Twist & Shout: Maximal Enstrophy Generation in 3-D Navier-Stokes

LU LU, Ph.D. (2006)
Wachovia Investments

CHARLES R. DOERING
University of Michigan

Outline

I. Enstrophy & regularity and uniqueness of solutions
II. Analytic estimate on rate of enstrophy generation
III. Variational formulation of maximal production
IV. Computational results & a reality check
V. Conclusions, remarks & laments

Navier-Stokes equations:

$$\dot{\vec{u}} + \vec{u}\cdot\vec{\nabla}\vec{u} + \vec{\nabla}p = \nu\Delta\vec{u}$$

$$0 = \vec{\nabla}\cdot\vec{u}$$

- Periodic box: $\vec{x} \in [0, L]^3$
- Initial condition: $\vec{u}(\vec{x},0) = \vec{u}_0(\vec{x})$

$$\left(\text{WLOG} \int \vec{u}_0(\vec{x})\, d^3x = 0\right)$$

Open* (\$1M) *question:

Does a smooth solution exist for all $t > 0$?

Some definitions & some things we know:

- Kinetic energy:

$$K(t) \equiv \tfrac{1}{2}\int \left|\vec{u}(\vec{x},t)\right|^2 d^3x = \tfrac{1}{2}\left\|\vec{u}(\vec{x},t)\right\|_2^2$$

- Vorticity:

$$\vec{\omega} = \nabla \times \vec{u} \quad \Rightarrow \quad \dot{\vec{\omega}} + \vec{u}\cdot\vec{\nabla}\vec{\omega} = \nu\Delta\vec{\omega} + \vec{\omega}\cdot\vec{\nabla}\vec{u}$$

- Enstrophy:

$$E(t) \equiv \left\|\vec{\omega}(\vec{x},t)\right\|_2^2 = \left\|\vec{\nabla}\vec{u}(\vec{x},t)\right\|_2^2 \geq \tfrac{8\pi^2}{L^2}K(t)$$

- If solution is *smooth* enough, ${}^{dK}\!/_{dt} = -\nu E$.

Global (in time) ***weak*** solutions exist:

- If $K_0 < \infty$, there are weak solutions with finite energy,

$$K(t) \le K_0 \qquad \forall t \ge 0$$

- … and with finite ***integrated enstrophy***,

$$\int_{t_a}^{t_b} E(t)\,dt \; < \; \infty \qquad \forall 0 \le t_a \le t_b$$

- … but only known to satisfy an energy ***inequality***,

$$K(t_b) \le K(t_a) - \nu \int_{t_a}^{t_b} E(t)\,dt \quad \text{for a.e. } t_a > 0$$

- … and there is ***no*** assurance that they are unique.

Local (in time) ***strong*** solutions exist:

- For $(8\pi^2/L^2)K_0 \le \boldsymbol{E_0} < \infty$,

$$\exists\, T(K_0, E_0, \nu) > 0 \ \ni\ E(t) < \infty \text{ for } 0 \le t < T.$$

- **Fact:**

$$E(t) < \infty \text{ for } t_a \le t \le t_b$$

$$\Updownarrow$$

$$\vec{u}(\cdot,t) \in C^\infty([0,L]^3) \text{ for } t_a < t \le t_b.$$

- And strong solutions are ***unique***.

As long as the enstrophy is finite ...

$$\frac{dK}{dt} = -\nu E$$

$$\frac{dE}{dt} = -2\nu \left\| \vec{\nabla} \vec{\omega} \right\|_2^2 + 2 \int \vec{\omega} \cdot \vec{\nabla} \vec{u} \cdot \vec{\omega} \, d^3 x$$

$$= \underbrace{-2\nu \left\| \Delta \vec{u} \right\|_2^2 + 2 \int \vec{u} \cdot \vec{\nabla} \vec{u} \cdot \Delta \vec{u} \, d^3 x}$$

$$\Uparrow$$

Enstrophy generation rate $G\{\boldsymbol{u}\}$ = ***production*** – ***dissipation***

Vortex stretching & enstrophy production:

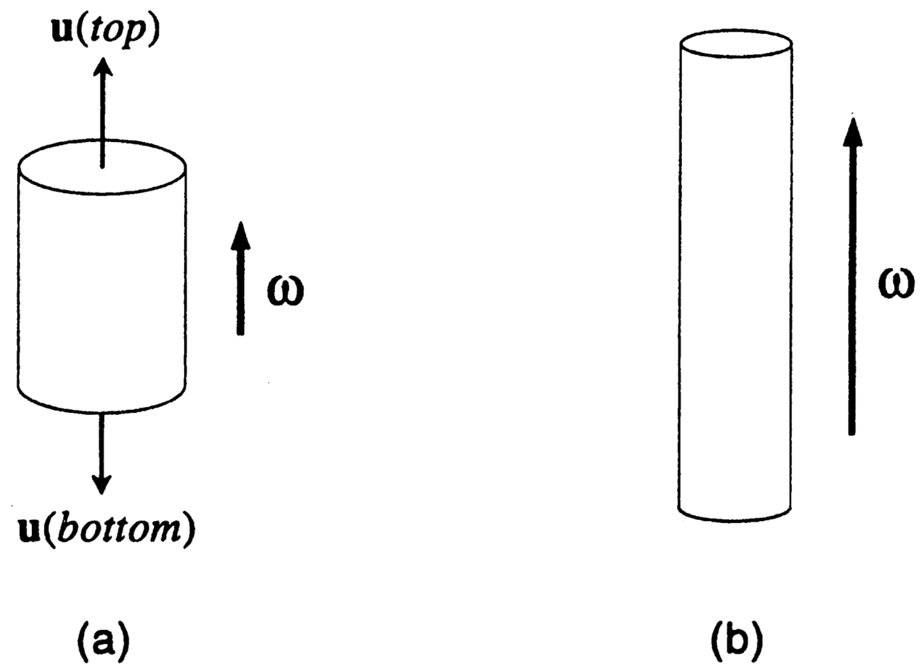


Fig. 1.4. The vortex stretching mechanism. When $\boldsymbol{\omega}\cdot\nabla\mathbf{u}$ has a component parallel to $\boldsymbol{\omega}$, as in (*a*), the fluid element is stretched in the direction of the vorticity. The resulting decrease in the element's moment of inertia, illustrated in (*b*), leads to an increase in the amplitude of the vorticity.

Vorticity can be amplified; enstrophy can be produced.

Does this nonlinear process get out of control?

Can $G\{u\}$ be estimated in terms of K and E?

$$G\{\vec{u}\} = -2\nu\left\|\Delta\vec{u}\right\|_2^2 + 2\int \vec{u}\cdot\vec{\nabla}\vec{u}\cdot\Delta\vec{u}\; d^3x$$

Hölder & Cauchy-Schwarz

$\Downarrow$

$$\leq -2\nu\left\|\Delta\vec{u}\right\|_2^2 + 2\left\|\vec{u}\right\|_\infty \left\|\vec{\nabla}\vec{u}\right\|_2 \left\|\Delta\vec{u}\right\|_2$$

$$= -2\nu\left\|\Delta\vec{u}\right\|_2^2 + 2\left\|\vec{u}\right\|_\infty \; E^{1/2} \left\|\Delta\vec{u}\right\|_2$$

Agmon-Sobolev-Gagliardo-Nirenberg (in 3D)

$\Downarrow$

$$\left\|\vec{u}\right\|_\infty \leq \tilde{c}\left\|\vec{\nabla}\vec{u}\right\|_2^{1/2}\left\|\Delta\vec{u}\right\|_2^{1/2} = \tilde{c}\,E^{1/4}\left\|\Delta\vec{u}\right\|_2^{1/2}$$

$$G\{\vec{u}\} \leq -2\nu \left\| \Delta \vec{u} \right\|_2^2 + 2\tilde{c}\, E^{3/4} \left\| \Delta \vec{u} \right\|_2^{3/2}$$

Hölder-Young ($ab \leq a^p/p + b^q/q$ when $1/p + 1/q = 1$)

$$\Downarrow$$

$$2\tilde{c}\, E^{3/4} \left\| \Delta \vec{u} \right\|_2^{3/2} \leq \frac{c}{\nu^3} E^3 + \nu \left\| \Delta \vec{u} \right\|_2^2$$

Integration by parts & Cauchy-Schwarz

$$\Downarrow$$

$$E = \left\| \vec{\nabla} \vec{u} \right\|_2^2 = -\int \vec{u} \cdot \Delta \vec{u}\, d^3x \leq \left\| \vec{u} \right\|_2 \left\| \Delta \vec{u} \right\|_2 = \sqrt{2K} \left\| \Delta \vec{u} \right\|_2$$

System of differential inequations:

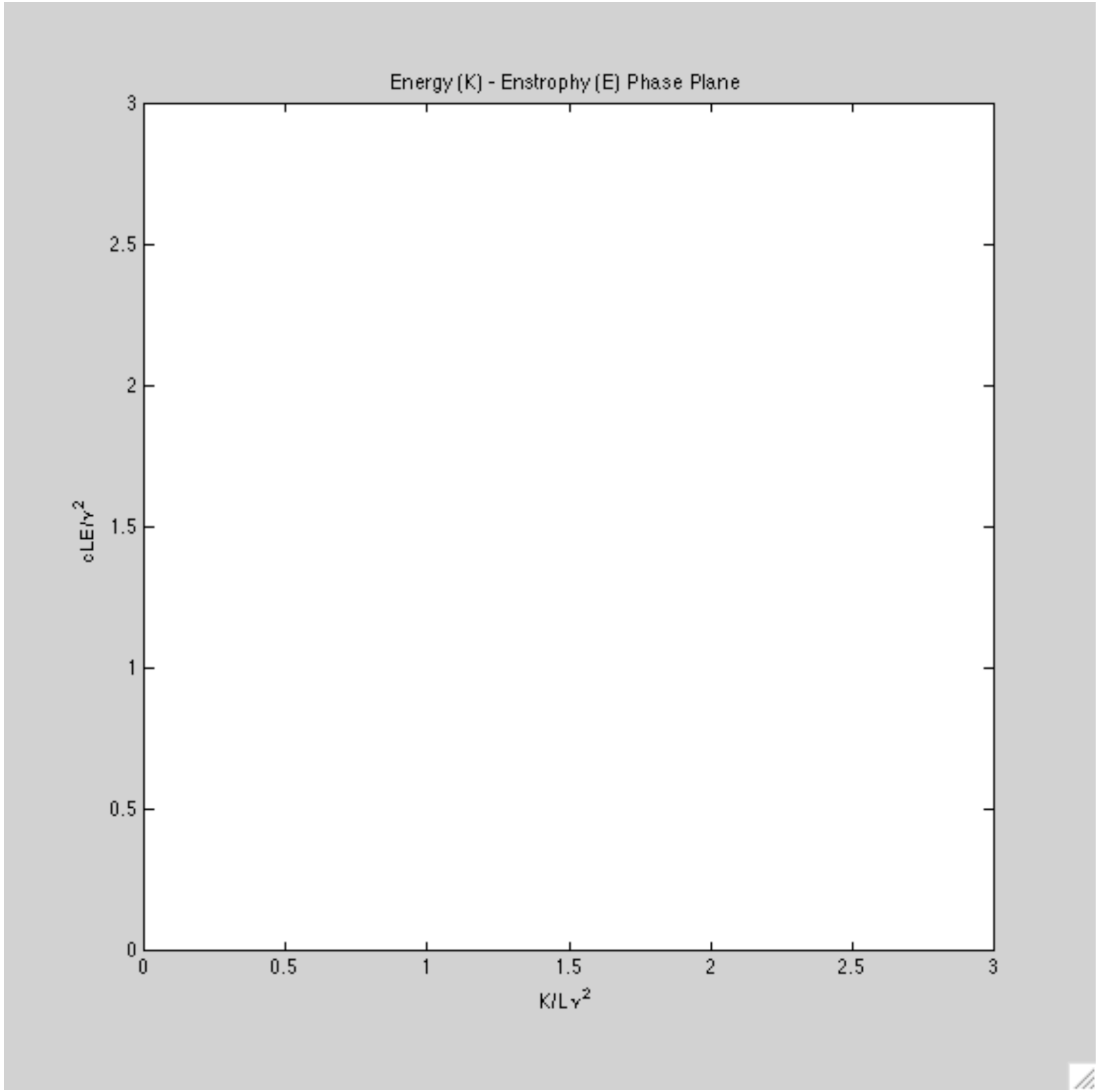
$$\frac{dK}{dt} = -\nu E$$

$$\frac{dE}{dt} \leq -\frac{\nu}{2}\frac{E^2}{K} + \frac{c}{\nu^3}E^3$$

$$\therefore \quad E \leq E_0\left(\frac{K}{K_0}\right)^{\frac{1}{2}}\left(1 + \frac{2cK_0E_0}{\nu^4}\left[\left(\frac{K}{K_0}\right)^{\frac{3}{2}} - 1\right]\right)^{-1}$$

... as long as RHS $\geq$ 0.

$$\frac{dK}{dt} = -\nu E \qquad \frac{dE}{dt} \le -\frac{\nu}{2}\frac{E^2}{K} + \frac{c}{\nu^3}E^3$$



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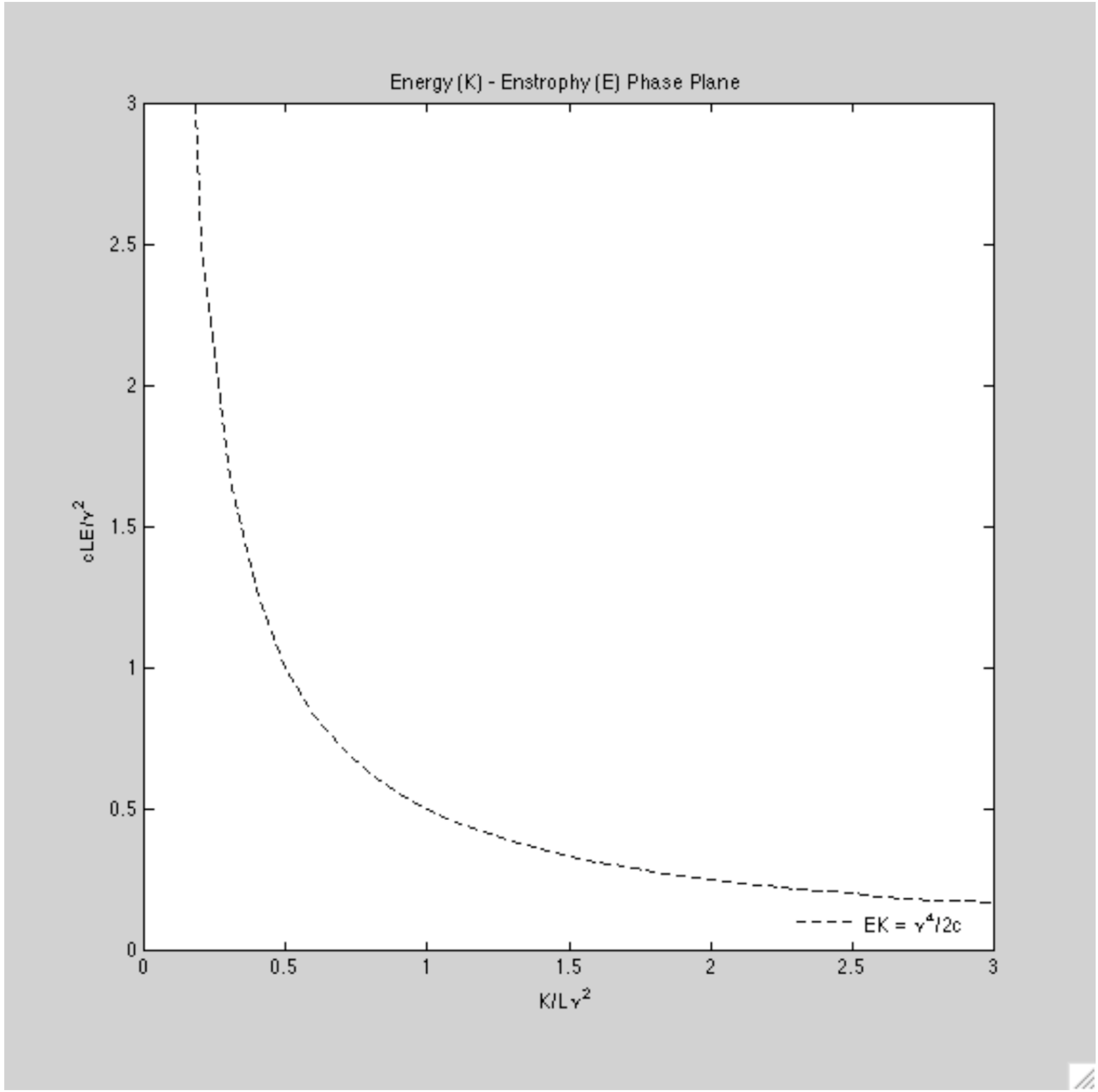

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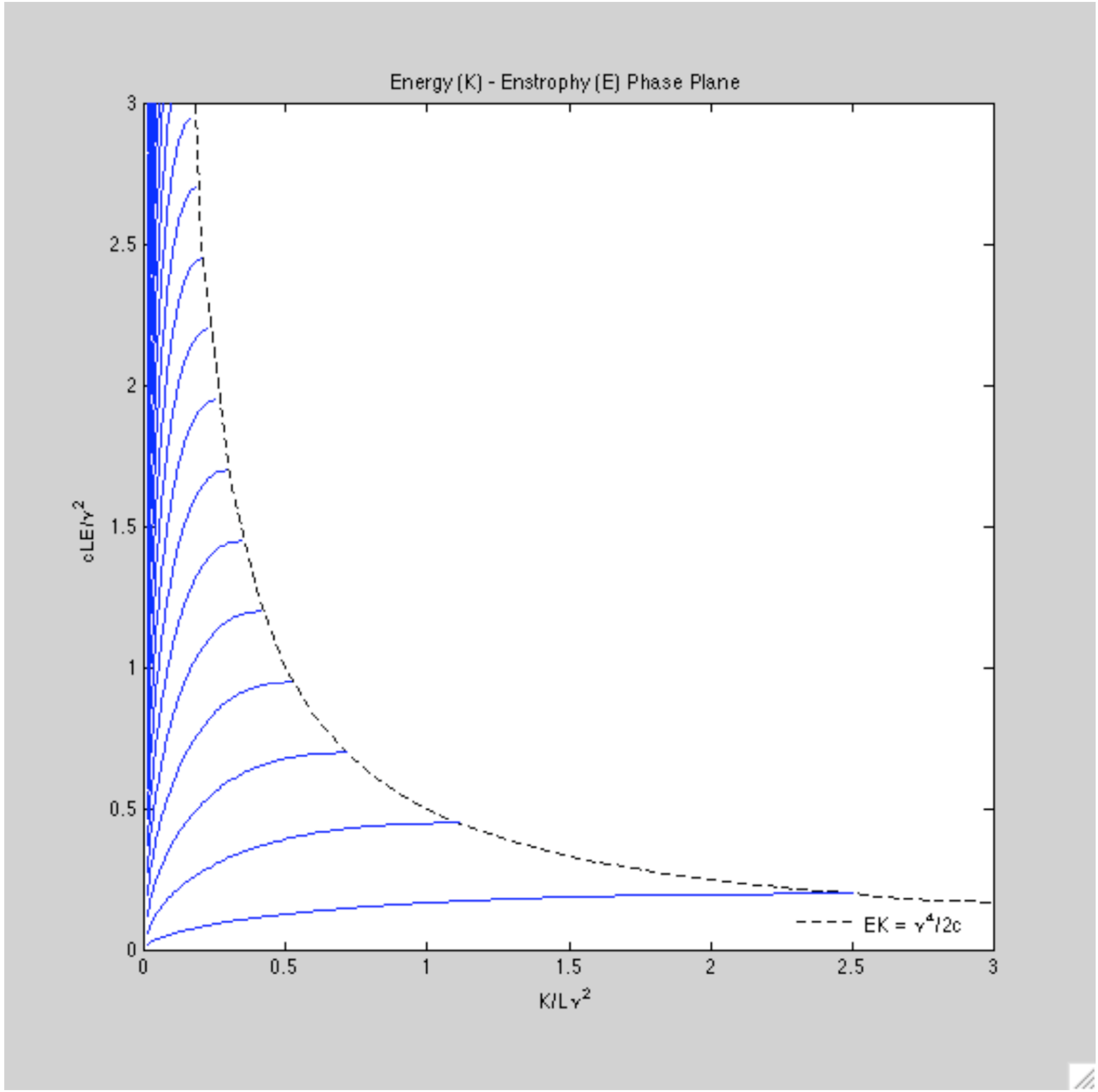
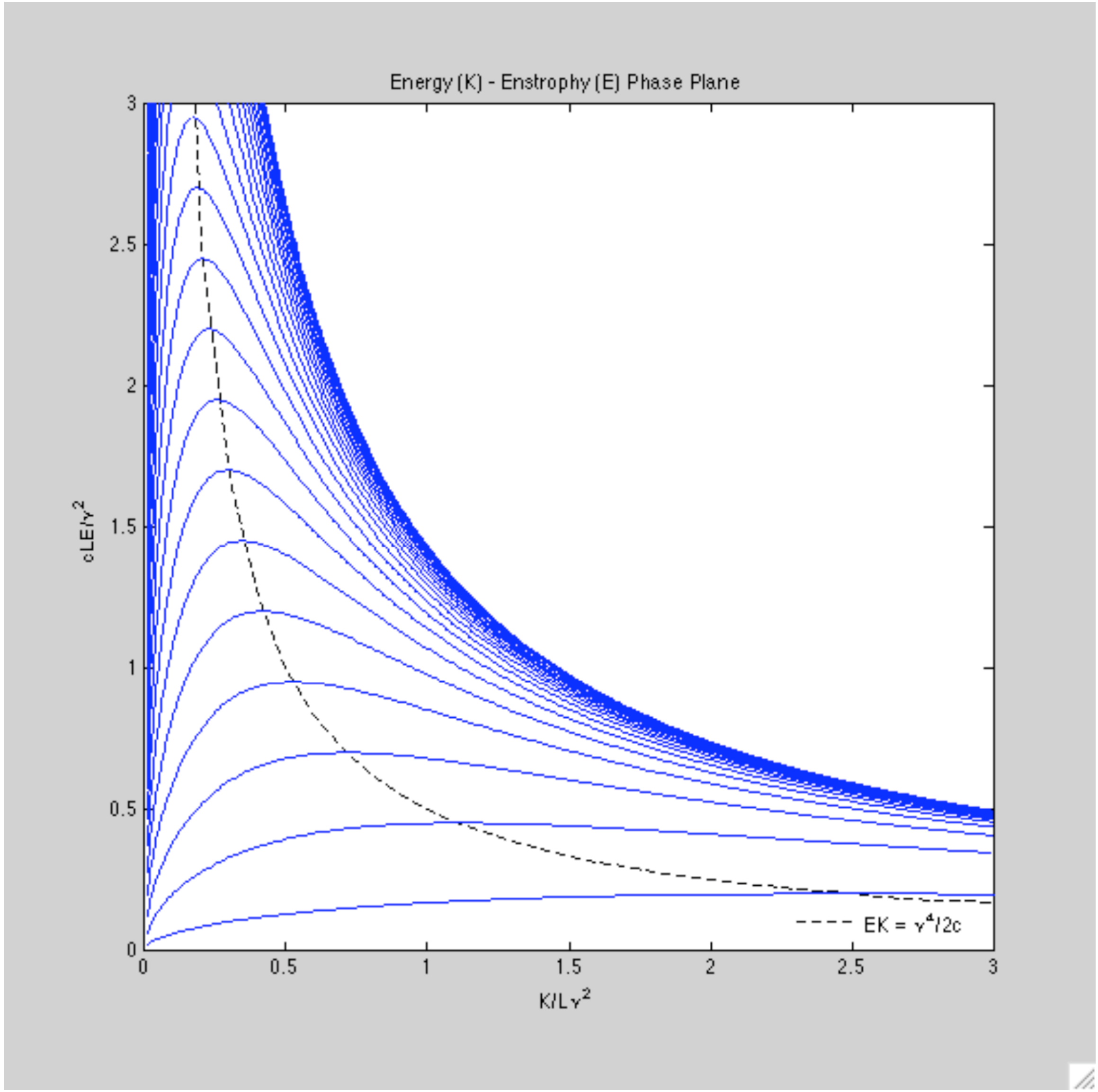
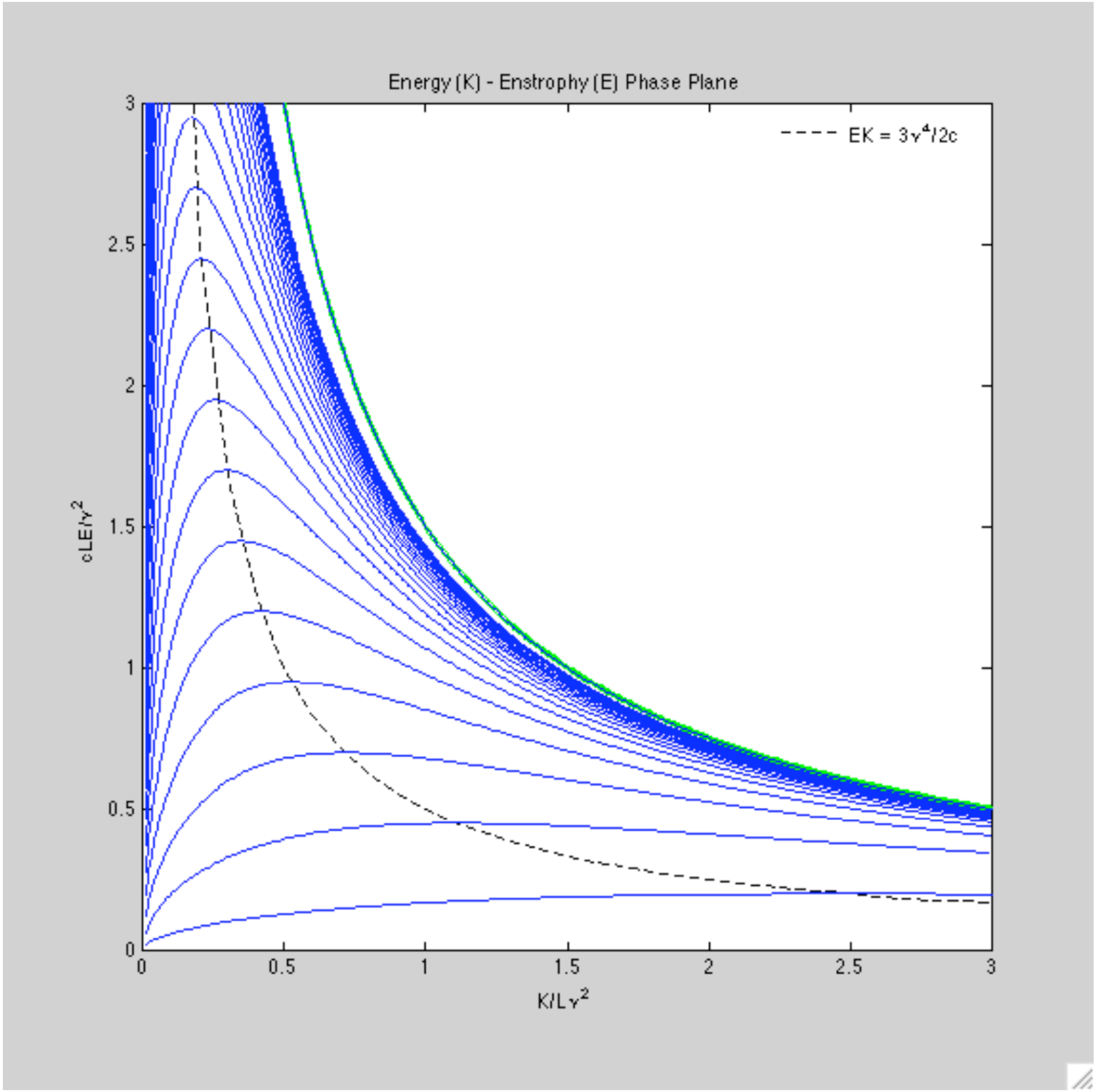


Enstrophy ***decreases*** if $E_0 K_0 \le \nu^4/2c$.

$$\frac{dK}{dt} = -\nu E \qquad \frac{dE}{dt} \le -\frac{\nu}{2}\frac{E^2}{K} + \frac{c}{\nu^3}E^3$$

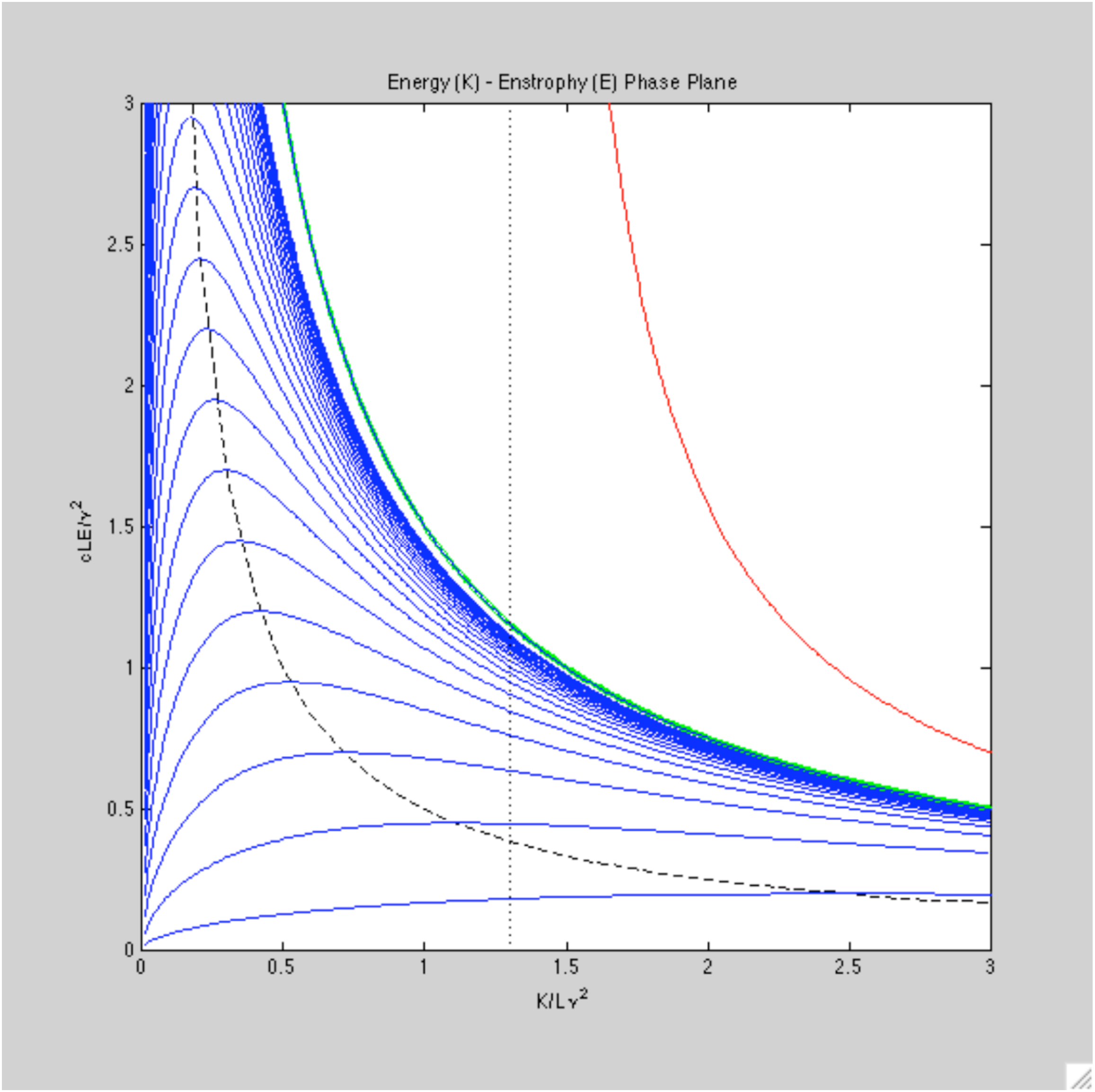


$$\frac{dK}{dt} = -\nu E \qquad \frac{dE}{dt} \leq -\frac{\nu}{2}\frac{E^2}{K} + \frac{c}{\nu^3}E^3$$



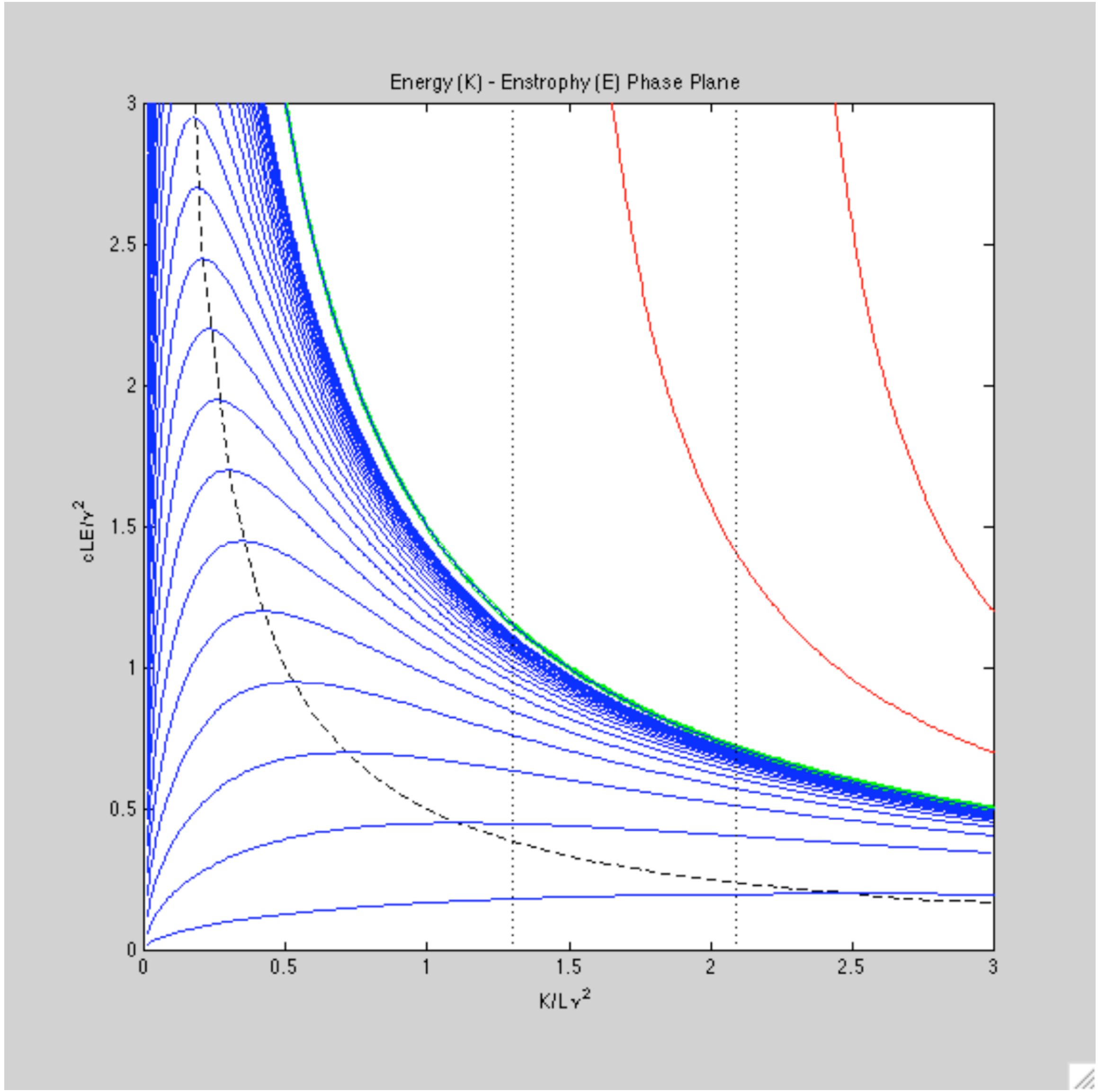
Global existence and uniqueness ***if*** $E_0 K_0 \leq 3\nu^4/2c$.

$$\frac{dK}{dt} = -\nu E \qquad \frac{dE}{dt} \leq -\frac{\nu}{2}\frac{E^2}{K} + \frac{c}{\nu^3}E^3$$



But does ***not*** prevent finite-time singularity if $E_0K_0 > 3\nu^4/2c$.

$$\frac{dK}{dt} = -\nu E \qquad \frac{dE}{dt} \leq -\frac{\nu}{2}\frac{E^2}{K} + \frac{c}{\nu^3}E^3$$



But does ***not*** prevent finite-time singularity if $E_0 K_0 \geq 3\nu^4/2c$.

Question:

How big can $G\{\boldsymbol{u}\}$ ***really*** get in terms of K and E?

- Analytic estimates ***don't*** account for div $\boldsymbol{u} = 0$ …
- or ***total*** competition between production & dissipation.
- Would like to solve the variational problem for max rate:

$$M(K,E;\nu,L) = \sup_{\vec{\nabla}\cdot\vec{u}=0}\left\{G\{\vec{u}\} \mid \tfrac{1}{2}\left\|\vec{u}\right\|_2^2 = K \text{ and } \left\|\vec{\nabla}\vec{u}\right\|_2^2 = E\right\}$$

Settle for slightly less:

$$\Re(E;\nu,L) = \sup_{\vec{\nabla}\cdot\vec{u}=0} \left\{ G\{\vec{u}\} \mid \left\|\vec{\nabla}\vec{u}\right\|_2^2 = E \right\}$$

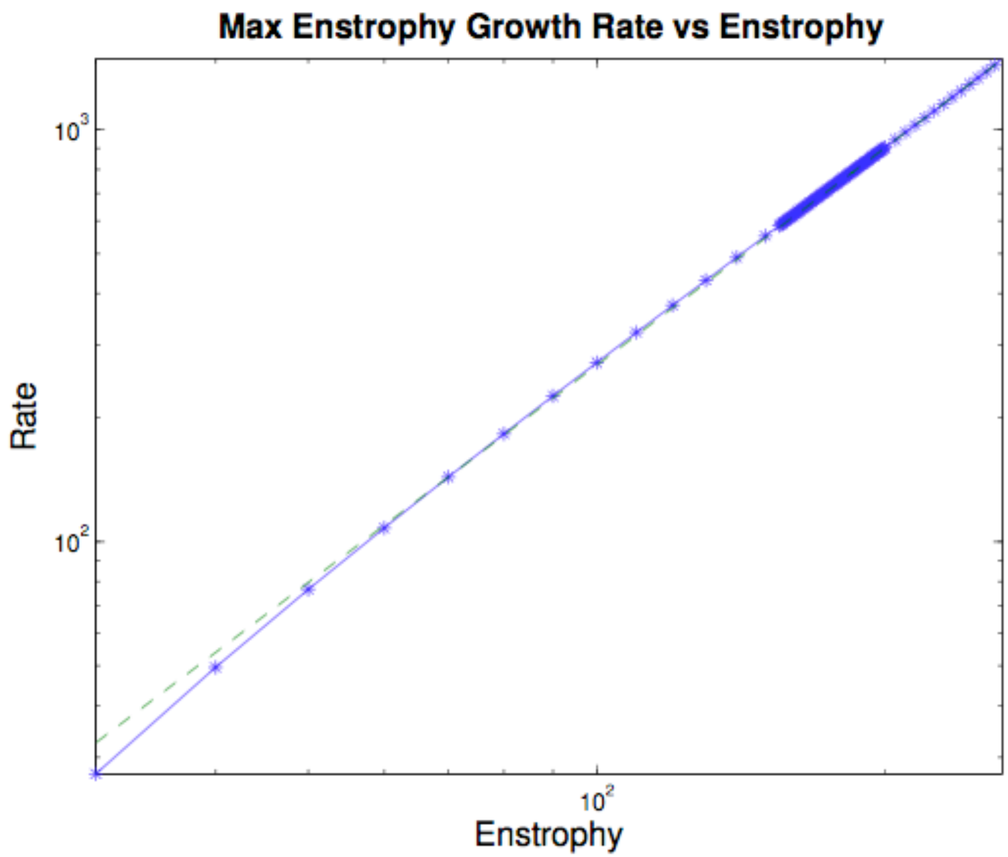
$$\text{so} \quad \frac{dE}{dt} \le \Re(E)$$

- We know that $\Re \le cE^3/\nu^3$ … but that ≠ **$1M**.
- "*Critical*" behavior is $\Re \sim E^2$ as $E \to \infty$.
- Solve the Euler-Lagrange equations:

$$0 = \frac{\delta}{\delta\vec{u}} \left\{ G\{\vec{u}\} + \int p\vec{\nabla}\cdot\vec{u}\, d^3x + \lambda \int \left|\vec{\nabla}\vec{u}\right|^2 d^3x \right\}$$

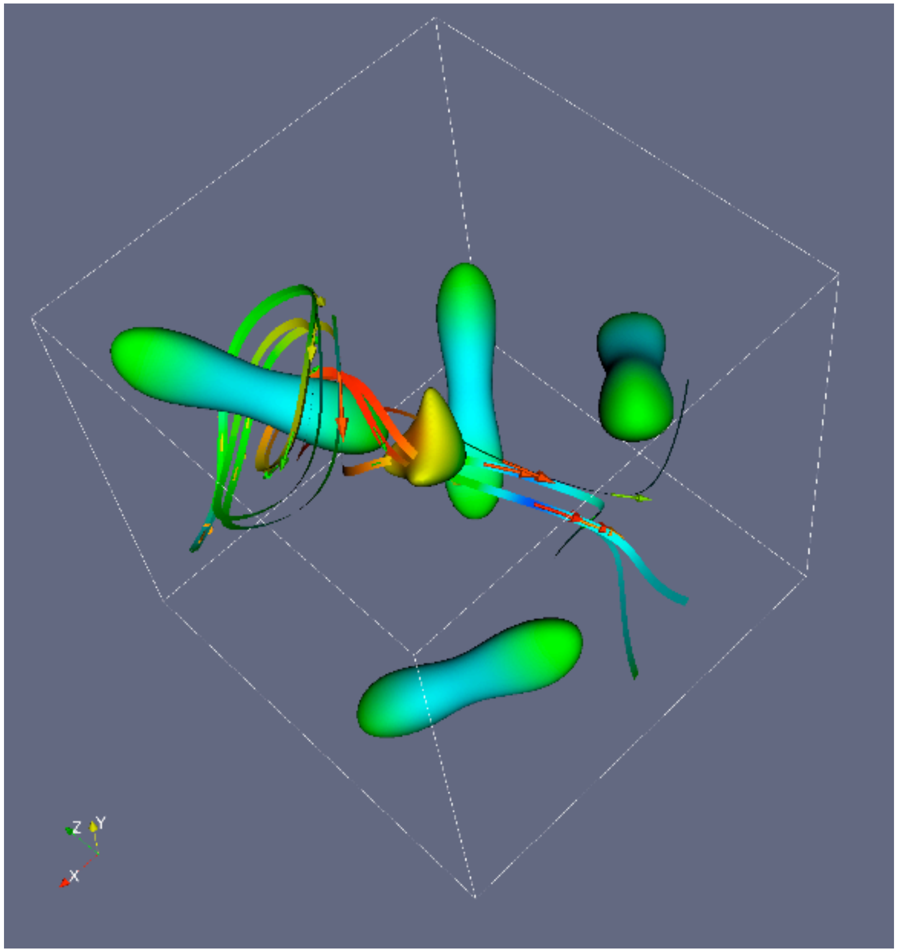
Computationally … via ***gradient ascent method***.

Starting from exact solution as $E \to 0$ …



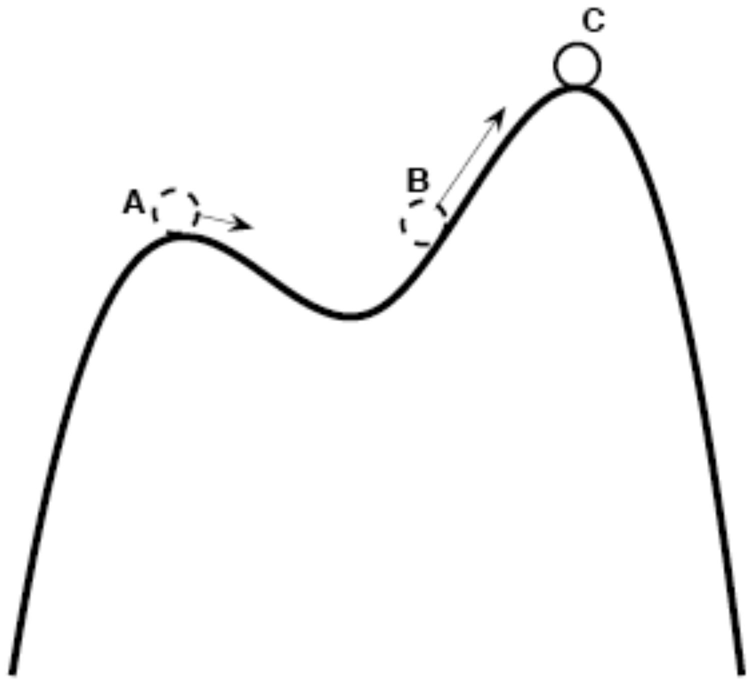
- Large E behavior is $\Re \sim E^{1.78\ (=7/4?)}$ … ***subcritial!***

What do the maximizers look like?



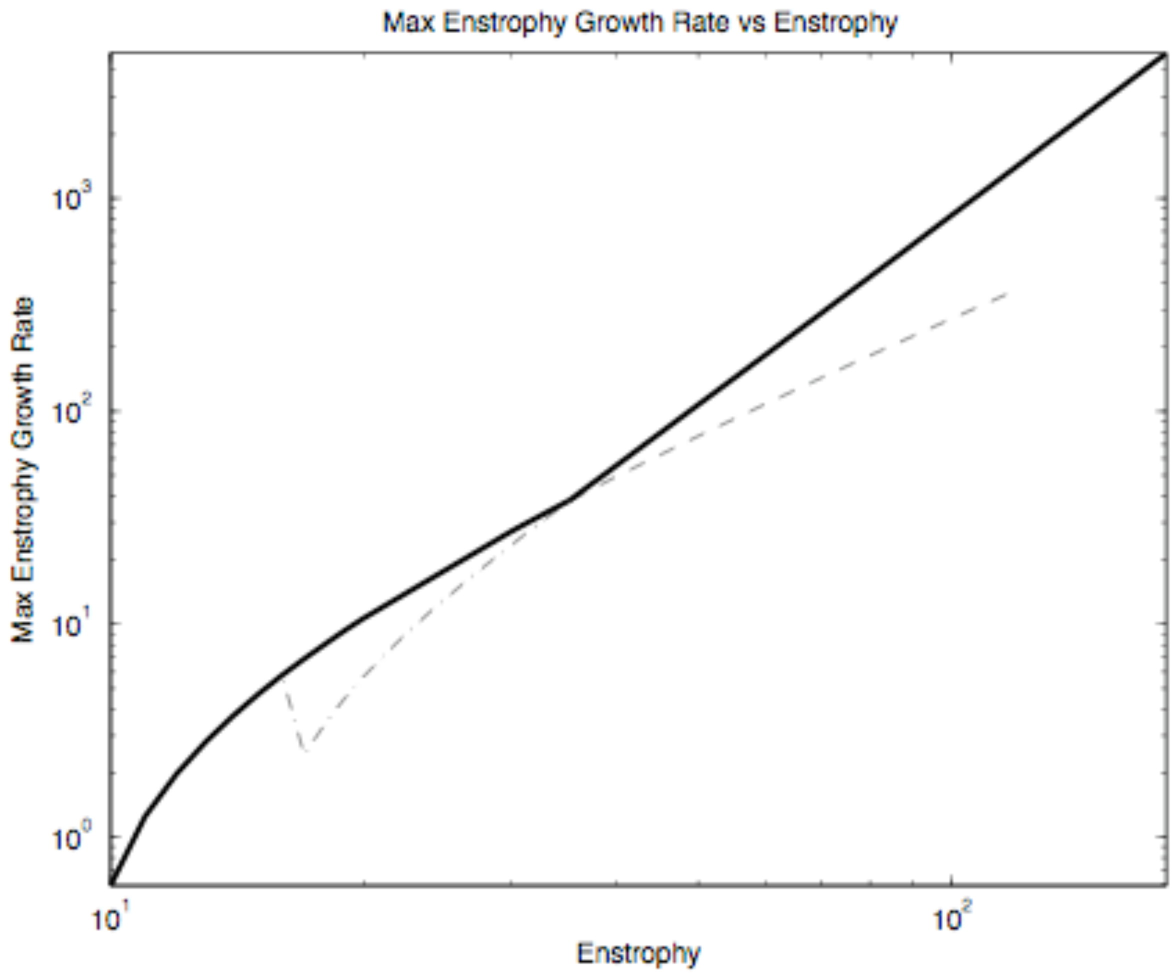
periodic array of “vortex stretchers”

But this is a non-convex variational problem



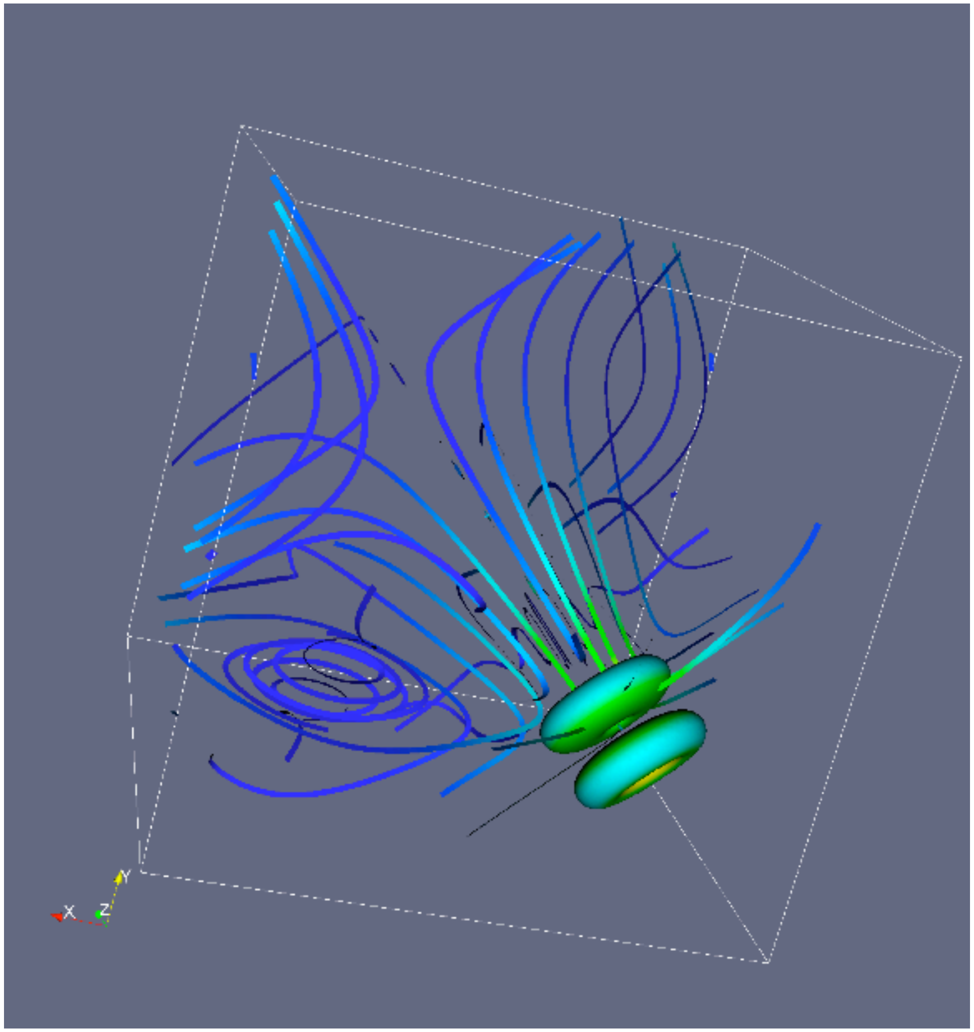
- Euler-Lagrange solutions are local extrema …
- So must see if there are other, global, maxima.

... another branch emerges at high E:



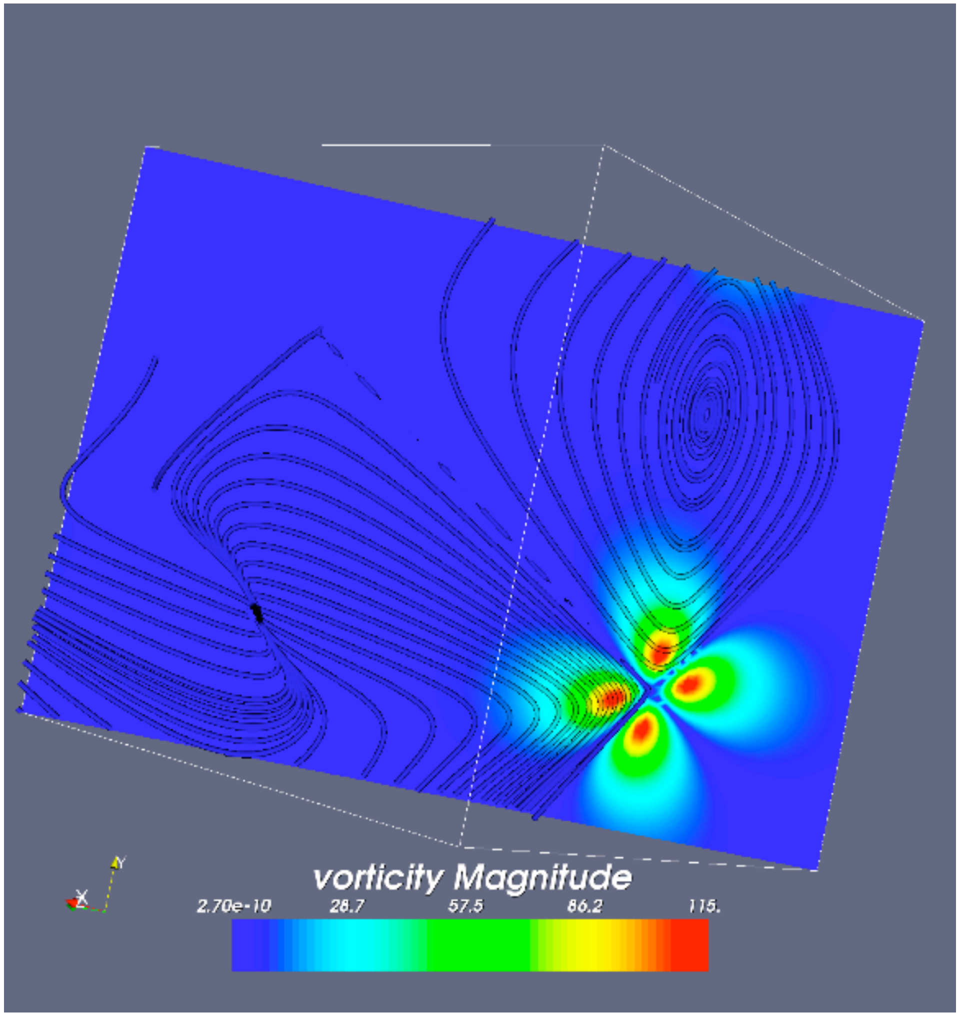
- Large E behavior is $\Re \sim E^{2.997\ (=3?)}$ ... ***as estimated***.

What do these maximizers look like?



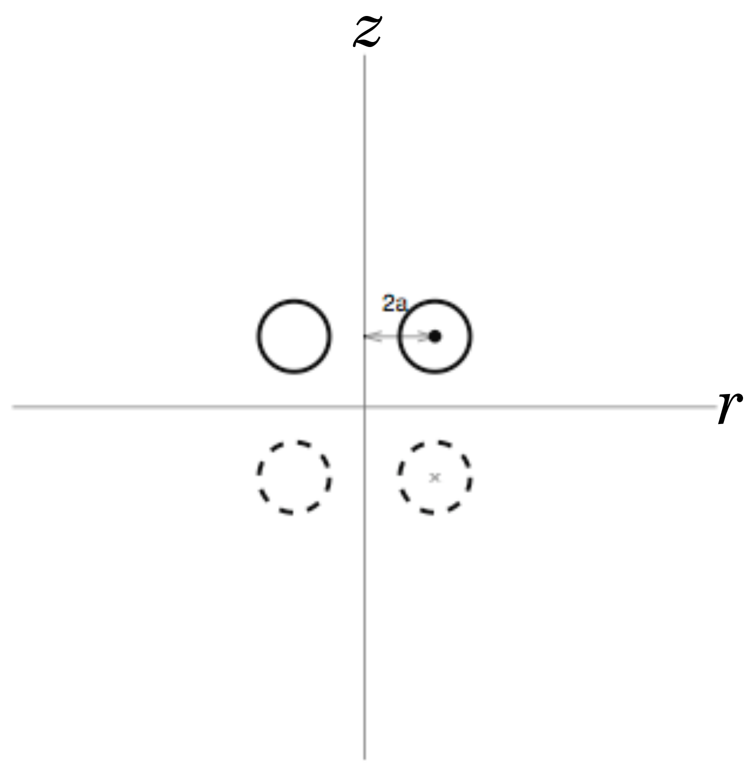
colliding vortex rings

Another view ...



Vorticity in a plane slice

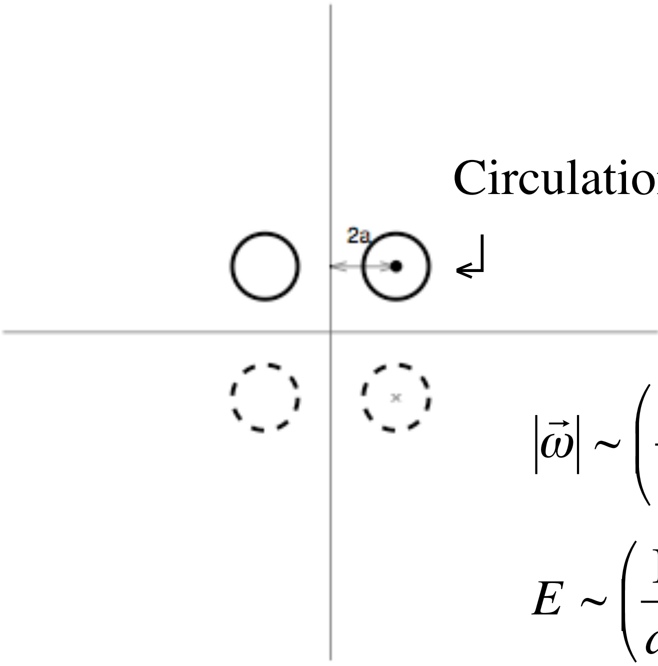
Reality check:



For velocity fields with cylindrical symmetry …

$$G\{\vec{u}\} = -2\nu \left\| \vec{\nabla}\vec{\omega} \right\|_2^2 + 2\int \omega_\theta^2 \frac{u_r}{r}\ d^3x$$

Reality check (continued):



Circulation Γ

$$|\vec{\omega}| \sim \left(\frac{\Gamma}{a^2}\right)$$

$$E \sim \left(\frac{\Gamma}{a^2}\right)^2 a^3 = \frac{\Gamma^2}{a}$$

$$G\{\vec{u}\} = -\nu \left\|\vec{\nabla}\vec{\omega}\right\|_2^2 + 2\int \omega_\theta^2 \frac{u_r}{r}\ d^3x$$

$$\sim -\nu \left(\frac{1}{a}\frac{\Gamma}{a^2}\right)^2 a^3 + \left(\frac{\Gamma}{a^2}\right)^2 \left(\frac{1}{a}\frac{\Gamma}{a}\right) a^3$$

$$\sim -\nu \frac{E}{a^2} + \frac{E^{3/2}}{a^{3/2}}$$

Then maximize over a …

… max occurs at $a \sim \nu^2/E$

$$\Rightarrow \quad G\{\vec{u}\} \sim \frac{E^3}{\nu^3}$$

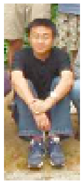
Conclusions, remarks & laments:

- **Conclusion #1**: The analytic asymptotic high-E estimate $\Re(E;\nu,L) \le cE^3/\nu^3$ ***can*** be saturated by divergence-free fields.
- **Lament #1**: no **$1M** to be found down this road!

- **Conclusion #2**: This "most dangerous" velocity field will ***not*** produce a singularity in N-S.
- **Lament #2**: no **$1M** to be found down that road!

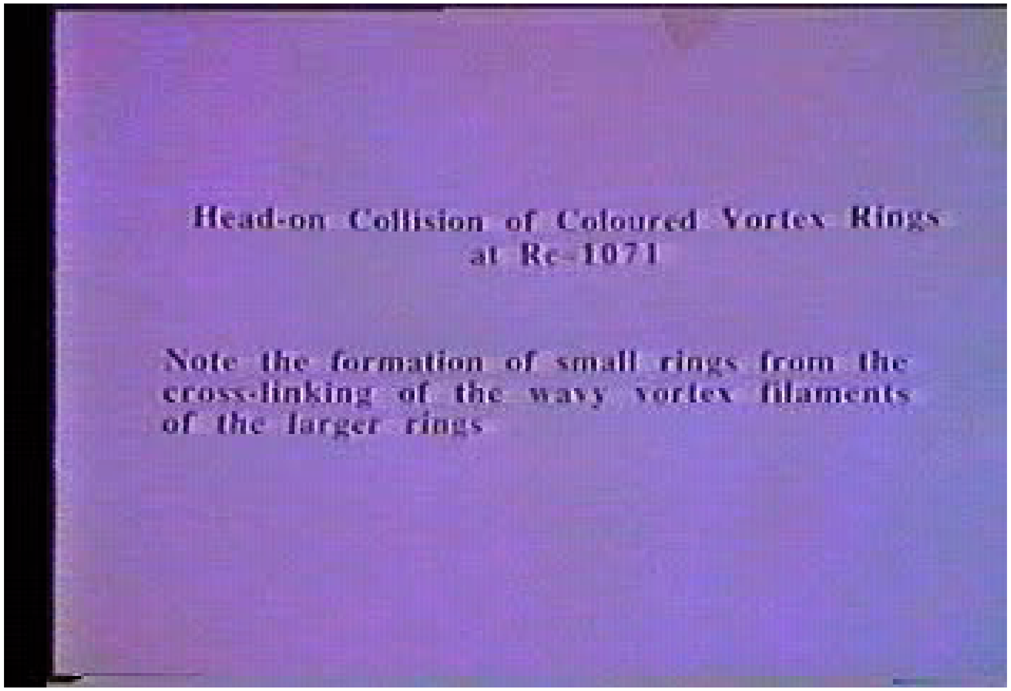
- **Conclusion #3**: $K \sim 1/E$ for the optimizer, so we're not sure if knowing the full upper limit $\boldsymbol{M(K,E;\nu,L)}$ will help …
- **Lament #3**: so **$1M** not ***clearly*** down that road, either!

Maybe Lu will find $1M in Manhattan ...



Just for fun … what *do* colliding vortex rings do?

(from website of Dr. T.B. Nickels <http://www2.eng.cam.ac.uk/~tbn22/Mov.html>)



The End

Twist & shout:

Well, shake it up, baby, now
Twist and shout.
C'mon c'mon, c'mon, c'mon, baby, now,
Come on and work it on out.

Well, work it on out, honey.
You know you look so good.
You know you got me goin', now,
Just like I knew you would.

Well, shake it up, baby, now,
Twist and shout.
C'mon, c'mon, c'mon, c'mon, baby, now,
Come on and work it on out.

You know you twist your little girl,
You know you twist so fine.
Come on and twist a little closer, now,
And let me know that you're mine.

Well, shake it, shake it, shake it, baby, now.
Well, shake it, shake it, shake it, baby, now.
Well, shake it, shake it, shake it, baby, now.
Ahhhhhhhhhh(low) Ahhhhhhhhhh(higher)
Ahhhhhhhhhh(higher) Ahhhhhhhhhhh(high)