I. Introduction

A New Method for Inverting Integrals

- Attenuated Radon Transform (SPECT)
- D to N map for Moving Boundary Value Problems

$$F(k) = \int_0^T e^{k^2 t + ikl(t)} f(t) dt, \quad k \in \mathbb{C}.$$

Integrable Nonlinear PDEs in $4+2$ and $3+1$

- DS and KP type generalizations.

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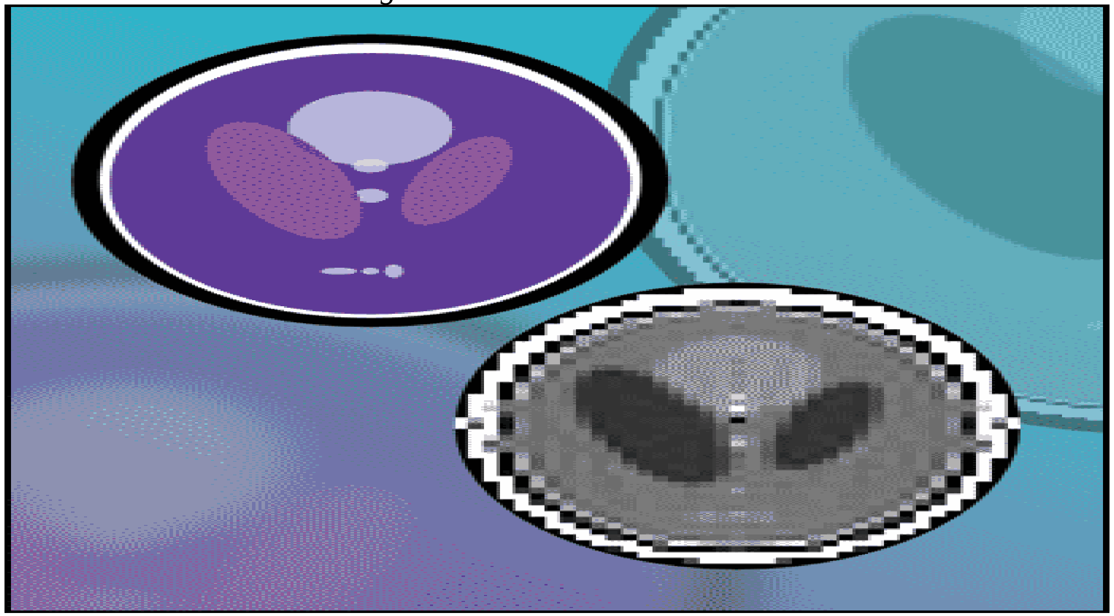
Feature Article

Generalized Fourier Transforms, Their Nonlinearization and the Imaging of the Brain

A. S. Fokas and L.-Y. Sung



II.Inversion of Integrals-Imaging

Ablowitz-F, Beals and Coifman (1982)

$$\frac{\partial \mu}{\partial x_1} + i\sigma_3 \frac{\partial \mu}{\partial x_2} - k[\sigma_3, \mu] = Q\mu, \quad k \in \mathbb{C}, \quad (x_1, x_2) \in \mathbb{R}^2,$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & q(x_1, x_2) \\ \overline{q}(x_1, x_2) & 0 \end{pmatrix}$$

Nonlinear FT in 2D via the $\bar{\partial}$ formalism
F-Gelfand (1992)

$$(\partial_{x_1} + i\partial_{x_2} - k)\, \mu(x_1, x_2, k) = q(x_1, x_2)$$

Novel derivation of 2D FT via $\bar{\partial}$

F-Novikov (1992)

$$\left[\frac{1}{2}\left(k+\frac{1}{k}\right)\partial_{x_1}+\frac{1}{2\mathrm{i}}\left(k-\frac{1}{k}\right)\partial_{x_2}\right]\mu(x_1,x_2,k)=f(x_1,x_2)$$

Novel derivation of Radon transform

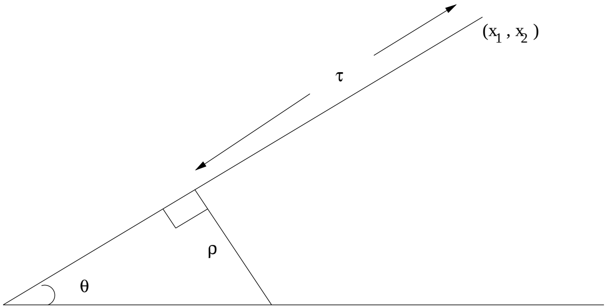
Novikov (2003), F (2004)

$$\left[\frac{1}{2}\left(k+\frac{1}{k}\right)\partial_{x_1}+\frac{1}{2\mathrm{i}}\left(k-\frac{1}{k}\right)\partial_{x_2}\right]\mu(x_1,x_2,k) \\ +f(x_1,x_2)\mu(x_1,x_2,k)=g(x_1,x_2)$$

Derivation of attenuated Radon transform

II.1 RADON TRANSFORM

Reconstruct f from its line integrals.



$$x_1 = \tau \cos\theta - \rho \sin\theta \qquad \tau = x_1 \cos\theta + x_2 \sin\theta$$
$$\leftrightarrow$$
$$x_2 = \tau \sin\theta + \rho \cos\theta \qquad \rho = -x_1 \sin\theta + x_2 \cos\theta$$

$$F(\tau, \rho, \theta) = f(\tau \cos\theta - \rho \sin\theta, \tau \sin\theta + \rho \cos\theta).$$

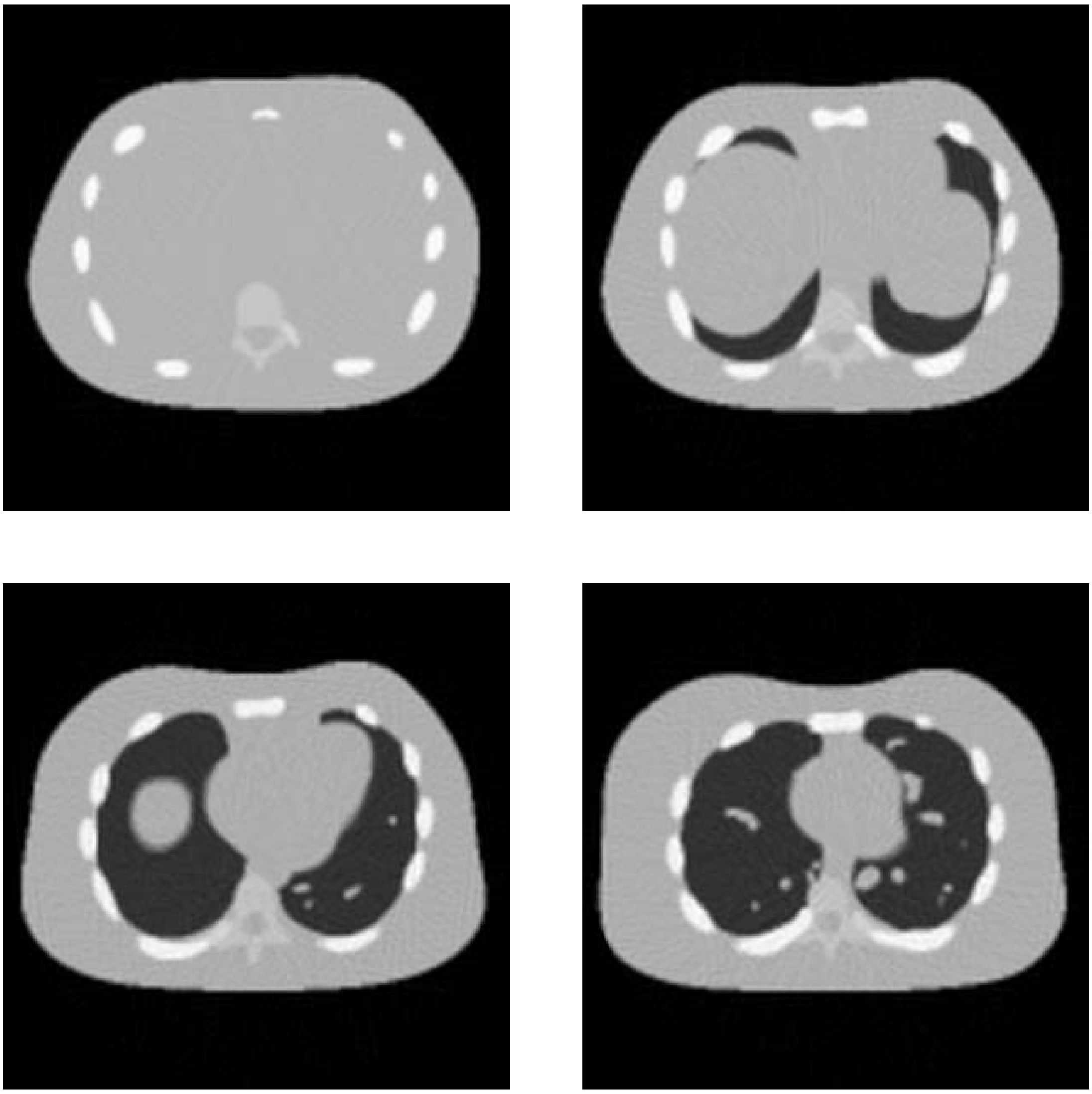
Direct Radon transform

$$\hat{f}(\rho, \theta) = \int_{-\infty}^{\infty} F(\tau, \rho, \theta) \mathrm{d}\tau.$$

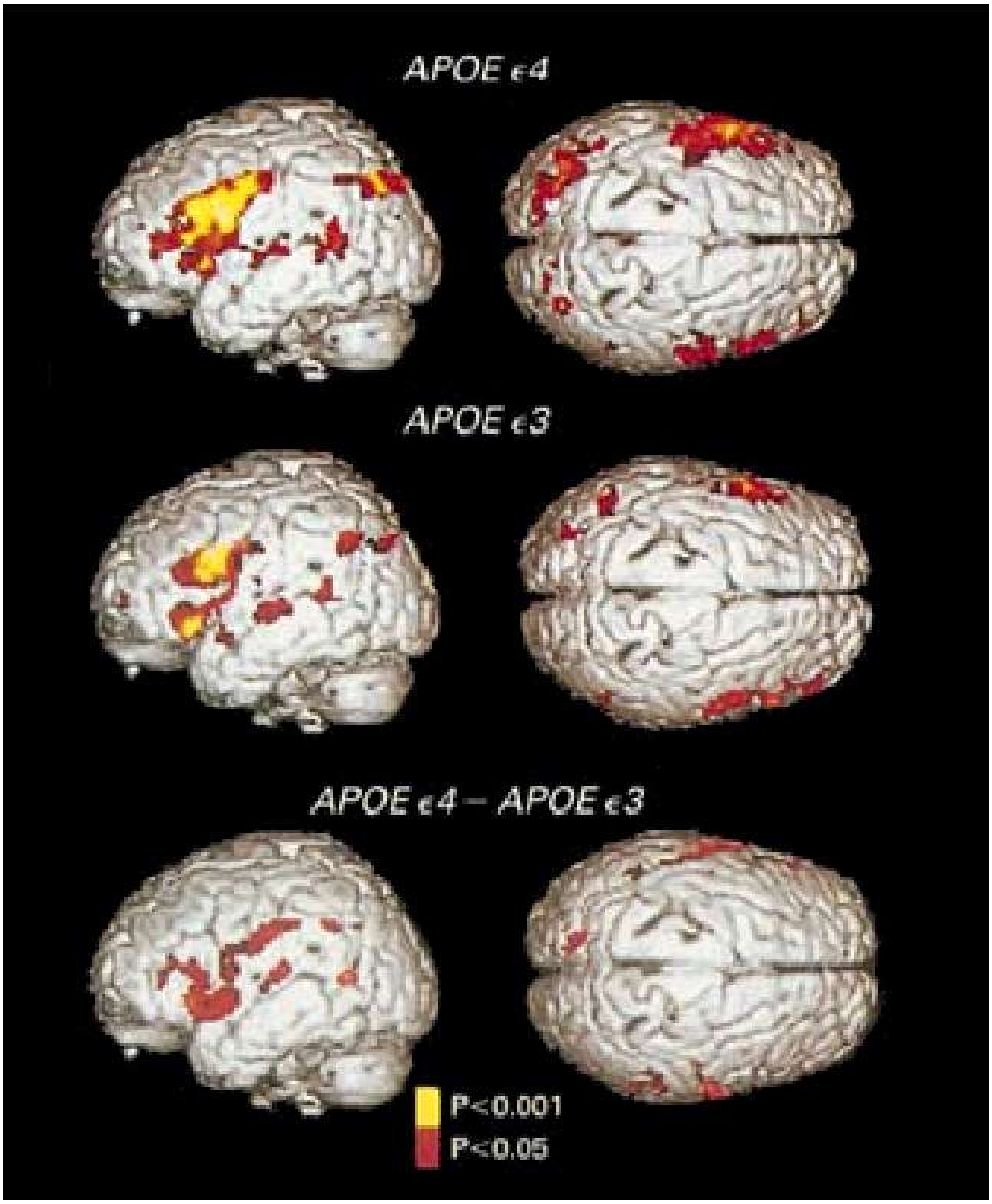
Inverse Radon transform(Filter Back Projection)

$$H(\theta, \rho) = \frac{1}{i\pi} \oint_{-\infty}^{\infty} \frac{\hat{f}(\rho', \theta)}{\rho' - \rho} \mathrm{d}\rho',$$

$$f(x_1, x_2) = \frac{1}{4\pi} (\partial_{x_1} - i\partial_{x_2}) \int_0^{2\pi} e^{i\theta} H(\theta, -x_1 \sin\theta + x_2 \cos\theta) \mathrm{d}\theta.$$



The reconstruction of the phantoms before the filtering procedure.


APOE ϵ4
APOE ϵ3
APOE ϵ4 – APOE ϵ3
P<0.001
P<0.05

II.2 MATHEMATICS OF SPECT

Reconstruct g from its weighted line integrals

$$I = \int_L e^{-\int_{L(x)} f ds} g d\tau.$$

Direct Attenuated Radon transform

$$\hat{g}_f(\rho, \theta) = \int_{-\infty}^{\infty} e^{-\int_\tau^\infty F(s,\rho,\theta) ds} G(\tau, \rho, \theta) d\tau$$

Inverse Attenuated Radon transform

$$P^{\pm} g(\rho) = \pm \frac{g(\rho)}{2} + \frac{1}{2i\pi} \oint_{-\infty}^{\infty} \frac{g(\rho')}{\rho' - \rho} d\rho'$$

$$H(\theta,\tau,\rho)=e^{\int_\tau^\infty F(s,\rho,\theta)ds}\Big\{e^{P^-\hat{f}(\rho,\theta)}P^-e^{-P^-\hat{f}(\rho,\theta)}+$$

$$+e^{-P^+\hat{f}(\rho,\theta)}P^+e^{P^+\hat{f}(\rho,\theta)}\Big\}\,\hat{g}_f(\rho,\theta),$$

$$g(x_1,x_2)=\frac{1}{4\pi}(\partial_{x_1}-i\partial_{x_2})\cdot$$

$$\cdot\int_0^{2\pi}e^{i\theta}H(\theta,x_1\cos\theta+x_2\sin\theta,x_2\cos\theta-x_1\sin\theta)d\theta.$$

Spectral analysis of a SINGLE equation→ Analytic inversion of integrals

$$\left\{\frac{1}{2}\left(k+\frac{1}{k}\right)\partial_{x_1}+\frac{1}{2i}\left(k-\frac{1}{k}\right)\partial_{x_2}\right\}\mu(x_1,x_2,k)=f(x_1,x_2)$$

$$k\in\mathbb{C},\quad (x_1,x_2)\in\mathbb{R}^2,\quad f(x)\in S(\mathbb{R}^2)$$

(i) Solve for μ in terms of f for ALL $k\in\mathbb{C}$

$$z\doteqdot\frac{1}{2i}\left(k-\frac{1}{k}\right)x_1-\frac{1}{2}\left(k+\frac{1}{k}\right)x_2$$

$$\frac{1}{2i}\left(\frac{1}{|k|^2}-|k|^2\right)\frac{\partial\mu}{\partial\bar{z}}=f(x_1,x_2),\quad (x_1,x_2)\in\mathbb{R}^2$$

Impose: $\mu=O\left(\frac{1}{z}\right),\quad z\to\infty.$

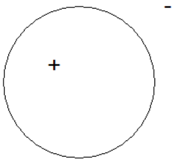
$$\boxed{\mu(x_1,x_2,k)=\frac{1}{2\pi i}\text{sgn}\left(\frac{1}{|k|^2}-|k|^2\right)\iint_{\mathbb{R}^2}\frac{f(x_1',x_2')dx_1'dx_2'}{z'-z},\ \ |k|\neq 1}$$

(ii) Solve for μ in terms of $\hat{f}$

$$\mu^{\pm}=\mp\left(P^{\mp}\hat{f}\right)-\int_{\tau}^{\infty}F(\rho,s,\theta)ds,\quad (\rho,\tau)\in\mathbb{R}^2,\quad \theta\in(0,2\pi)$$

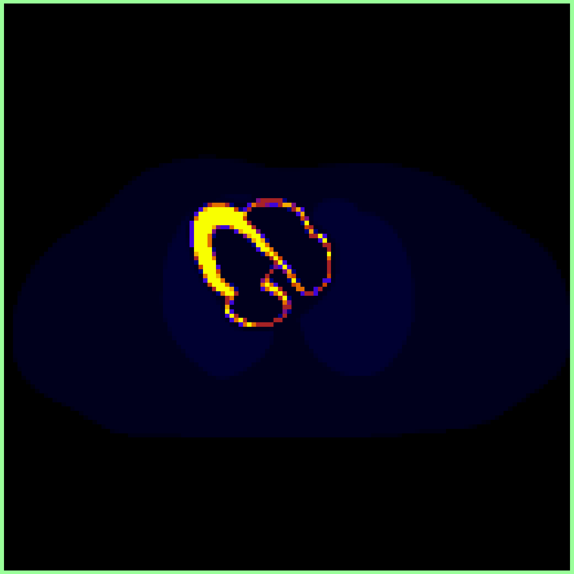
$$\boxed{\mu(x_1,x_2,k)=-\frac{1}{2i\pi^2}\int_0^{2\pi}\frac{e^{i\theta}}{e^{i\theta}-k}\left(\oint_{-\infty}^{\infty}\frac{\hat{f}(\rho,\theta)\mathrm{d}\rho}{\rho-(x_2\cos\theta-x_1\sin\theta)}\right)\mathrm{d}\theta}$$

complex k-plane :

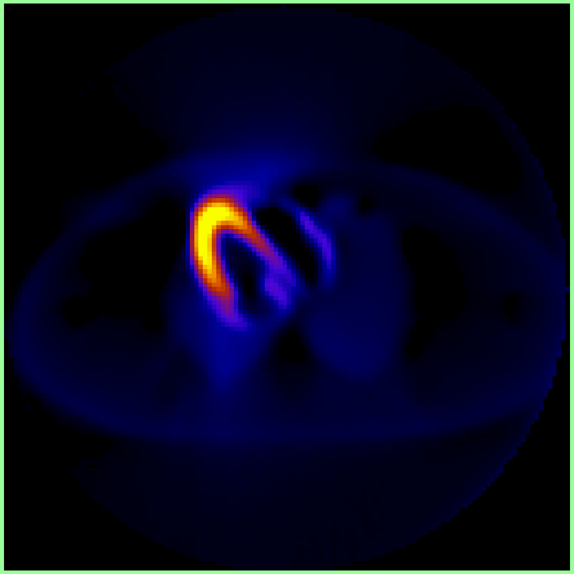


NCAT phantom

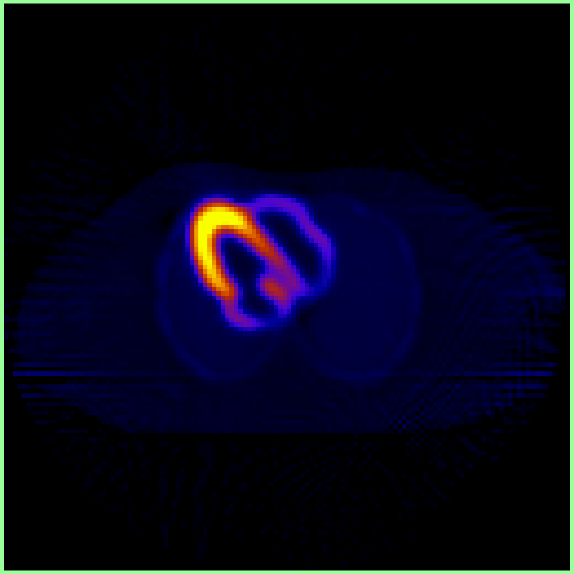
gaussian blur, σ = 0.021 (GP collimator), noise free, R=28cm, 200 projections



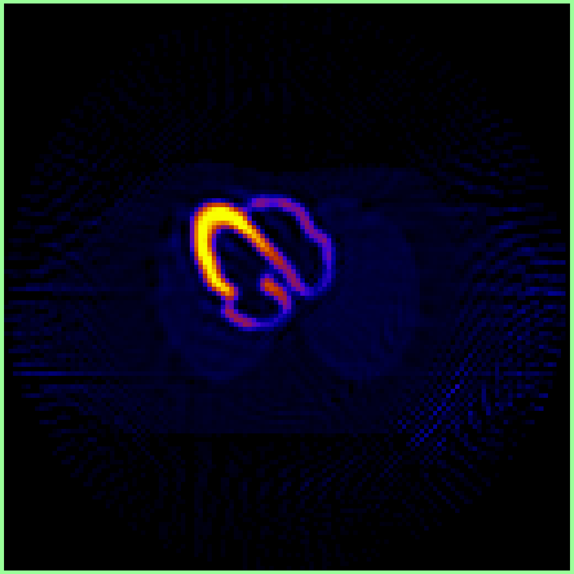
Original



FBP, no attenuation correction



IART, no deblurring



IART, with deblurring

III. Inverting Integrals: The D to N Map

$$q_t = q_{xx}, \quad 0 < x < \infty, 0 < t < T.$$

$$q(x,0) = 0, \quad q_x(0,t) = g_1(t).$$

$$q(0,t) = -\frac{1}{\sqrt{\pi}} \int_0^t \frac{g_1(s)}{\sqrt{t-s}} ds, \quad 0 < t < T.$$

$$q_t + q_{xxx} = 0, \quad 0 < x < \infty, 0 < t < T.$$

$$q(x,0) = 0, \quad q_{xx}(0,t) = g_2(t).$$

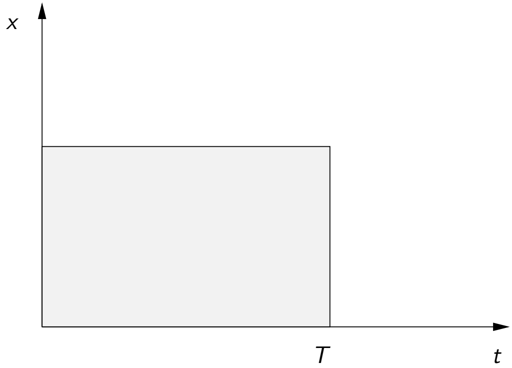
$$q(0,t) = c_1 \int_0^t \frac{g_2(s)}{(t-s)^{\frac{1}{3}}} ds, \quad q_x(0,t) = c_2 \int_0^t \frac{g_2(s)}{(t-s)^{\frac{2}{3}}} ds$$

The Global Relation and Lax pairs

$$\left[e^{-ikx+k^2t}q(x,t)\right]_t - \left[e^{-ikx+k^2t}(q_x(x,t)+ikq(x,t))\right]_x = 0, \quad k \in \mathbb{C}$$

$$\int_{\partial D} e^{-ikx+k^2t}[q(x,t)dx + (q_x(x,t)+ikq(x,t))dt] = 0, \quad k \in \mathbb{C}.$$

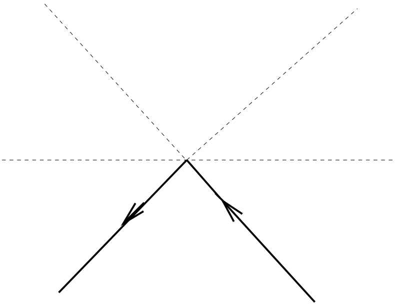
$\underline{0 < x < \infty}$



$$\int_0^T e^{k^2 s}[q_x(0,s) + ikq(0,s)]ds =$$
$$\int_0^\infty e^{-ikx} q(x,0)dx - e^{k^2 T}\int_0^\infty e^{-ikx} q(x,T)dx, \quad Imk \leq 0$$

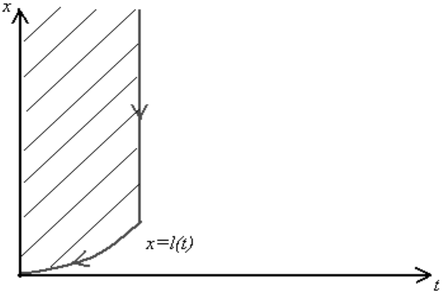
$e^{-k^2 t}$:

$$\int_0^T e^{k^2 s}\left[q_x(0,s) + ikq(0,s)\right]ds = -e^{k^2 T}\int_0^\infty e^{-ikx} q(x,T)dx, \quad Imk \leq 0$$



$$\mu_x + ik\mu = q$$
$$\mu_t + k^2\mu = q_x + ikq$$

$l(t) < x < \infty$



$$\ddot{l}(t) > 0, \quad 0 < t < T,$$

$$l(0) = 0$$

$$\int_0^T e^{k^2 s - ikl(s)} \left[q_x(l(s), s) + (\dot{l}(s) + ik) g_0(s) \right] ds =$$

$$\hat{q}_0(k) - e^{k^2 T} \int_{l(T)}^{\infty} e^{-ikx} q(x, T) dx.$$

(F, Pelloni, JMP 2007)

(Delillo , F, Inverse Problems 2007)

$$F(k) = \int_0^T e^{k^2 s - ikl(s)} f(s) ds, \qquad k \in \mathbb{C}.$$

$$\mu_t(t,k) + \left(k^2 - ik\dot{l}(t)\right) \mu(k,t) = kf(t), \quad k \in \mathbb{C}, \quad 0 < t < T.$$

$$\frac{3}{4}f(t) = -\frac{1}{2\pi i}\int_{\Gamma(t)} k e^{-k^2 t + ikl(t)} F(k)dk + \int_0^t f(s)K(s,t)ds, \quad 0 < t < T,$$

$\Gamma(t)$:

$$\left\{ k = k_R + ik_I,\ k_R^2 - k_I^2 + k_I \dot{l}(t) = 0;\ -\infty < k_R < \infty,\ k_I < 0;\ 0 < t < T \right\},$$

Figure: The curve $\Gamma(t)$

$$K(s,t) = -\frac{1}{2\pi}\int_0^\infty \left[1 - \frac{i}{2}\left(\frac{\nu}{\sqrt{\nu^2 - \dot{l}(s)\nu}} + \frac{\sqrt{\nu^2 - \dot{l}(s)\nu}}{\nu}\right) B(\nu,s,t)\right] d\nu,$$

$$B(\nu,s,t) = \left(\sqrt{\nu^2 - \dot{l}(s)\nu} + i\nu\right)\exp\Big[-\nu(\dot{l}(s) - \vartheta(t,s))(s-t)$$
$$+ i\sqrt{\nu^2 - \dot{l}(s)\nu}\,(2\nu - \vartheta(t,s))(s-t)\Big], \quad \vartheta(t,s) = \frac{l(t) - l(s)}{t-s}.$$

$$\mu(t,k) = \mu_j(t,k), \quad k \in \Omega_j(t), \quad 0 < t < T, \quad j = 1,2,3$$

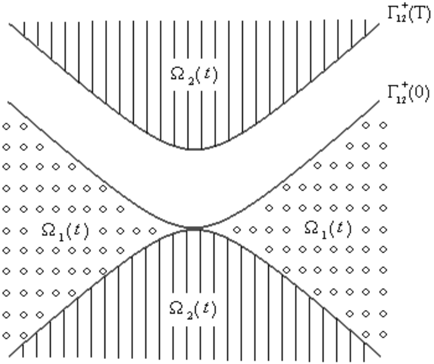


Figure: The domains $\Omega_1(t)$ and $\Omega_2(t)$.

$$\Omega_1(t) : \left\{ k_I > 0,\ k_R^2 - k_I^2 + k_I \dot{l}(0) > 0 \right\} \cup \left\{ k_I < 0,\ k_R^2 - k_I^2 + k_I \dot{l}(t) > 0 \right\},$$

$$\Omega_2(t) : \left\{ k_I > 0,\ k_R^2 - k_I^2 + k_I \dot{l}(T) < 0 \right\} \cup \left\{ k_I < 0,\ k_R^2 - k_I^2 + k_I \dot{l}(t) < 0 \right\}.$$

$$\Gamma_{12}^{+}(t) : \left\{ k_R^2 - k_I^2 + k_I \dot{l}(t) = 0,\ k_I > 0 \right\}.$$

$$\mu_j(t,k) = k\int_{t_j}^{t} e^{k^2(s-t)-ik(l(s)-l(t))} f(s)ds,\ 0<t<T,\ k\in\Omega_j,\ j=1,2,3.$$

$$t_1 = 0, \qquad t_2 = T, \qquad t_3 = S(k_R, k_I),$$

$$k_R^2 - k_I^2 + k_I \dot{l}(t) = 0,\ 0<t<T,\ k_I>0,\ -\infty<k_R<\infty:\ t=S(k_R,k_I).$$

$$\mu_j = O\left(\frac{1}{k}\right), \quad k\to\infty, \quad k\in\Omega_j, \quad j=1,2,3.$$

$$\mu(t,k,\bar{k}) = \frac{1}{2\pi i}\int_{\Gamma(t)} \frac{''jump''}{\lambda - k} d\lambda - \frac{1}{\pi}\iint_{\Omega(t)} \frac{\partial \mu(t,\lambda,\bar{\lambda})}{\partial \lambda} \frac{d\lambda_R d\lambda_I}{\lambda - k}$$

$$0 < t < T, \quad k \in \mathbb{C}$$

Above inversion and global relation imply:

$$\frac{3}{4}g_1(t) = \frac{1}{2\sqrt{\pi}}\left[\frac{1}{\sqrt{t}}\int_0^\infty e^{-\frac{(l(t)-x)^2}{4t}}\dot{q}_0(x)dx - \int_0^t \frac{e^{-\frac{(l(t)-l(s))^2}{4(t-s)}}}{\sqrt{t-s}}\dot{g}_0(s)ds\right] + \int_0^t g_1(s)K(s,t)ds, \qquad 0 < t < T$$

IV. Nonlinear PDEs in 4+2 and 3+1

(F, PRL, May 2006)

$$\frac{\partial \mu}{\partial \bar{x}} + \sigma_3 \frac{\partial \mu}{\partial \bar{y}} - k[\sigma_3, \mu] + Q\mu = 0, \qquad (*)$$

$$x = \frac{1}{2}(\xi + \eta), y = \frac{1}{2}(\xi - \eta), k = k_1 + ik_2,$$

$$\xi = \xi_1 + i\xi_2, \quad \eta = \eta_1 + i\eta_2$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & q_1(\xi_1, \xi_2, \eta_1, \eta_2) \\ q_2(\xi_1, \xi_2, \eta_1, \eta_2) & 0 \end{pmatrix}$$

Nonlinear FT in 4D

$$\{q_1, q_2\} \longrightarrow \{f_1, f_2\}$$

$$f_1(k_1, k_2, \lambda_1, \lambda_2) = c \int_{\mathbb{R}^4} e^{-4i(k_2\xi_1 - k_1\xi_2 + \lambda_2\eta_1 - \lambda_1\eta_2)} q_1 \mu_{22} d\xi_1 d\xi_2 d\eta_1 d\eta_2,$$

$$f_2(k_1, k_2, \lambda_1, \lambda_2) = c \int_{\mathbb{R}^4} e^{-4i(-\lambda_2\xi_1 + \lambda_1\xi_2 - k_2\eta_1 + k_1\eta_2)} q_2 \mu_{11} d\xi_1 d\xi_2 d\eta_1 d\eta_2,$$

$c = (2/\pi)^3$.
$\mu(\xi_1, \xi_2, \eta_1, \eta_2, k_1, k_2)$ is determined in terms of
$\{q_j(\xi_1, \xi_2, \eta_1, \eta_2)\}_1^2$ by $(*)$ with

$$\mu \sim I \quad \text{as} \quad |\xi_1|^2 + |\xi_2|^2 + |\eta_1|^2 + |\eta_2|^2 \to \infty$$

$$\{f_1, f_2\} \longrightarrow \{q_1, q_2\}$$

$$q_1(\xi_1, \xi_2, \eta_1, \eta_2) = \int_{\mathbb{R}^4} e^{4i(k_2\xi_1 - k_1\xi_2 + \lambda_2\eta_1 - \lambda_1\eta_2)} f_1 \mu_{11} dk_1 dk_2 d\lambda_1 d\lambda_2$$

$$q_2(\xi_1, \xi_2, \eta_1, \eta_2) = \int_{\mathbb{R}^4} e^{4i(-\lambda_2\xi_1 + \lambda_1\xi_2 - k_2\eta_1 + k_1\eta_2)} f_2 \mu_{22} dk_1 dk_2 d\lambda_1 d\lambda_2,$$

μ is determined in terms of $\{f_j(k_1, k_2, \lambda_1, \lambda_2)\}_1^2$ by

$$\frac{\partial \mu(k_1, k_2)}{\partial \bar{k}} = \int_{\mathbb{R}^2} \mu(\lambda_1, \lambda_2) \cdot$$

$$\cdot \begin{pmatrix} 0 & e^{-4i(k_2\xi_1 - k_1\xi_2 + \lambda_2\eta_1 - \lambda_1\eta_2)} f_1 \\ e^{-4i(-\lambda_2\xi_1 + \lambda_1\xi_2 - k_2\eta_1 + k_1\eta_2)} f_2 & 0 \end{pmatrix} d\lambda_1 d\lambda_2,$$

$$\mu \sim I + O(\frac{1}{k}), \quad k \to \infty$$

Integrable PDEs in 4+2

$$\left\{ q_j^{(0)}(\xi_1, \xi_2, \eta_1, \eta_2) \right\}_1^2 \longrightarrow \left\{ f_j^{(0)}(k_1, k_2, \lambda_1, \lambda_2) \right\}_1^2$$

$$\left\{ f_1^{(0)} E, f_2^{(0)} E^{-1} \right\} \longrightarrow \{ q_j(\xi_1, \xi_2, \eta_1, \eta_2, t_1, t_2) \}_1^2$$

$$E = e^{4i(\lambda_1 \lambda_2 + k_1 k_2) t_2 - 2i(\lambda_1^2 - \lambda_2^2 + k_1^2 - k_2^2) t_1}$$

Then,

$$(-1)^j \partial_{\bar{t}} q_j + \frac{1}{4}(\partial_{\bar{\xi}}^2 + \partial_{\bar{\eta}}^2) q_{\bar{j}} - q_j \partial_{\bar{\xi}}^{-1} (q_1 q_2)_{\bar{\eta}} = 0, \quad j = 1, 2,$$

$$\partial_{\bar{\xi}}^{-1} q = -\frac{1}{\pi} \int_{\mathbb{R}^2} \frac{q(\xi_1', \xi_2')}{\xi - \xi'} d\xi_1 d\xi_2.$$

Implicit Reductions to 3+1

independence of $t_1 \leftrightarrow \lambda_1\lambda_2 + k_1k_2 = 0$

Linear limit:

$$\frac{\partial^2 q}{\partial\xi_1\partial\xi_2} + \frac{\partial^2 q}{\partial\eta_1\partial\eta_2} = 0.$$

KP type

$$\frac{\partial q}{\partial\bar{t}} = \frac{1}{4}\frac{\partial^3 q}{\partial\bar{x}^3} - \frac{3}{2}q\frac{\partial q}{\partial\bar{x}} + \frac{3}{4}\partial_{\bar{x}}^{-1}\frac{\partial^2 q}{\partial\bar{y}^2}$$

independence of $t_2 \leftrightarrow k_1^3 - \lambda_1^3 + 3\lambda_1\lambda_2^2 - 3k_1k_2^2 = 0$

Explicit Reductions to 3+1

Potential KdV

$$q_{\bar t} = \frac{1}{4} q_{\bar x \bar x \bar x} - \frac{3}{4} q_{\bar x}^2, \quad t = t_1 + i t_2, \quad x = x_1 + i x_2, \tag{1}$$

To preserve reality:

$$q_t = \frac{1}{4} q_{xxx} - \frac{3}{4} q_x^2. \tag{2}$$

$(q_{\bar t})_t = (q_t)_{\bar t}$:

$$\left(q_{\bar x \bar x \bar x} - 3 q_{\bar x}^2\right)_t = \left(q_{xxx} - 3 q_x^2\right)_{\bar t}.$$

Use $\partial_{\bar{x}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$:

$$(\Delta q)(\Delta q_{x_2}) = 0, \quad \Delta = \partial_{x_1}^2 + \partial_{x_2}^2.$$

(1)$\pm$ (2):

$$q_{t_1} = \frac{1}{16}\left(\partial_{x_1}^3 - 3\partial_{x_1}\partial_{x_2}^2\right) q - \frac{3}{8}\left(q_{x_1}^2 - q_{x_2}^2\right), \tag{3}$$

$$q_{t_2} = \frac{1}{16}\left(-\partial_{x_2}^3 + 3\partial_{x_2}\partial_{x_1}^2\right) q - \frac{3}{4} q_{x_1} q_{x_2}. \tag{4}$$

$$\boxed{q_{t_1} = \frac{1}{4} q_{x_1 x_1 x_1} - \frac{3}{8}\left(q_{x_1}^2 - q_{x_2}^2\right), \quad \Delta q = 0,} \tag{5}$$

$$\boxed{q_{t_2} = -\frac{1}{4} q_{x_2 x_2 x_2} - \frac{3}{4} q_{x_1} q_{x_2}, \quad \Delta q = 0.} \tag{6}$$

Claim : $\Delta q_{t_1} = \Delta q_{t_2} = 0$

Let $q(x_1, x_2, 0) = q_0(x_1, x_2)$ be a harmonic function. Then $q(x_1, x_2, t_1)$ and $q(x_1, x_2, t_2)$ satisfy the integrable systems (5) and (6) respectively. If q_0 is real, then q remains real.

Potential KP

$$q_{\bar{t}} = \frac{1}{4} q_{\bar{x}\bar{x}\bar{x}} - \frac{3}{4} q_{\bar{x}}^2 + \frac{3}{4} \bar{L} q, \quad L = \partial_x^{-1} \partial_y^2 \,,$$

where

$$y = y_1 + i y_2, \quad \partial_{\bar{x}}^{-1} f = \frac{1}{\pi} \int_{\mathbb{R}^2} f(x_1', x_2') \frac{dx_1' dx_2'}{x - x'}.$$

$$(\Delta q)(\Delta q_{x_2}) + q_{\bar{x}} L q_{\bar{x}} - q_x \bar{L} q_x + \frac{1}{2} \left(\bar{L} q_x^2 - L q_{\bar{x}}^2 \right) = 0.$$

$$\Delta q = q_{\bar{x}y} = 0, \quad \Rightarrow \quad \Delta q = 0, \quad q_{y_2} = -\partial_{x_1}^{-1} q_{x_2 y_1}$$

$$\boxed{q_{t_1} = \tfrac{1}{4} q_{x_1 x_1 x_1} - \tfrac{3}{8}\left(q_{x_1}^2 - q_{x_2}^2\right) + \tfrac{3}{4}\partial_{x_1}^{-1} q_{y_1 y_1}, \quad \Delta q = 0,}$$

$$\boxed{q_{t_2} = -\tfrac{1}{4} q_{x_2 x_2 x_2} - \tfrac{3}{4} q_{x_1} q_{x_2} - \tfrac{3}{4}\partial_{x_2}^{-1} q_{y_1 y_1}, \quad \Delta q = 0.}$$

$$\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$$

Example: $-\infty < x_1 < \infty, \quad x_2 > 0.$

$q(x_1, 0, y_1, 0) = Q_0(x_1, y_1).$

$q_{x_2}(x_1, 0, y_1, t) = Hq_{x_1}(x_1, 0, y_1, t)$

$q(x_1, 0, y_1, t) = Q(x_1, y_1, t)$

$$Q_{t_1} = \frac{1}{4} Q_{x_1 x_1 x_1} - \frac{3}{8}\left[Q_{x_1}^2 - (HQ_{x_1})^2\right] + \frac{3}{4}\partial_{x_1}^{-1} Q_{y_1 y_1},$$
$$Q(x_1, y_1, 0) = Q_0(x_1, y_1).$$

Laplace's equation with $q(x_1, 0, y_1, t) = Q(x_1, y_1, t)$, yields $q(x_1, x_2, y, t)$