higher order accurate algorithms.

People: Bou-Rabee and JM

AVI: Asynchronous Variational Integrators





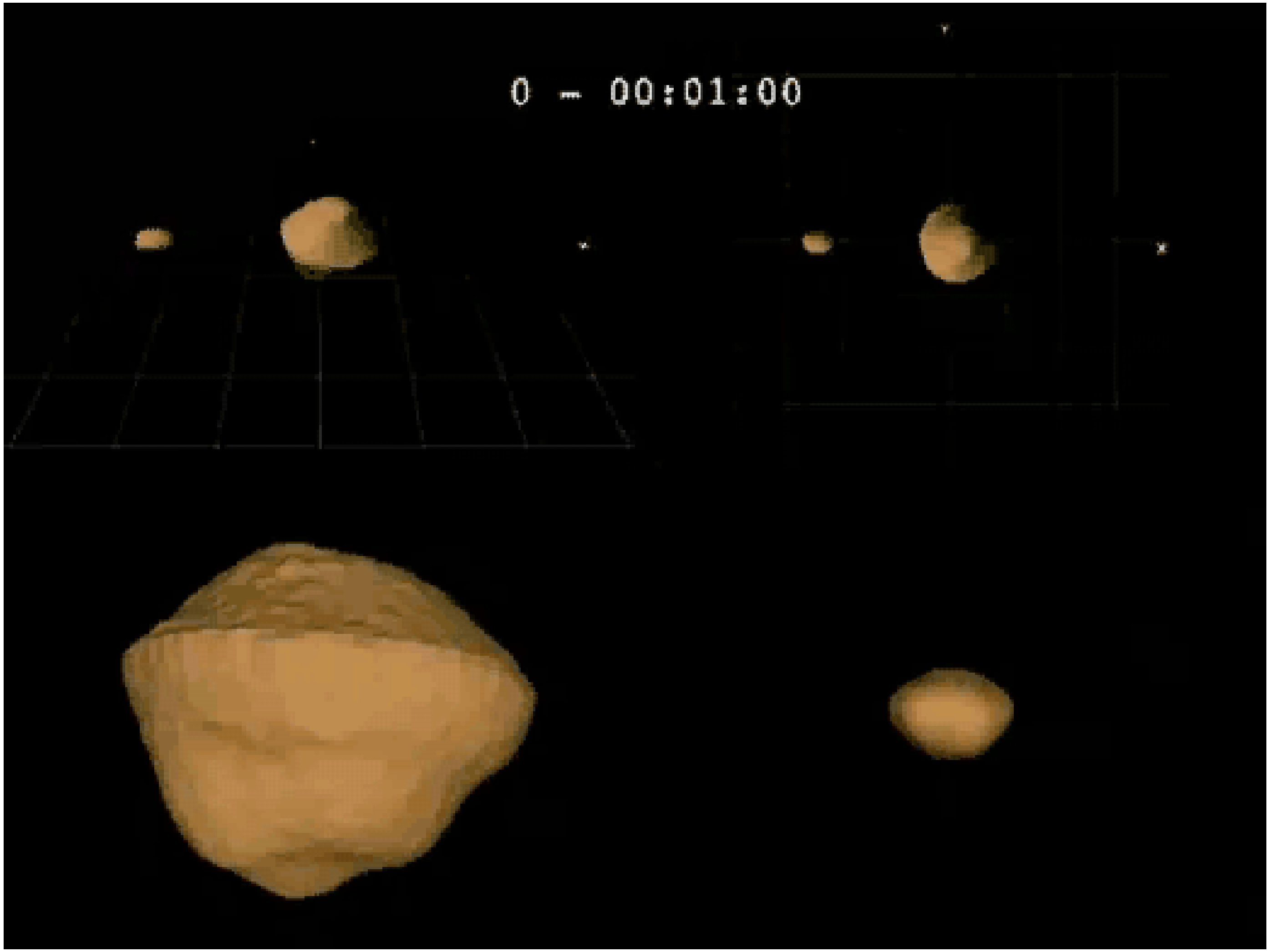


Excellent energy behavior, even after millions of temporal updates.

Can take different time steps with different elements in the mesh.

People: Lew, West, Ortiz, Cirac and JM

State of the Art



People: Scheeres, Leok et al

Electromagnetism

- For electromagnetism on a fixed background spacetime X with metric g, the Lagrangian density is
$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \sqrt{-g}\, d^4x, \quad F = dA.$$
- Legendre transformation
$$\mathfrak{F}^{\mu\nu} = F^{\mu\nu}\sqrt{-g} \qquad \text{and} \qquad p = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \sqrt{-g}.$$
- Cartan form $\Theta_{\mathcal{L}}$
$$\Theta_{\mathcal{L}} = \sqrt{-g} F^{\nu\mu} dA_\nu \wedge d^3x_\mu + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \sqrt{-g}\, d^4x$$
- The 5-form $\Omega_{\mathcal{L}} = -\mathbf{d}\Theta_{\mathcal{L}}$
- Euler–Lagrange equations: $dF = 0$, $\delta F = 0$ (same as Maxwell).

Symmetry

- Symmetry groups $\mathcal{G}$, automorphism group of Y.
- Can cover diffeomorphisms on the base X
- Lift to dual of the jet bundle using the analog of cotangent lift
- Associated multimomentum map

$$J : J^1Y^* \to \mathfrak{g}^* \otimes \Lambda^n J^1Y^* = L(\mathfrak{g}, \Lambda^n J^1Y^*)$$

given in coordinates by

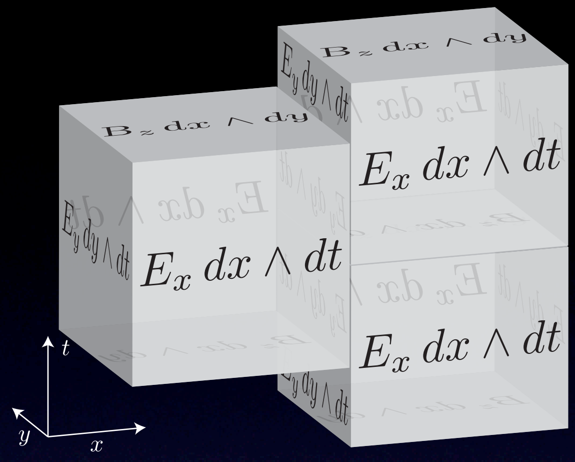
$$\langle J, \xi \rangle = (p_A{}^\mu \xi^A + p\, \xi^\mu)\, d^n x_\mu - p_A{}^\mu \xi^\nu dy^A \wedge d^{n-1} x_{\mu\nu},$$

- If the Lagrangian is covariant then there is an associated Noether theorem giving a conservation identity for J.

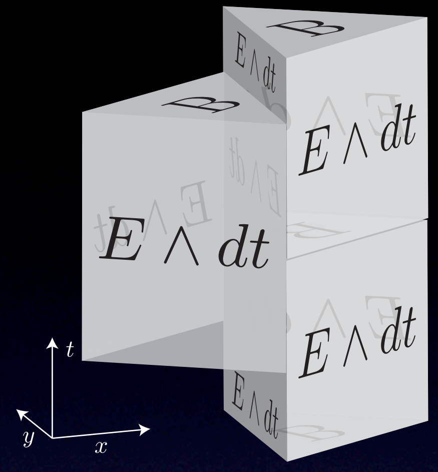
What about symmetry?

- Examples like electromagnetism have a symmetry
- Gravity too, but...
- Associated conservation laws, such as div E = 0.
- For discrete integrators to respect that, need the discrete action to have the same symmetry
- Symmetry in E and M is of course the usual gauge symmetry: A —> A + df
- For E and M, this problem is cured using DEC (Discrete Exterior Calculus)

AVIs, DEC and Computational Electromagnetism

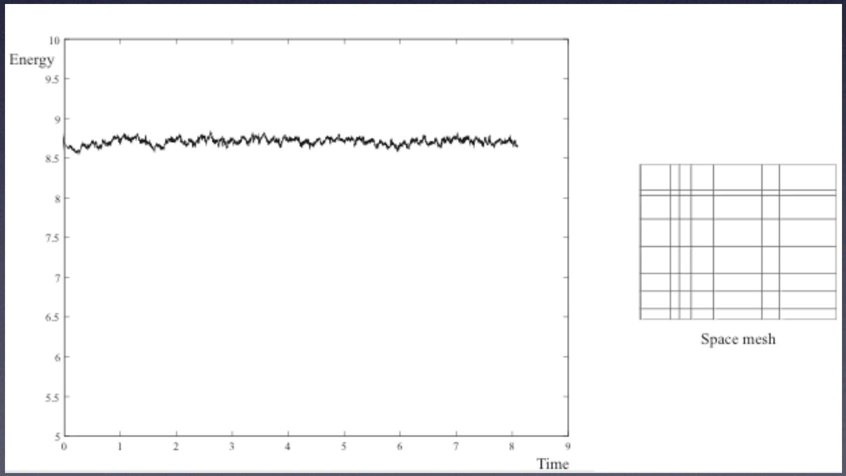


Rectangular spatial mesh



Unstructured spatial mesh

AVI methods generalize the Yee/ Bossavit/ Kettunen scheme and allow for asynchronous time stepping (yet obey the geometry)
Still multisymplectic and with good respect for the mechanics



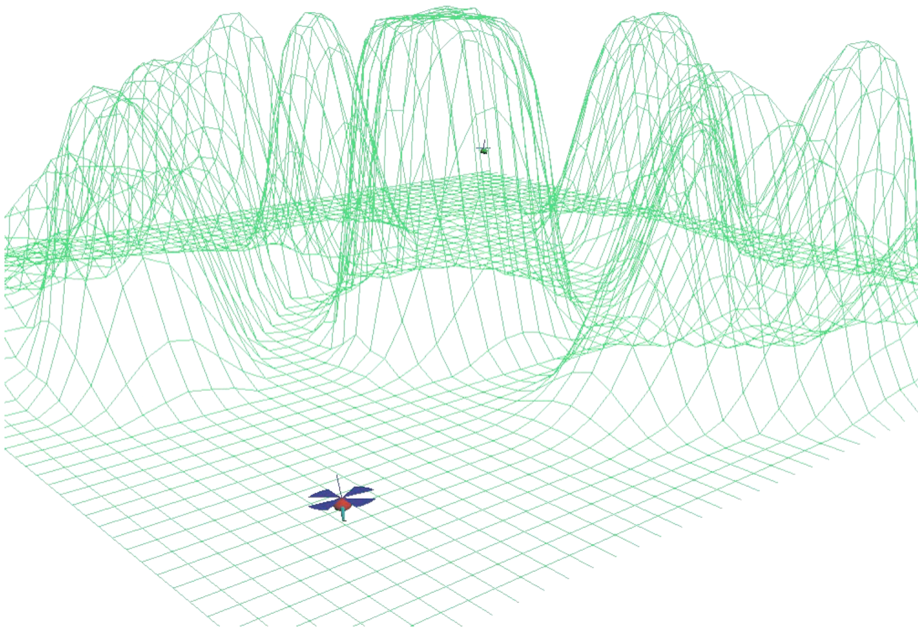
People: Stern, Desbrun, Tong and JM

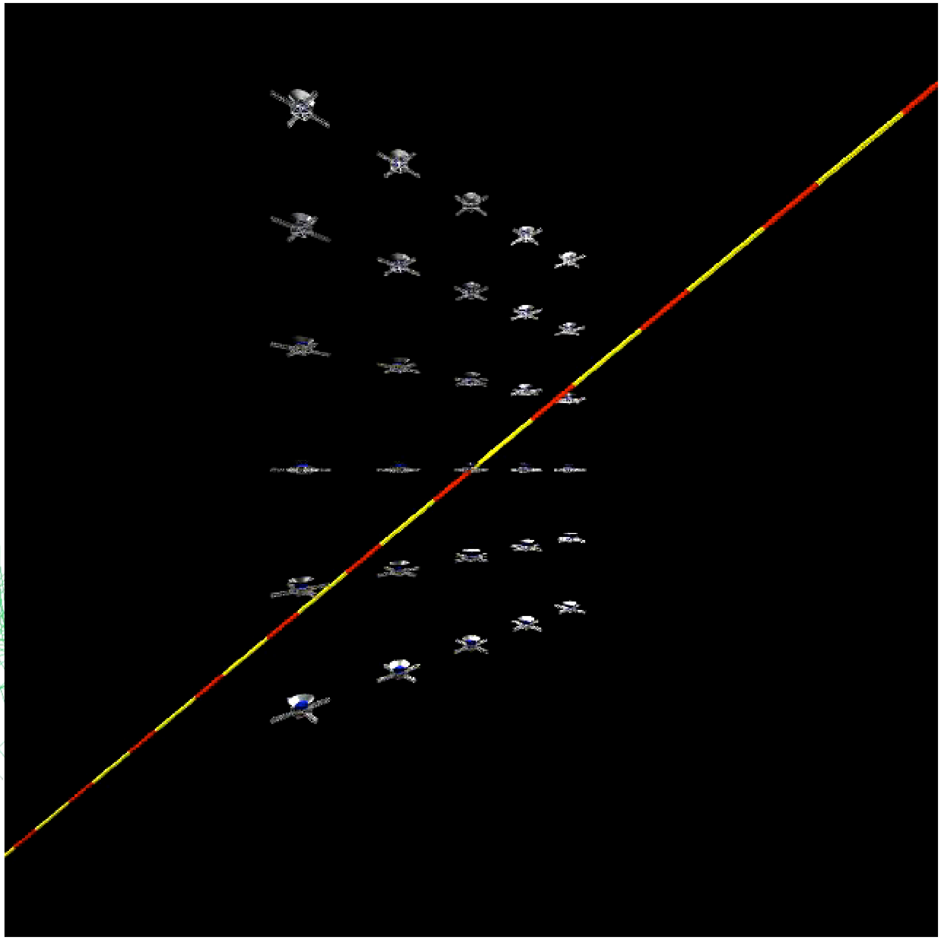
Discrete Mechanics Break

- **Discrete mechanics is great for doing optimal control problems in addition of doing time stepping in field theories.**
- **We give a few examples of DMOC (Discrete Mechanics and Optimal Control) in action to wet your appetite**

DMOC—Discrete Mechanics and Optimal Control

- Same variational discretization methods as time stepping. Flexibility of variational methods enable hierarchical and parallelizable methodologies

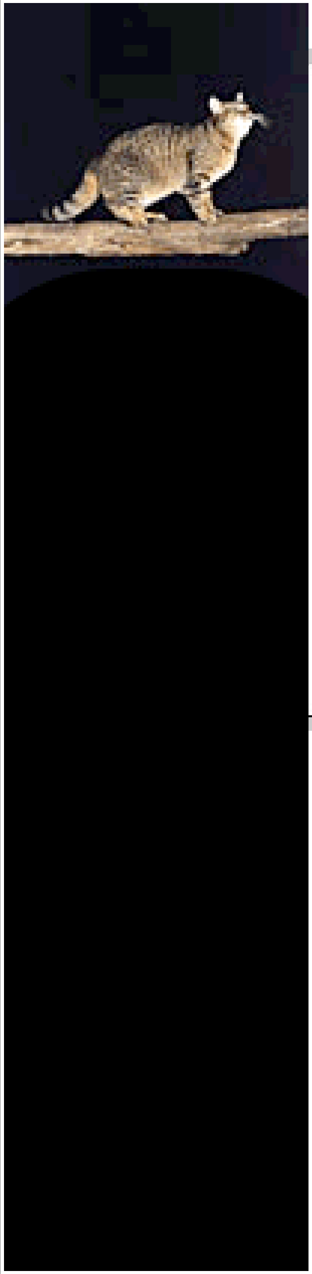


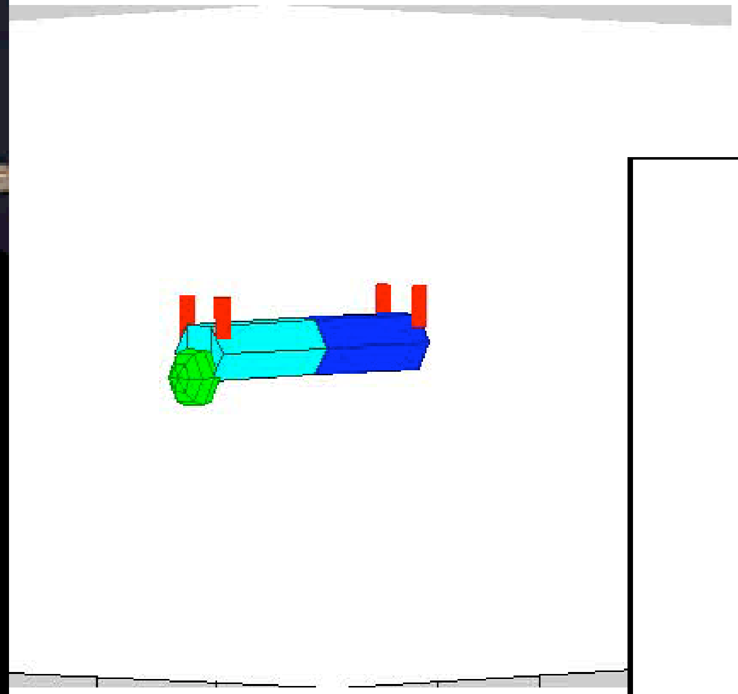


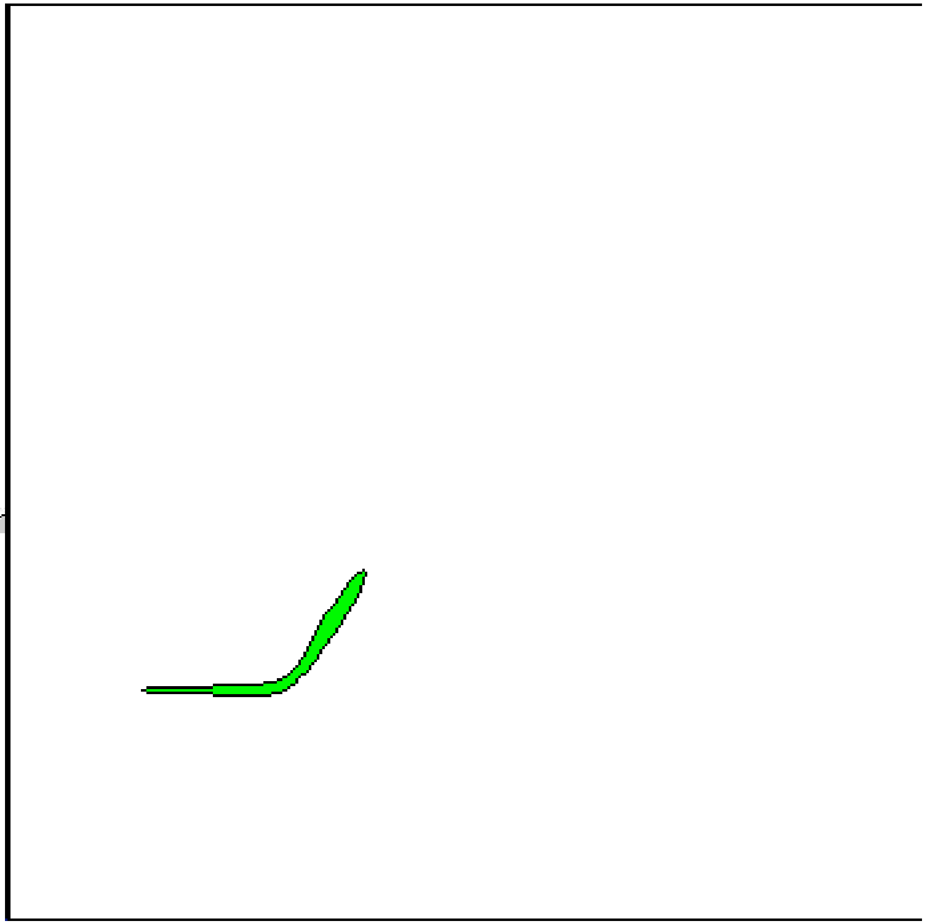
- Many problems can be parallelized
- Applies to many sorts of problems, such as optimal walking and other robotic problems

People: Ober-Bloebaum, Junge, Kobilarov and JM

Falling Cats, Optimal Swimming

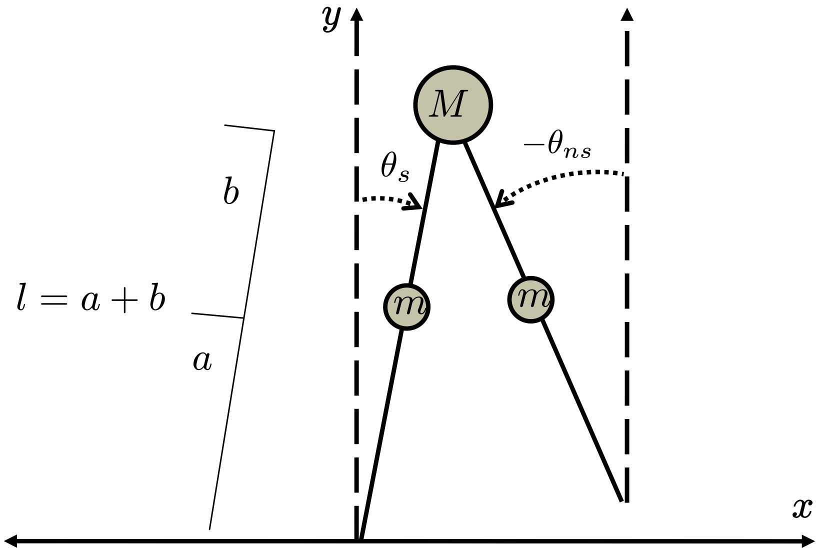


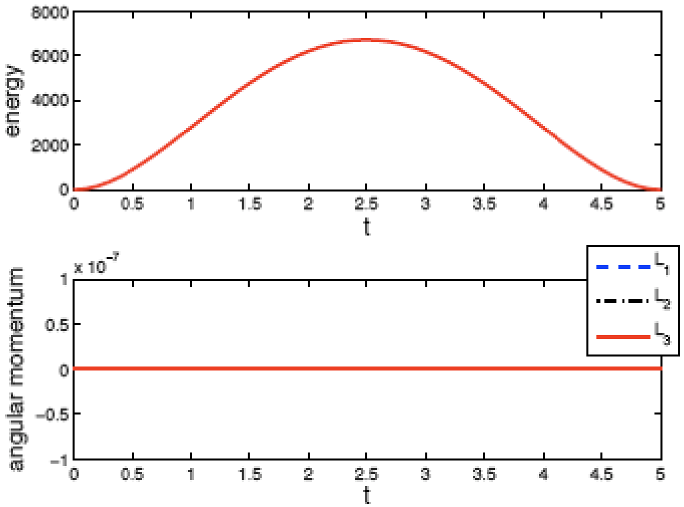


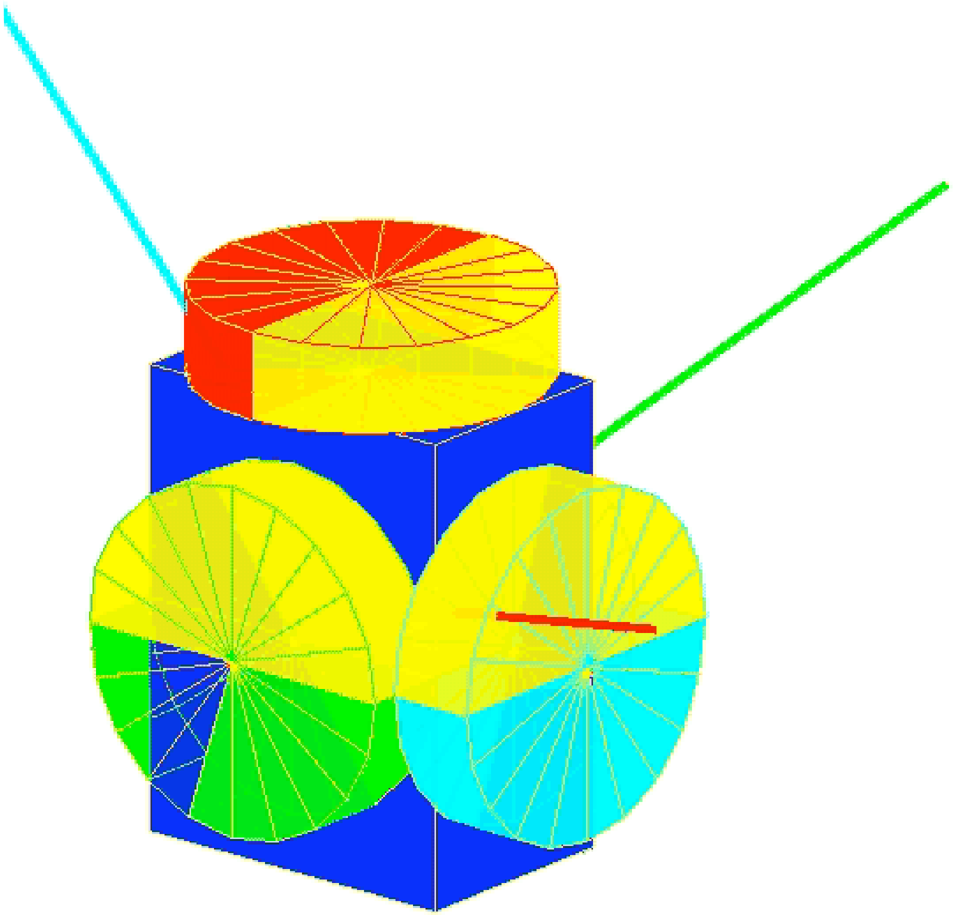


People: Kanso, Martin, Ober-Bloebaum, Leyendecker and JM

Optimal Walking and Spacecraft Reorientation







People: Ober-Bloebaum, Leyendecker, Ortiz, Pekarek and JM

SEM Tensor

- **Theorem.** (Gotay and JM, 1992). Consider a $\mathcal{G}$-covariant Lagrangian field theory and suppose $\mathcal{G}$ covers $\mathrm{Diff}(X)$. For each section $\phi : X \to Y$ there exists a unique $(1,1)$-tensor density $\mathfrak{T}(\phi)$ on X such that

$$\int_\Sigma i_\Sigma^*(j^1\phi)^* J^{\mathcal{L}}(\xi_Y) = \int_\Sigma T^\mu{}_\nu(\phi)\xi^\nu d^n x_\mu$$

  for all $\xi \in \mathfrak{X}_c(X)$ and all hypersurfaces Σ, where $i_\Sigma : \Sigma \to X$ is the inclusion.
- This defines the SEM tensor in terms of fluxes of the multimomentum map. In coordinates T is given by the "canonical stress energy tensor" one finds in the books plus correction terms (Belinfante-Rosenfeld formula).

Major Glitch and a Cure

- If a field theory is coupled to gravity, then the symmetry assumption will hold.
- But what if the spacetime metric is a background, as in electromagnetism?
- Use the Karel Kuchar trick: suppressing the jet dependence, write $\mathcal{L}(\phi, G)$ for the given Lagrangian, depending on a metric G on X. Assume there is another metric g on another copy $\tilde{X}$ of X.
- Introduce a *new field* and a *new Lagrangian*:

$$\widetilde{\mathcal{L}}(\phi, \eta) = \mathcal{L}(\phi, \eta^* g)$$

for the new Lagrangian, where $\eta : X \to \tilde{X}$. This will cure the ills.

Covariant Lagrangian

- Assume that the given Lagrangian $\mathcal{L}$ has the eminently reasonable covariance property for a diffeomorphism $\psi : X \to X$:

$$\psi^* \left(\mathcal{L}(\phi, G)\right) = \mathcal{L}(\psi^* \phi, \psi^* G)$$

  where, as earlier, we assume that there is a natural way to lift the diffeomorphism action to Y.

- The new Lagrangian is covariant and so will have a momentum map and we use the above theorem to define the SEM tensor.
- But we seem to have introduced another disease! What are the Euler–Lagrange equations for the new field?

The Cure

- *Three wonderful things now happen*
- First, using the techniques in the 1992 Gotay and JM paper, we get the "Hilbert formula":
- $T^{\mu\nu} = 2\frac{\partial L}{\partial G_{\mu\nu}}$
- Denoting the momentum conjugate to the derivatives ${\eta^a}_{,\mu}$ by ${p_a}^\mu$, we get a version of the Piola-Kirchhoff SEM tensor in nonlinear elasticity:

$$ {\rho_a}^\mu = T^{\mu\nu} {\eta^b}_{,\nu} g_{ab} $$

- Finally, the Euler–Lagrange equations for the extra field show that the divergence of the stress energy momentum tensor is zero!

The SEM part was done with these two guys

Mark Gotay

Marco Castrillon-Lopez

Our paper on these things is our birthday gift to you, Darryl