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Analytical and Numerical Study of the Two-dimensional Navier-Stokes-α and Leray-α models of turbulence

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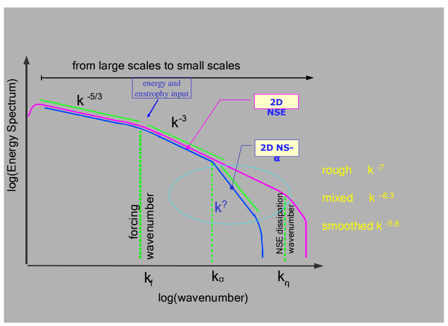
Summary

The Navier-Stokes-α model of turbulence is a mollification of the Navier-Stokes equations in which the vorticity is advected and stretched by a smoothed velocity field. The smoothing is performed by filtering the velocity field over spatial scales of size smaller than α. This is achieved by convolution with a kernel associated with the Green's function of the Helmholtz operator scaled by a parameter α. The statistical properties of the smoothed velocity field are expected to match those of Navier-Stokes turbulence for scales larger than α, thus providing a more computable model for those scales.

For wavenumbers k such that $k\alpha \gg 1$, corresponding to spatial scales smaller than α, there are three candidate power laws for the energy spectrum, corresponding to three possible characteristic time scales in the model equations: one from the smoothed field, the second from the rough field and the third from a special combination of the two. In two dimensions, the second time scale may be understood to characterize the dynamics of the conserved enstrophy.

We measure the scaling of the energy spectra from high-resolution simulations of the two-dimensional Navier-Stokes-α model, in the limit as $\alpha \to \infty$. The energy spectrum of the smoothed velocity field scales as k^{-7} in the direct enstrophy cascade regime, consistent with dynamics dominated by the timescale associated with the *rough* velocity field. We are thus able to deduce that the dynamics of the dominant cascading conserved quantity, namely the enstrophy of the rough velocity, governs the scaling of all derived statistical quantities.

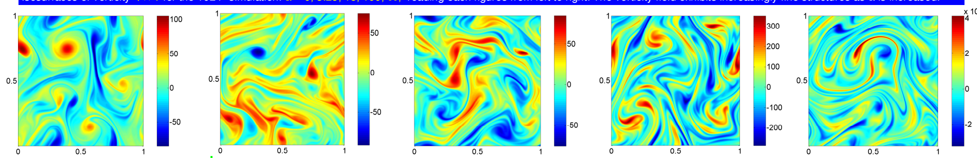


For the two-dimensional Leray-α model, the time scale coming from the special combination of the rough and smoothed field is understood to characterize the dynamics of the conserved enstrophy.

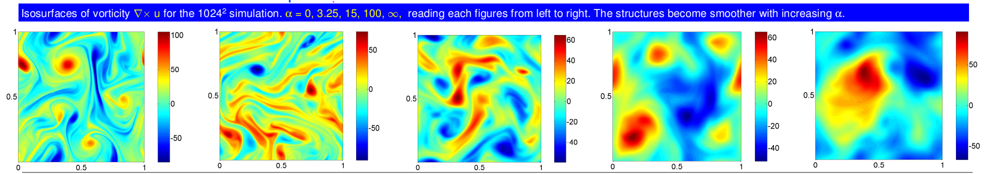
We measure the scaling of the energy spectra from high-resolution simulations of the two-dimensional Leray-α model, in the limit as $\alpha \to \infty$. The energy spectrum of the smoothed velocity field scales as k^{-5} in the direct enstrophy cascade regime, consistent with dynamics dominated by the timescale associated with the special combination of the rough and smoothed velocity field. We are thus able to deduce that the dynamics of the dominant cascading conserved quantity, namely the enstrophy produced by the special combination of the rough and smoothed velocity, governs the scaling of all derived statistical quantities.

This result is consistent with our result above, on the behaviour of the spectral slope of the two-dimensional Navier--Stokes-α model in the wavenumber regime $k\alpha \gg 1$.

Isosurfaces of vorticity $\nabla\times v$ for the 1024^2 simulation. α = 0, 3.25, 15, 100, ∞, reading each figures from left to right. The vorticity field exhibits increasingly fine structures as α is increased.



Isosurfaces of vorticity $\nabla\times u$ for the 1024^2 simulation. α = 0, 3.25, 15, 100, ∞, reading each figures from left to right. The structures become smoother with increasing α.



The NS-α analytic sub-grid scale model of turbulence

Foias, Holm, Titi (J. Dyn. Diff. Eqns. 2001)

$$\frac{\partial}{\partial_t}v - \nu\Delta v + (u\cdot\nabla)v + \sum_{j=1}^{3} v_j \nabla u_j + \nabla p = f$$
$$\nabla\cdot v = \nabla\cdot u = 0$$
$$v = u - \alpha^2\Delta u$$

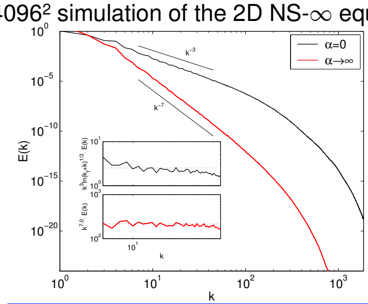
The Leray-α analytic sub-grid scale model of turbulence

Cheskidov, Holm, Olsen, Titi (Royal Soc. A, MPES 2005)

$$\frac{\partial}{\partial_t}v - \nu\Delta v + (u\cdot\nabla)v + \nabla p = f$$
$$\nabla\cdot v = \nabla\cdot u = 0$$
$$v = u - \alpha^2\Delta u$$

Leray Acta Math. 1934 – Regularized NSE
$u = \phi_\alpha * v$
ϕ_α - the Green's function associated with $(1-\alpha^2\Delta)$

4096^2 simulation of the 2D NS-∞ equation



The black curve is the spectrum for the DNS, the red curve is the spectrum for $\alpha\to\infty$.
The black curve in the inset corresponds to the NSE energy spectrum compensated by $k^3\ln(kf+k)^{1/3}$.
The red curve in the inset is the energy spectrum $E^u(k)$ for NS-∞ compensated by k^7.
The region $6 < k < 40$ is flat indicating the nominal range over which the k^{-7} scaling holds.

Energy conserved* = .5 (u,v)
Enstrophy conserved* = .5 ($\nabla\times$ v, $\nabla\times$ v)

Power Laws

vv: $k^{-21/3} = k^{-7}$
uv: $k^{-19/3} \sim k^{-6.3}$
uu: $k^{-17/3} \sim k^{-5.6}$

Numerical Results: ($\alpha\to\infty$)

1024^2: $k^{-7.4}$
2048^2: $k^{-7.1}$
4096^2: k^{-7}

* -- in the absence of viscosity and forcing

THE NS-∞ EQUATION

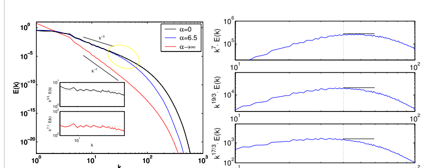
Scale (to prevent trivial dynamics)

$$\partial_t v - \nu\Delta v - u\times\nabla\times v = -(\alpha/L)^2\nabla p + (\alpha/L)^2 f$$
$$\nabla\cdot u = \nabla\cdot v = 0$$
$$v = u - \alpha^2\Delta u$$

Take the limit $\alpha\to\infty$

$$\partial_t v - \nu\Delta v - u\times\nabla\times v = -\nabla p + f$$
$$\nabla\cdot u = \nabla\cdot v = 0$$
$$v = -L^2\Delta u$$

2048^2 simulation of the 2D NS-α for α finite

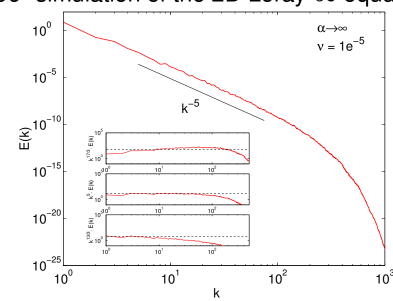


Compensated energy spectra for 2048^2 simulation for α =6.5 (k_α=39.75; vertical dashed line).

The energy spectrum is compensated by k^7, $k^{19/3}$, and, $k^{17/3}$ respectively.

The region $39 < k < 70$ in the first subplot follows a flat regime which indicates the nominal range over which the k^{-7} scaling holds.

4096^2 simulation of the 2D Leray-∞ equation



The red curve is the 2D Leray-α spectrum for $\alpha\to\infty$. The red curves in the inset are the energy spectrum compensated by $k^{17/3}$, $k^{15/3}$, $k^{13/3}$, respectively.
The region $7 < k < 70$ in the second subplot follows a flat regime which indicates the nominal range over which the k^{-5} scaling holds.

Energy conserved* = .5 (u,v)
Enstrophy conserved* = .5 (∇u, ∇v)
~.5($\nabla\times$ u, $\nabla\times$ v)

Power Laws

vv: $k^{-17/3} = k^{-5.6}$
uv: $k^{-15/3} \sim k^{-5}$
uu: $k^{-13/3} \sim k^{-4.3}$

Numerical Results: ($\alpha\to\infty$)

4096^2: k^{-5}

* -- in the absence of viscosity and forcing