Quaternions and particle dynamics in the Euler fluid equations

J. D. Gibbon

Mathematics Department

Imperial College London, London SW7 2AZ

j.d.gibbon@imperial.ac.uk

D^{2}H-Fest

Lausanne, July 2007

In honour of Darryl Holm on his 60th birthday

Some papers with Darryl on this topic

Darryl Holm & JDG 2007 : *Lagrangian particle paths & ortho-normal quaternion frames,* Nonlinearity **20,** 1745-1759, 2007.

Darryl Holm & JDG : *Lagrangian analysis of alignment dynamics for isentropic compressible magnetohydrodynamics,* http://arxiv.org/nlin.CD/0608009, New J. Physics focus issue: "MHD & the dynamo problem", (in proof) 2007.

JDG, **Darryl Holm**, Kerr & Roulstone : *Quaternions and particle dynamics in the Euler fluid equations,* Nonlinearity **19**, 1969-83, 2006

JDG : *Ortho-normal quaternion frames, Lagrangian evolution equations and the 3D Euler equations,* article to appear in Russian Math Surveys 2007

Summary of this talk

Motivation : Do the 3D Euler equations possess some subtle geometric structure that guides the growth & direction of vorticity? (Peter Constantin, *Geometric statistics in turbulence*, SIAM Rev. **36**, 73–98, 1994).

1. **The Euler singularity problem : Beale-Kato Majda (BKM) Thm.** **Numerical studies :** A history of investigations on the development of a finite time singularity in $\boldsymbol{\omega}$ in 3D Euler. Work on the **direction of vorticity.**
2. **Quaternions:** what are they & why are they now considered to be important?
3. **Lagrangian particle dynamics :** Explicit equations are displayed for the Lagrangian derivatives of an ortho-normal co-ordinate system for a particle
4. **The 3D-Euler equations :** Ertel's Theorem shows how these & other problems fit naturally into this framework (JDG, Holm et al 2006).
5. **Modelling the pressure Hessian :** Restricted Euler equations; tetrad model.

3D Euler equations – still the same after 250 years!

1. The 3D incompressible Euler equations in terms of the **velocity field** $\boldsymbol{u}(\boldsymbol{x},\, t)$:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)\boldsymbol{u} = -\nabla p\,; \qquad \text{div}\,\boldsymbol{u} = 0$$

div $\boldsymbol{u} = 0$ constrains the pressure p to obey $\{S = \frac{1}{2}(u_{i,j} + u_{j,i})$ strain matrix$\}$

$$-\Delta p = TrS^2 - \tfrac{1}{2}\boldsymbol{\omega}^2\,.$$

2. 3D incompressible Euler in terms of the **vorticity field** $\boldsymbol{\omega} = \text{curl}\,\boldsymbol{u}$:

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla\right)\boldsymbol{\omega} = \boldsymbol{\omega}\cdot\nabla\boldsymbol{u} = S\boldsymbol{\omega}\,.$$

3. **Why the interest in singularities?**

- **Physically** their formation may signify the onset of turbulence & may be a mechanism for energy transfer to small scales.
- **Numerically** they require very special methods – a great challenge to CFD.
- **Mathematically** their onset would rule out a global existence result.

The Beale-Kato-Majda-Theorem

Beale-Kato-Majda Theorem (1984) : *There exists a global solution of the 3D Euler equations* $\boldsymbol{u} \in C([0,\,\infty]; H^s) \cap C^1([0,\,\infty]; H^{s-1})$ *for* $s \geq 3$ *if*

$$\int_0^T \|\boldsymbol{\omega}(\cdot\,,\tau)\|_{L^\infty(\Omega)}\,d\tau < \infty\,, \qquad \textit{for every } T > 0.$$

Remark: See also Kozono & Taniuchi (2000) for a version using the BMO norm.

Corollary to BKM Thm : If a singularity is observed in a numerical experiment

$$\|\boldsymbol{\omega}(\cdot\,,t)\|_{L^\infty(\Omega)} \sim (T-t)^{-\beta}$$

then it is necessary to have $\beta \geq 1$ for the singularity to be genuine & not an artefact of the numerics.

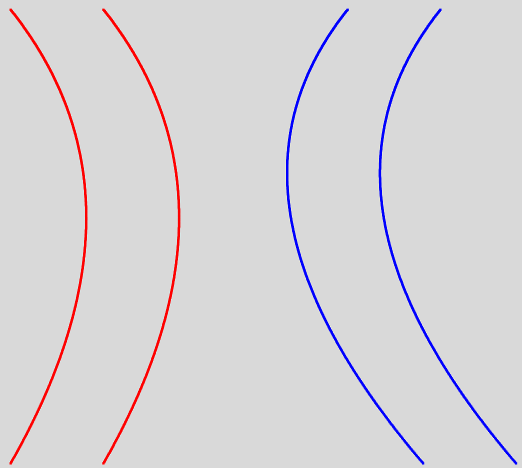
Numerical search for singularities

(a revised & up-dated version of a list originally compiled by Rainer Grauer)

1. Morf, Orszag & Frisch (1980): Padé-approximation, complex time singularity of 3D Euler {see also Bardos *et al* (1976)}: Singularity: yes. (Pauls, Matsumoto, Frisch & Bec (2006) on complex singularities of 2D Euler).
2. Chorin (1982): Vortex-method. Singularity: yes.
3. Brachet, Meiron, Nickel, Orszag & Frisch (1983): Taylor-Green calculation. Saw vortex sheets and the suppression of singularity. Singularity: no.
4. Siggia (1984): Vortex-filament method; became anti-parallel. Singularity: yes.
5. Ashurst & Meiron/Kerr & Pumir (1987): Singularity: yes/no.
6. Pumir & Siggia (1990): Adaptive grid. Singularity: no.
7. Bell & Marcus (1991): Projection method. Singularity: yes.
8. Brachet, Meneguzzi, Vincent, Politano & P-L Sulem (1992): pseudospectral code, Taylor-Green vortex. Singularity: no.

9. Kerr (1993, 2005): Used Chebyshev polynomials with anti-parallel initial conditions; resolution $512^2 \times 256$. Found amplification of vorticity by 18. Observed $\|\boldsymbol{\omega}\|_{L^\infty(\Omega)} \sim (T-t)^{-1}$. Singularity: yes.
10. Grauer & Sideris (1991): 3D axisymmetric swirling flow. Singularity: yes.
11. Boratav & Pelz (1994, 1995): Kida's high symmetry. Singularity: yes.
12. Pelz & Gulak (1997): Kida's high symmetry. Singularity: yes.
13. Grauer, Marliani & Germaschewski (1998): Singularity: yes.
14. Pelz (2001, 2003): Singularity: yes.
15. Kida has edited a memorial issue for Pelz in Fluid Dyn. Res., **36**, (2005) :
    - Cichowlas & Brachet: Singularity: no.
    - Pelz & Ohkitani: Singularity: no.
    - Gulak & Pelz: Singularity: yes.
16. Hou & Li (2006): Agrees with Kerr (1993) until the final stage and then growth slows. Singularity: no.

Direction of vorticity: the work of CFM & DHY



a) Constantin, Fefferman & Majda (1996) discussed the idea of vortex lines being "smoothly directed" in a region of greatest curvature. They argued that if the velocity is finite in a ball $(B_{4\rho})$ & $\lim_{t\to T}\sup_{\boldsymbol{W}_0}\int_0^t \|\nabla\hat{\boldsymbol{\omega}}(\cdot\,,\tau)\|^2_{L^\infty(B_{4\rho})}\,d\tau < \infty$ then there can be no singularity at time T.

b) Deng, Hou & Yu (2006) take the arc length $L(t)$ of a vortex line L_t with $\hat{\boldsymbol{n}}$ the unit normal and κ the curvature. If $M(t) \equiv \max\left(\|\nabla\cdot\hat{\boldsymbol{\omega}}\|_{L^\infty(L_t)}, \|\kappa\|_{L^\infty(L_t)}\right)$ they argue that there will be no blow-up at time T if

1. $U_{\hat{\omega}}(t) + U_{\hat{n}}(t) \lesssim (T-t)^{-A}$ $\qquad A + B = 1,$
2. $M(t)L(t) \leq$ const > 0
3. $L(t) \gtrsim (T-t)^{B}$.

Lord Kelvin (William Thompson) once said:

> **Quaternions came from Hamilton after his best work had been done, & though beautifully ingenious, they have been an unmixed evil to those who have touched them in any way.**

O'Connor, J. J. & Robertson, E. F. 1998 *Sir William Rowan Hamilton*,

http://www-groups.dcs.st-and.ac.uk/ history/Mathematicians/Hamilton.html

Kelvin was wrong because quaternions are now used in the avionics & robotics industries to track objects undergoing sequences of tumbling rotations.

Quaternions & rotation Sequences: a Primer with Applications to Orbits, Aerospace & Virtual Reality, J. B. Kuipers, Princeton University Press, 1999.

They are also used in the computer animation business:

See **"Visualizing quaternions"**, by Andrew J. Hanson, MK-Elsevier, 2006.

A quote from Hanson's introduction:

> Although the advantages of the quaternion forms for the basic equations of attitude control – clearly presented in Cayley (1845), Hamilton (1853, 1866) & especially Tait (1890) – had been noticed by the aeronautics & astronautics community, the technology did not penetrate the computer animation community until the land-mark Siggraph 1985 paper of Shoemake.
>
> The importance of Shoemake's paper is that it took the concept of the orientation frame for moving 3D objects & cameras ... exposed the deficiencies of the then-standard Euler-angle methods & introduced quaternions to animators as a solution.

What are quaternions? (Hamilton 1843)

Quaternions are constructed from a scalar p & a 3-vector $\boldsymbol{q}$ by forming the tetrad

$$\mathfrak{p} = [p,\, \boldsymbol{q}] = pI - \boldsymbol{q}\cdot\boldsymbol{\sigma}\,, \qquad\qquad \boldsymbol{q}\cdot\boldsymbol{\sigma} = \sum_{i=1}^{3} q_i\,\sigma_i$$

based on the Pauli spin matrices that obey the relations $\sigma_i\sigma_j = -\delta_{ij} - \epsilon_{ijk}\sigma_k$

$$\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Thus quaternions obey the multiplication rule (associative but non-commutative)

$$\mathfrak{p}_1 \circledast \mathfrak{p}_2 = [p_1p_2 - \boldsymbol{q}_1\cdot\boldsymbol{q}_2\,,\; p_1\boldsymbol{q}_2 + p_2\boldsymbol{q}_1 + \boldsymbol{q}_1\times\boldsymbol{q}_2]\;.$$

They have a close connection with the Cayley-Klein parameters & thus the Euler angles of a rotation (Whittaker 1944).

Quaternions, Cayley-Klein parameters & Rotations

Let $\hat{\mathfrak{p}} = [p,\, \boldsymbol{q}]$ be a unit quaternion with inverse $\hat{\mathfrak{p}}^* = [p,\, -\boldsymbol{q}]$ with $p^2 + q^2 = 1$. For a pure quaternion $\mathfrak{r} = [0,\, \boldsymbol{r}]$ the transformation $\mathfrak{r} \to \mathfrak{R} = [0,\, \boldsymbol{R}]$

$$\mathfrak{R} = \hat{\mathfrak{p}} \circledast \mathfrak{r} \circledast \hat{\mathfrak{p}}^* = [0,\, (p^2 - q^2)\boldsymbol{r} + 2p(\boldsymbol{q} \times \boldsymbol{r}) + 2\boldsymbol{q}(\boldsymbol{r} \cdot \boldsymbol{q})] \equiv O(\theta\,, \hat{\boldsymbol{n}})\boldsymbol{r}\,,$$

gives the **Euler-Rodrigues** formula for the rotation $O(\theta\,, \hat{\boldsymbol{n}})$ by an angleθ of $\boldsymbol{r}$ about its normal $\hat{\boldsymbol{n}}$. **Cayley-Klein parameters** $(SU(2))$ are the elements of

$$\hat{\mathfrak{p}} = \pm[\cos\tfrac{1}{2}\theta,\, \hat{\boldsymbol{n}}\sin\tfrac{1}{2}\theta]$$

If $\hat{\mathfrak{p}} = \hat{\mathfrak{p}}(t)$ then

$$\dot{\mathfrak{R}}(t) = (\dot{\hat{\mathfrak{p}}} \circledast \hat{\mathfrak{p}}^*) \circledast \mathfrak{R} - ((\dot{\hat{\mathfrak{p}}} \circledast \hat{\mathfrak{p}}^*) \circledast \mathfrak{R})^*\,,$$

$$\dot{\boldsymbol{R}} = \boldsymbol{\Omega}_0(t) \times \boldsymbol{R}$$

The angular velocity is $\boldsymbol{\Omega}_0(t) = 2\mathrm{Im}\,(\dot{\hat{\mathfrak{p}}} \circledast \hat{\mathfrak{p}}^*)$: **the rigid body result: see Marsden-Ratiu 2003**.

It don't mean a thing if it ain't got that swing! (Duke Ellington)

Consider the general Lagrangian evolution equation for a 3-vector $\boldsymbol{w}$ such that

$$\frac{D\boldsymbol{w}}{Dt} = \boldsymbol{a}(\boldsymbol{x},\, t) \qquad\qquad \frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla$$

transported by a velocity field $\boldsymbol{u}$. Define the scalar α_a the 3-vector $\boldsymbol{\chi}_a$ as

$$\alpha_a = |\boldsymbol{w}|^{-1}(\hat{\boldsymbol{w}}\cdot\boldsymbol{a})\,, \qquad\qquad \boldsymbol{\chi}_a = |\boldsymbol{w}|^{-1}(\hat{\boldsymbol{w}}\times\boldsymbol{a})\,,$$

for $|\boldsymbol{w}| \neq 0$. Using the parallel/perp decomposition

$$\boldsymbol{a} = \alpha_a\boldsymbol{w} + \boldsymbol{\chi}_a\times\boldsymbol{w}\,,$$

$$\frac{D|\boldsymbol{w}|}{Dt} = \alpha_a|\boldsymbol{w}|\,, \qquad\qquad \frac{D\hat{\boldsymbol{w}}}{Dt} = \boldsymbol{\chi}_a\times\hat{\boldsymbol{w}}\,.$$

- α_a **is the 'growth rate'** (Constantin 1994)
- **Note that** $\chi_a = 0$ **when** w **and** a **align.**
- **For Euler when** $w = \omega$ **aligns with** $S\omega$ **(straight tube/sheet)** $\Rightarrow \chi = 0$**.**

Define the quaternions

$$\mathfrak{q}_a = [\alpha_a,\, \boldsymbol{\chi}_a]\,, \qquad\qquad \mathfrak{w} = [0,\, \boldsymbol{w}]\,.$$

The above decomposition allows us to write $D\boldsymbol{w}/Dt = \boldsymbol{a}$ as

$$\frac{D\mathfrak{w}}{Dt} = \mathfrak{q}_a \circledast \mathfrak{w}\,,$$

which is associated with an ortho-normal frame – **the "quaternion frame"**

$$\{\hat{\boldsymbol{w}}\,,\quad \hat{\boldsymbol{\chi}}_a\,,\quad (\hat{\boldsymbol{w}} \times \hat{\boldsymbol{\chi}}_a)\}\,.$$

- The Lagrangian evolution of this frame can be explicitly found – next slide.
- It provides us with an ortho-normal basis in which to expand other vectors – see later for a Theorem regarding the pressure.
- The frame collapses when $\boldsymbol{a}$ and $\boldsymbol{w}$ align; i.e., when $\boldsymbol{\chi}_a = 0$.

Theorem : (JDG/Holm 06) *If $\boldsymbol{a}$ is differentiable in the Lagrangian sense s.t.*

$$\frac{D\boldsymbol{a}}{Dt} = \boldsymbol{b}(\boldsymbol{x},\, t)\,,$$

(i) $\mathfrak{q}_a$ *and* $\mathfrak{q}_b$ *satisfy the Riccati equation* $(|\boldsymbol{w}| \neq 0)$,

$$\frac{D\mathfrak{q}_a}{Dt} + \mathfrak{q}_a \circledast \mathfrak{q}_a = \mathfrak{q}_b\,;$$

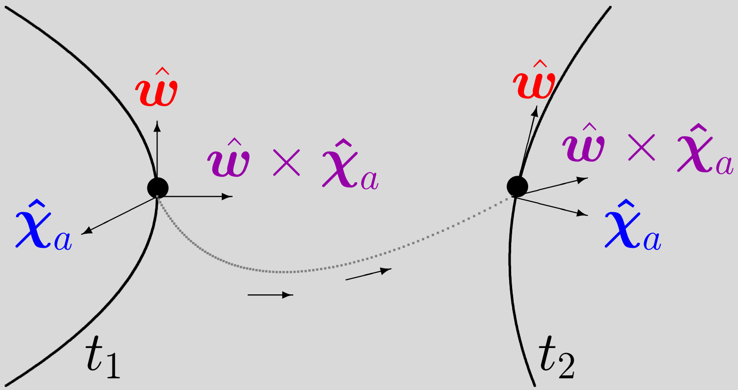
(ii) *The Lagrangian time derivative of the ortho-normal frame* $(\hat{\boldsymbol{w}},\, \hat{\boldsymbol{\chi}}_a,\, \hat{\boldsymbol{w}} \times \hat{\boldsymbol{\chi}}_a) \in SO(3)$ *is expressed as*

$$\begin{aligned} \frac{D\hat{\boldsymbol{w}}}{Dt} &= \boldsymbol{\mathcal{D}}_{ab} \times \hat{\boldsymbol{w}}\,, \\ \frac{D(\hat{\boldsymbol{w}} \times \hat{\boldsymbol{\chi}}_a)}{Dt} &= \boldsymbol{\mathcal{D}}_{ab} \times (\hat{\boldsymbol{w}} \times \hat{\boldsymbol{\chi}}_a)\,, \\ \frac{D\hat{\boldsymbol{\chi}}_a}{Dt} &= \boldsymbol{\mathcal{D}}_{ab} \times \hat{\boldsymbol{\chi}}_a\,, \end{aligned}$$

where the Darboux angular velocity vector $\boldsymbol{\mathcal{D}}_{ab}$ *is defined as*

$$\boldsymbol{\mathcal{D}}_{ab} = \boldsymbol{\chi}_a + \frac{c_b}{\chi_a}\hat{\boldsymbol{w}}\,, \qquad\qquad c_b = \hat{\boldsymbol{w}} \cdot (\hat{\boldsymbol{\chi}}_a \times \boldsymbol{\chi}_b)\,.$$

The orientation frame of a particle



The dotted line represents a particle ($\bullet$) trajectory moving from $(\boldsymbol{x}_1, t_1)$ to $(\boldsymbol{x}_2, t_2)$. The orientation of the orthonormal unit vectors

$$\{\hat{\boldsymbol{w}}\,,\ \ \hat{\boldsymbol{\chi}}_a\,,\ \ (\hat{\boldsymbol{w}} \times \hat{\boldsymbol{\chi}}_a)\}$$

is driven by the Darboux vector $\boldsymbol{\mathcal{D}}_{ab} = \boldsymbol{\chi}_a + \frac{c_b}{\chi_a}\hat{\boldsymbol{w}}$ where $c_b = \hat{\boldsymbol{w}} \cdot (\hat{\boldsymbol{\chi}}_a \times \boldsymbol{\chi}_b)$.
Thus we need the 'quartet' of vectors to make this process work

$$\{\boldsymbol{u},\ \boldsymbol{w},\ \boldsymbol{a},\ \boldsymbol{b}\}\ .$$

The frame orientation is a visual diagnostic in addition to the path.

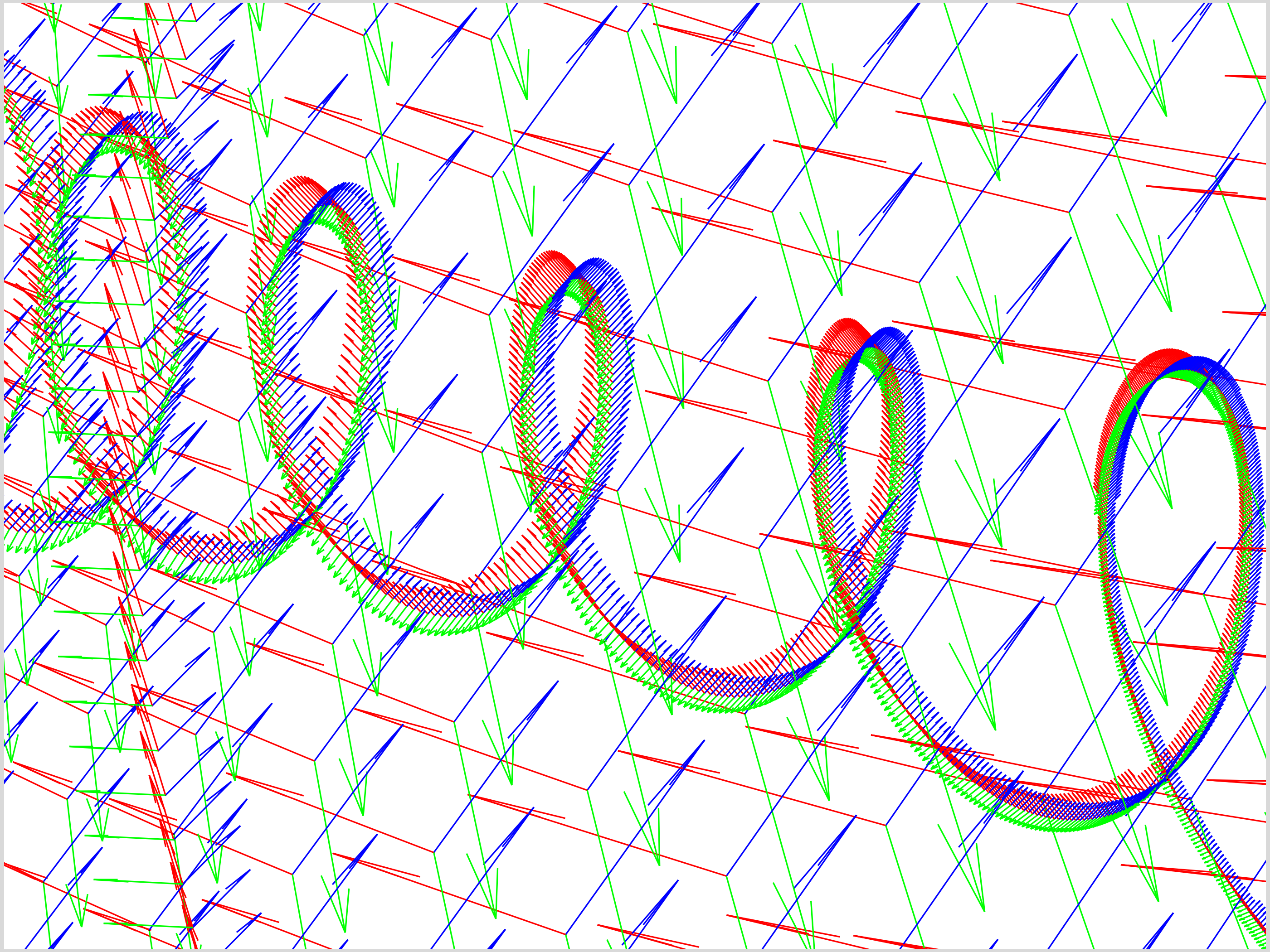


Figure 1: **$\boldsymbol{u}$-field is Arter-flow: $\boldsymbol{b}(\mathbf{x}(t)) = [\sin(k_1 x(t));\ \sin(k_2 y(t));\ \sin(k_3 z(t))]$; $\boldsymbol{k} = [1, 2, 0.5]$. The initial $\boldsymbol{a}(\mathbf{x}_0, t_0) = [0.5, 0.2, 0.6]$ and the initial $\boldsymbol{w}(\mathbf{x}_0, t_0) = [0.1, 0.2, 0.3]$ with $\mathbf{x}_0 = [0.1, 0.2, 0.3]$ as the initial particle position. Computation by Matthew Dixon.**

The 3D Euler equations and Ertel's Theorem

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega}\cdot\nabla\boldsymbol{u} = S\boldsymbol{\omega} \qquad \text{Euler in vorticity format}$$

Theorem: (Ertel 1942) If $\boldsymbol{\omega}$ satisfies the 3D incompressible Euler equations then any arbitrary differentiable μ satisfies

$$\frac{D}{Dt}(\boldsymbol{\omega}\cdot\nabla\mu) = \boldsymbol{\omega}\cdot\nabla\left(\frac{D\mu}{Dt}\right) \quad \Longrightarrow \quad \left[\frac{D}{Dt}, \boldsymbol{\omega}\cdot\nabla\right] = 0\,.$$

Proof: Consider $\boldsymbol{\omega}\cdot\nabla\mu \equiv \omega_i\,\mu_{,i}$

$$\begin{aligned}\frac{D}{Dt}(\omega_i\,\mu_{,i}) &= \frac{D\omega_i}{Dt}\mu_{,i} + \omega_i\left\{\frac{\partial}{\partial x_i}\left(\frac{D\mu}{Dt}\right) - u_{k,i}\mu_{,k}\right\}\\ &= \underbrace{\{\omega_j u_{i,j}\,\mu_{,i} - \omega_i u_{k,i}\,\mu_{,k}\}}_{zero\ under\ summation} + \omega_i\frac{\partial}{\partial x_i}\left(\frac{D\mu}{Dt}\right)\end{aligned}$$

In characteristic (Lie-derivative) form, $\boldsymbol{\omega}\cdot\nabla(t) = \boldsymbol{\omega}\cdot\nabla(0)$ is a Lagrangian invariant (Cauchy 1859) and is "frozen in".

Various references

- Ertel; *Ein Neuer Hydrodynamischer Wirbelsatz*, Met. Z. **59**, 271-281, (1942).
- Truesdell & Toupin, Classical Field Theories, *Encyclopaedia of Physics III/1*, ed. S. Flugge, Springer (1960): gives the connection with Cauchy.
- Klainerman (1984 unpublished).
- Hoskins, McIntyre, & Robertson; *On the use & significance of isentropic potential vorticity maps*, Quart. J. Roy. Met. Soc., **111**, 877-946, (1985).
- Ohkitani ; Phys. Fluids, **A5**, 2576, (1993).
- Kuznetsov & Zakharov; *Hamiltonian formalism for nonlinear waves*, Physics Uspekhi, **40** (11), 1087– 1116 (1997).
- Viudez; *On the relation between Beltrami's material vorticity and Rossby-Ertel's Potential*, J. Atmos. Sci. (2001).

The pressure Hessian

Define the Hessian matrix of the pressure

$$P = \{p_{,ij}\} = \left\{\frac{\partial^2 p}{\partial x_i\,\partial x_j}\right\}$$

Ohkitani (1993) & Klainerman (1984) took $\mu = u_i$.

Result: The vortex stretching vector $\boldsymbol{\omega}\cdot\nabla\boldsymbol{u} = S\boldsymbol{\omega}$ obeys

$$\frac{D(\boldsymbol{\omega}\cdot\nabla\boldsymbol{u})}{Dt} = \frac{D(S\boldsymbol{\omega})}{Dt} = \boldsymbol{\omega}\cdot\nabla\left(\frac{D\boldsymbol{u}}{Dt}\right) = -P\,\boldsymbol{\omega}$$

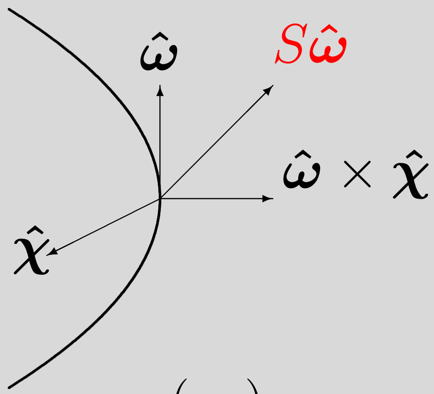
Thus for Euler, via Ertel's Theorem, we have the identification:

$$\boldsymbol{w} \equiv \boldsymbol{\omega} \qquad \boldsymbol{a} \equiv \boldsymbol{\omega}\cdot\nabla\boldsymbol{u} = S\boldsymbol{\omega} \qquad \boldsymbol{b} \equiv -P\boldsymbol{\omega}$$

with a quartet

$$(\boldsymbol{u},\,\boldsymbol{w},\,\boldsymbol{a},\,\boldsymbol{b}) \equiv (\boldsymbol{u},\,\boldsymbol{\omega},\,S\boldsymbol{\omega},\,-P\boldsymbol{\omega})\,.$$

Euler: the variables $\alpha(\boldsymbol{x},t)$ and $\boldsymbol{\chi}(\boldsymbol{x},t)$



See JDG, Holm, Kerr & Roulstone 2006.

$$S\hat{\boldsymbol{\omega}} = \alpha\,\hat{\boldsymbol{\omega}} + \boldsymbol{\chi}\times\hat{\boldsymbol{\omega}}$$

$$(\alpha_a) \qquad \alpha = \hat{\boldsymbol{\omega}}\cdot S\hat{\boldsymbol{\omega}} \qquad\qquad \boldsymbol{\chi} = \hat{\boldsymbol{\omega}}\times S\hat{\boldsymbol{\omega}} \qquad (\boldsymbol{\chi}_a)$$

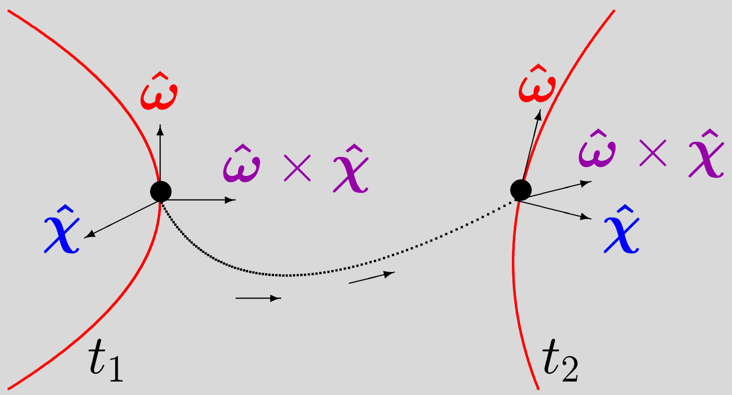
$$(-\alpha_b) \qquad \alpha_p = \hat{\boldsymbol{\omega}}\cdot P\hat{\boldsymbol{\omega}} \qquad\qquad \boldsymbol{\chi}_p = \hat{\boldsymbol{\omega}}\times P\hat{\boldsymbol{\omega}} \qquad (-\boldsymbol{\chi}_b)$$

$$\mathfrak{q} = [\alpha,\,\boldsymbol{\chi}] \qquad\qquad \mathfrak{q}_b = -[\alpha_p,\,\boldsymbol{\chi}_p]$$

$$\frac{D\mathfrak{q}}{Dt} + \mathfrak{q}\circledast\mathfrak{q} + \mathfrak{q}_p = 0\,,$$

constrained by div $\boldsymbol{u} = 0 \quad \Longrightarrow \quad -\,TrP = \Delta p = u_{i,j}u_{j,i} = TrS^2 - \frac{1}{2}\omega^2$.

Lagrangian frame dynamics of an Euler fluid particle



The dotted line represents the fluid packet ($\bullet$) trajectory moving from $(\boldsymbol{x}_1, t_1)$ to $(\boldsymbol{x}_2, t_2)$. The orientation of the orthonormal unit vectors

$$\{\hat{\boldsymbol{\omega}}\,, \quad \hat{\boldsymbol{\chi}}\,, \quad (\hat{\boldsymbol{\omega}} \times \hat{\boldsymbol{\chi}})\}$$

is driven by the Darboux vector

$$\boldsymbol{\mathcal{D}} = \boldsymbol{\chi} + \frac{c_p}{\chi}\hat{\boldsymbol{\omega}}\,, \qquad\qquad c_p = -\hat{\boldsymbol{\omega}} \cdot (\hat{\boldsymbol{\chi}} \times \boldsymbol{\chi}_p)\,.$$

Thus the pressure Hessian within c_1 drives the Darboux vector $\boldsymbol{\mathcal{D}}$.

Remark: the α and $\boldsymbol{\chi}$ equations

In terms of α and $\boldsymbol{\chi}$, the Riccati equation for $\mathfrak{q}$

$$\frac{D\mathfrak{q}}{Dt} + \mathfrak{q} \circledast \mathfrak{q} + \mathfrak{q}_p = 0\,;$$

becomes

$$\frac{D\alpha}{Dt} = \chi^2 - \alpha^2 - \alpha_p\,, \qquad \frac{D\boldsymbol{\chi}}{Dt} = -2\alpha\boldsymbol{\chi} - \boldsymbol{\chi}_p\,.$$

Stationary values are

$$\alpha = \gamma_0\,, \qquad \boldsymbol{\chi} = \mathbf{o}\,, \qquad \alpha_p = -\gamma_0^2$$

which correspond to **Burgers'-like vortices**.

When tubes & sheets bend & tangle then $\boldsymbol{\chi} \neq 0$ and $\mathfrak{q}$ becomes a full tetrad driven by $\mathfrak{q}_p$ which is coupled back through the elliptic pressure condition.

Note: Off-diagonal elements of P change rapidly near intense vortical regions across which χ_p and α_p change rapidly.

A summary of quartets $\{\boldsymbol{u},\ \boldsymbol{w},\ \boldsymbol{a},\ \boldsymbol{b}\}$ for other 3D problems

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u}\cdot\nabla$$

System	$\boldsymbol{u}$	$\boldsymbol{w}$	$\boldsymbol{a}$	$\boldsymbol{b}$	BKM
incom Euler	$\boldsymbol{u}$	$\boldsymbol{\omega}$	$S\boldsymbol{\omega}$	$-P\boldsymbol{\omega}$	$\int_0^T \lVert\boldsymbol{\omega}\rVert_\infty\, dt$
baro-Euler	$\boldsymbol{u}$	$\boldsymbol{\omega}/\rho$	$(\boldsymbol{\omega}/\rho)\cdot\nabla\boldsymbol{u}$	$-(\omega_j/\rho)\partial_j(\rho\partial_i p)$	
MHD	$\boldsymbol{v}^{\pm}$	$\boldsymbol{v}^{\mp}$	$\boldsymbol{B}\cdot\nabla\boldsymbol{v}^{\mp}$	$-P\boldsymbol{B}$	$\int_0^T (\lVert\boldsymbol{\omega}\rVert_\infty + \lVert\boldsymbol{J}\rVert_\infty)\, dt$
Mixing	$\boldsymbol{u}$	$\boldsymbol{\delta\ell}$	$\boldsymbol{\delta\ell}\cdot\nabla\boldsymbol{u}$	$-P\boldsymbol{\delta\ell}$	$\int_0^T \lVert\boldsymbol{\delta\ell}\rVert_\infty\, dt$

For MHD, $\boldsymbol{v}^{\pm} = \boldsymbol{u} \pm \boldsymbol{B}$, $\boldsymbol{J} = \mathrm{curl}\,\boldsymbol{B}$ and

$$\frac{D^{\pm}}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{v}^{\pm}\cdot\nabla$$

The $\int_0^T (\lVert\boldsymbol{\omega}\rVert_\infty + \lVert\boldsymbol{J}\rVert_\infty)\, dt < \infty$ MHD result is due to Caflisch, Klapper & Steel 1998.

Restricted Euler equations: modelling the Hessian P

The gradient matrix $M_{ij} = \partial u_j / \partial x^i$ satisfies $(\mathrm{tr}\, P = -\mathrm{tr}\,(M^2))$

$$\frac{DM}{Dt} + M^2 + P = 0\,, \qquad \mathrm{tr}\, M = 0\,.$$

Several attempts have been made to model the Lagrangian averaged pressure Hessian by introducing **a constitutive closure.** This idea goes back to

Léorat (1975); Vieillefosse (1984); Cantwell (1992)

who assumed that the Eulerian pressure Hessian P is isotropic. This results in the *restricted Euler equations* with

$$P = -\tfrac{1}{3} I\, \mathrm{tr}\,(M^2)\,, \qquad \mathrm{tr}\, I = 3\,.$$

- **Constantin's distorted Euler equations (1986) :** Euler can be written as

$$\frac{\partial M}{\partial t} + M^2 + Q(t) Tr(M^2) = 0\,, \qquad Q_{ij} = R_i R_j$$

with the Riesz transform $R_i = (-\Delta)^{-1/2}\partial_i$. 'Distorted Euler equns' appear with $Q(t)$ replaced by $Q(0) \Rightarrow$ rigorous blow up.

- **Tetrad model** of Chertkov, Pumir & Shraiman (1999), recently developed by Chevillard & Meneveau (2006). Underlying its mean flow features is the assumption that the Lagrangian pressure Hessian is isotropic.

For P to transform as a Riemannian metric and satisfy $\mathrm{tr}\, P = -\mathrm{tr}\,(M^2)$

$$P = -\frac{G}{\mathrm{tr}\, G}\mathrm{tr}\,(M^2)\,, \qquad \mathsf{G}(t) = I \quad \Rightarrow \quad \text{restricted Euler}\,.$$

$\mathrm{tr}\, P = -\mathrm{tr}\,(M^2)$ is satisfied for *any* choice of $G = G^T$

$$P = -\left[\sum_{\beta=1}^{N} c_\beta \frac{\mathsf{G}_\beta}{\mathrm{tr}\,\mathsf{G}_\beta}\right]\mathrm{tr}\,(M^2)\,, \qquad \text{with} \qquad \sum_{\beta=1}^{N} c_\beta = 1\,,$$

so long as an evolutionary flow law is provided for each of the symmetric tensors $\mathsf{G}_\beta = \mathsf{G}_\beta^T$ with $\beta = 1, \ldots, N$. Any choice of flow laws for G would also determine the evolution of the driving term $\mathfrak{q}_b$ in the Riccati equation.

Frame dynamics & the Frenet-Serret equations

With $\hat{\boldsymbol{w}}$ as the unit tangent vector, $\hat{\boldsymbol{\chi}}$ as the unit bi-normal and $\hat{\boldsymbol{w}} \times \hat{\boldsymbol{\chi}}$ as the unit principal normal, the matrix N can be formed

$$N = \left(\hat{\boldsymbol{w}}^T,\, (\hat{\boldsymbol{w}} \times \hat{\boldsymbol{\chi}})^T,\, \hat{\boldsymbol{\chi}}^T\right)\,,$$

with

$$\frac{DN}{Dt} = GN\,, \qquad G = \begin{pmatrix} 0 & -\,\chi_a & 0 \\ \chi_a & 0 & -\,c_1\chi_a^{-1} \\ 0 & c_1\chi_a^{-1} & 0 \end{pmatrix}.$$

The Frenet-Serret equations for a space-curve are

$$\frac{dN}{ds} = FN \qquad \text{where} \qquad F = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix},$$

where κ is the curvature and τ is the torsion.

The arc-length derivative d/ds is defined by

$$\frac{d}{ds} = \boldsymbol{w} \cdot \nabla \,.$$

The evolution of the curvature κ and torsion τ may be obtained from Ertel's theorem expressed as the commutation of operators $\left[\frac{D}{Dt},\, \boldsymbol{w} \cdot \nabla\right] = 0$

$$\alpha_a \frac{d}{ds} + \left[\frac{D}{Dt},\, \frac{d}{ds}\right] = 0 \,.$$

This commutation relation immediately gives

$$\alpha_a F + \frac{DF}{Dt} = \frac{dG}{ds} + [G,\, F] \,.$$

Thus Ertel's Theorem gives explicit evolution equations for the curvature κ and torsion τ that lie within the matrix F and relates them to c_1, χ_a and α_a.