EPDiff and Optimal Control of Shapes

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Talk Outline

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- How does EPDiff arise in the analysis of shape? A Darryl-eyed view

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- Particle-mesh discretisation

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- Particle-mesh discretisation
- Numerical examples

EP equation and optimal control

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see Bloch, Crouch, Marsden & Ratiu, 1998

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$$\begin{aligned} \tfrac{d}{dt} I\Omega &= \dot{Q}^T P + Q^T \dot{P}, \\ &= (Q\Omega)^T P + Q^T P\Omega, \\ &= \Omega^T (Q^T P) + (Q^T P)\Omega, \\ &= \Omega^T (I\Omega) + (I\Omega)\Omega. \end{aligned}$$

When in general can P and Q be eliminated from the dynamical equations?

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Hamiltonian $H = \ell(\xi(P, Q))$

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and $P \diamond Q$ is a cotangent-lifted momentum map

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- The resulting numerical method can be reduced to a discrete EP equation provided that the discretisation is left-invariant
- See: Marsden and Bou-Rabee (2007) and Cotter and Holm (submitted 2007)

Variational image matching

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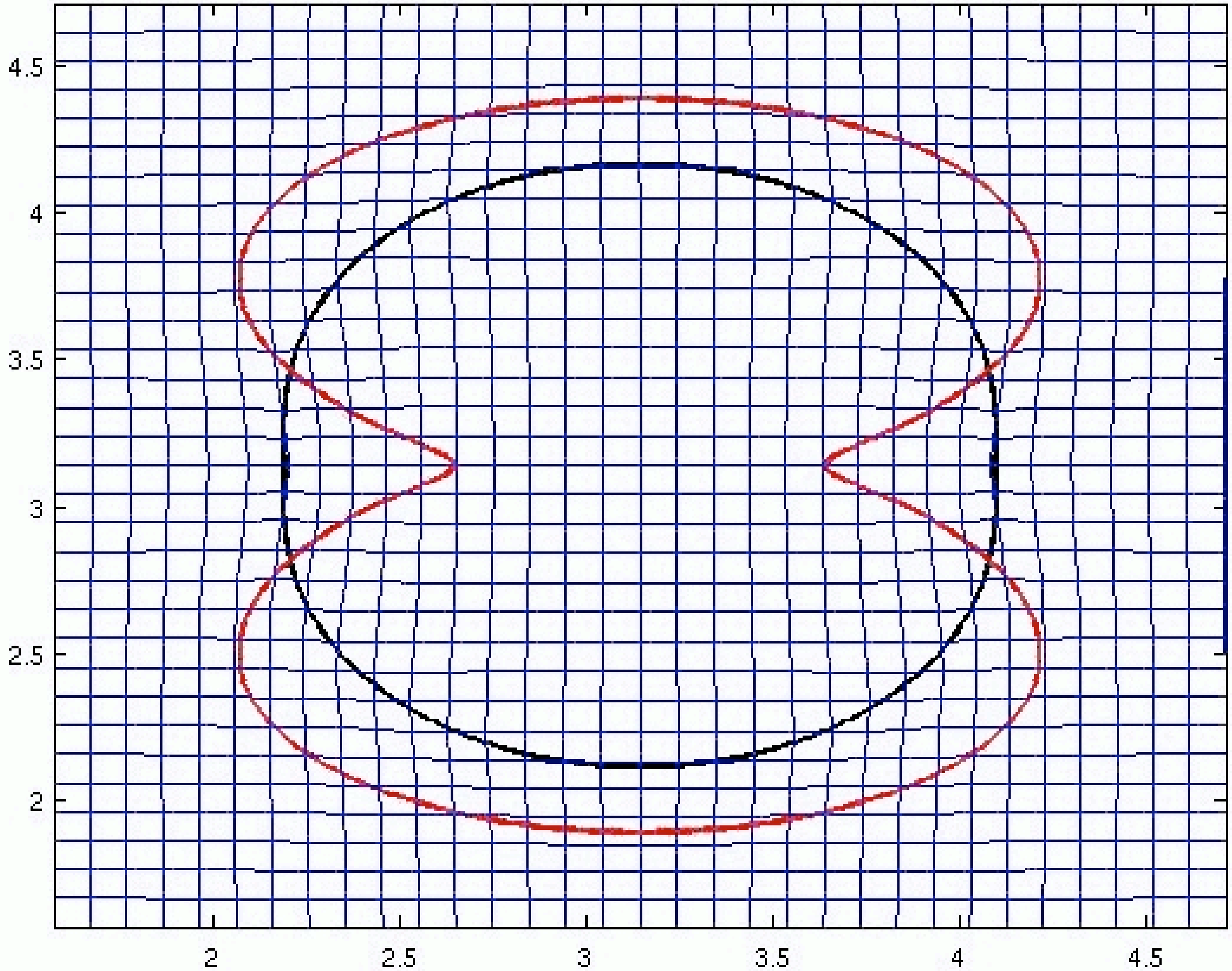
For the rest of the talk specialise to embeddings

What does the shortest path between two shapes look like?


4.5
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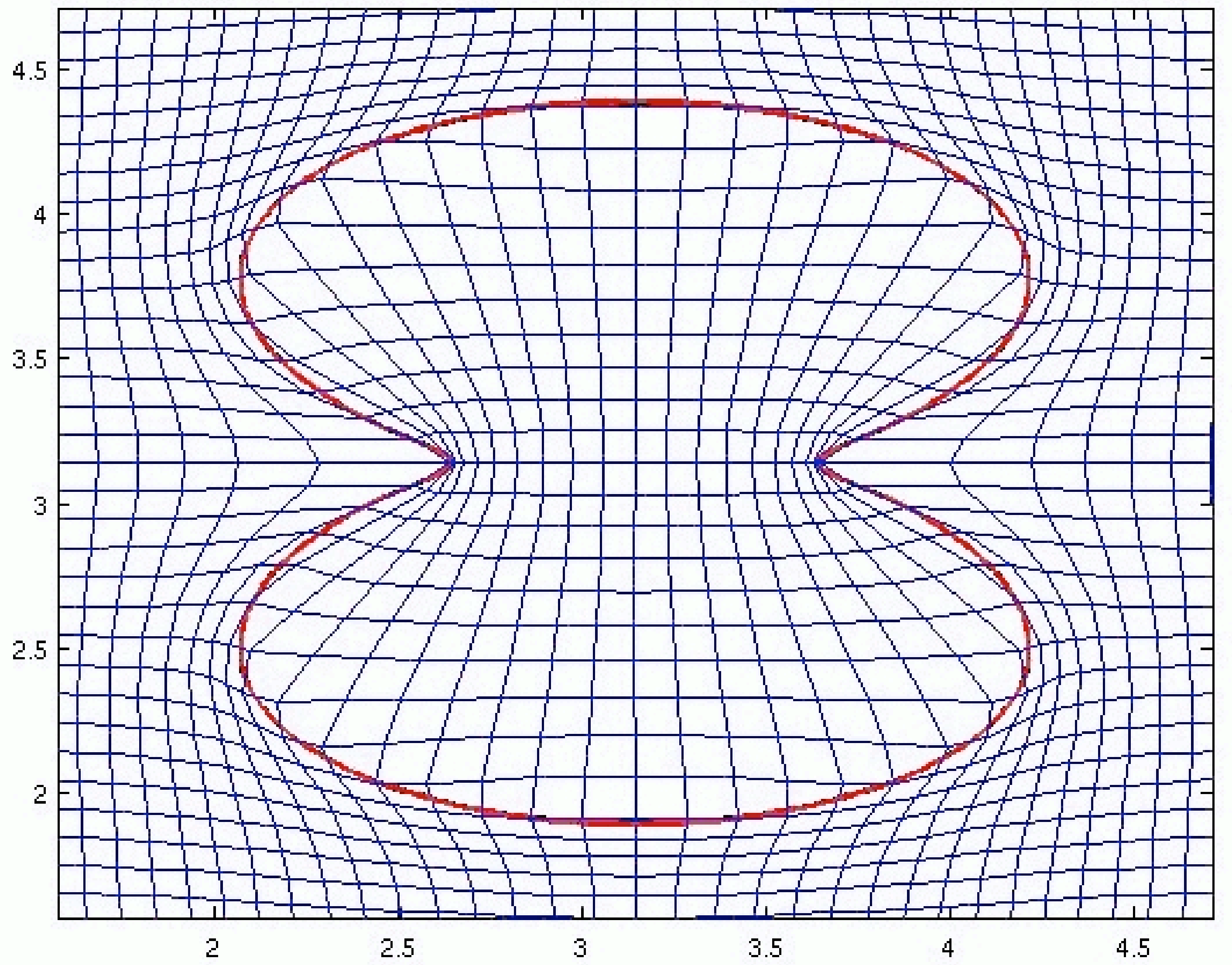
4.5
4
3.5
3
2.5
2
2
2.5
3
3.5
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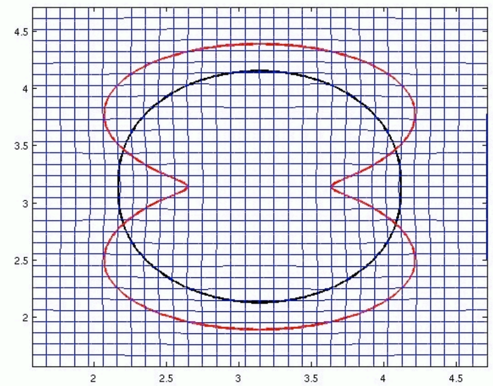
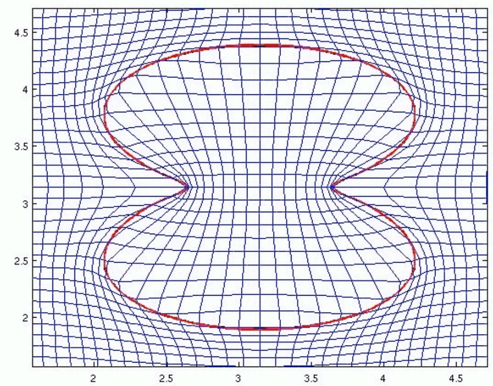
4.5

What can we do with these paths?

- Length along path measures amount of **deformation** from one shape to another
- A way of **comparing** and **classifying** shapes
- Deformation from one shape to another is encoded in **initial conditions** for deformation velocity field, where we can perform linear statistics





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and

$$\boldsymbol{Q}_0(s) = \boldsymbol{Q}^A(s), \quad \forall s \in (0, 2\pi], \qquad \boldsymbol{Q}_1 \equiv \boldsymbol{Q}^B$$

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- For now, matching condition is a “soft constraint” enforced *via* penalty term

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where

$$\|\boldsymbol{u}_t + \epsilon \hat{\boldsymbol{u}}\|^2 = \|\boldsymbol{u}_t\|^2 + \epsilon \int_\Omega \boldsymbol{m}_t \cdot \hat{\boldsymbol{u}} \mathrm{d}V(\boldsymbol{x}) + \mathcal{O}(\epsilon^2)$$

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Equations become

$$\begin{aligned}
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Elimination

For any choice of norm, we can eliminate $\boldsymbol{P}$ and $\boldsymbol{Q}$ to get an equation purely in terms of $\boldsymbol{u}$

$$\frac{\partial}{\partial t}\boldsymbol{m} + \mathrm{ad}^*_{\boldsymbol{u}}\,\boldsymbol{m} = \frac{\partial}{\partial t}\boldsymbol{m} + \nabla\cdot(\boldsymbol{u}\boldsymbol{m}) + (\nabla\boldsymbol{u})^T\boldsymbol{m} = 0$$

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In 1-dimension this becomes (Camassa and Holm, 1993)

$$m_t + \frac{\partial m}{\partial x}u + 2\frac{\partial u}{\partial x}m = 0, \qquad m = u - \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

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$$\mathrm{d}\mu_A(\boldsymbol{x}) = \int_S \hat{\mu}_A(s)\delta(\boldsymbol{x} - \boldsymbol{Q}^A(s))\,\mathrm{d}s\,\mathrm{d}V(\boldsymbol{x})$$

$$\mathrm{d}\mu_B(\boldsymbol{x}) = \int_S \hat{\mu}_B(s)\delta(\boldsymbol{x} - \boldsymbol{Q}^B(s))\,\mathrm{d}s\,\mathrm{d}V(\boldsymbol{x})$$

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Penalty functional is

$$f[\boldsymbol{Q}_1] = \int_\Omega \int_\Omega K(\boldsymbol{x}, \boldsymbol{y})\mathrm{d}\nu(\boldsymbol{x})\mathrm{d}\nu(\boldsymbol{y}), \quad \mathrm{d}\nu = \mathrm{d}\mu_B - \mathrm{d}\mu_1$$

Momentum

- Geodesic is determined by initial conditions for shape coordinates Q and conjugate momenta P
- Given a reference shape, have an isomorphism between any topologically equivalent shape and initial conditions for P
- P is in a linear space so we can apply linear statistical techniques to this representation

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- We have relaxed the endpoint condition by requiring that embeddings are equivalent rather than setting boundary condition for each parameter value
- The optimal solution has momentum which is normal to the shape (see Miller, Trouvé and Younes (2003))

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This is closely linked to the Clebsch representation of fluid dynamics, and the Kelvin circulation theorem

Particle-mesh discretisation

Particle-mesh method

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- Lagrangian **particles** moving with the flow represent the movement of the embedded curve/surface

Particle-mesh method

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- Velocity represented on a fixed **mesh**
- Closely related to Hamiltonian particle-mesh method (HPM) for shallow-water equations (Frank, Gottwald, Reich)

Interpolation

Given a set of velocities $\{\boldsymbol{u}_k\}_{k=1}^{n_g}$ stored at grid points $\{\boldsymbol{x}_k\}_{k=1}^{n_g}$ define a velocity field everywhere on Ω

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Now restrict to a finite set of points $\{\boldsymbol{Q}_\beta\}_{\beta=1}^{n_p}$

$$\dot{\boldsymbol{Q}}_\beta = \sum_k \boldsymbol{u}_k \psi_k(\boldsymbol{Q}_\beta)$$

Particle-mesh equations

We adopt a **geometric approach** by trying to exactly optimise a discretised functional

would be momentum map if action

Particle-mesh equations

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$$\min_{\boldsymbol{u},\boldsymbol{Q},\boldsymbol{P}}\left(\int_0^1\left(\|\boldsymbol{u}_t\|^2+\int_S \boldsymbol{P}_t(s)\cdot\left(\frac{\partial}{\partial t}\boldsymbol{Q}_t(s)-\int_{\boldsymbol{\Omega}}\boldsymbol{u}_t(\boldsymbol{x})\delta(\boldsymbol{x}-\boldsymbol{Q}_t(s))\mathrm{d}V(\boldsymbol{x})\right)\mathrm{d}s\right)\mathrm{d}t+\frac{1}{\sigma^2}f[\boldsymbol{Q}_1]\right)$$

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We adopt a **geometric approach** by trying to exactly optimise a discretised functional

$$\min_{\boldsymbol{u},\boldsymbol{Q},\boldsymbol{P}} \left(\int_0^1 \left(\|\boldsymbol{u}\|_g^2 + \sum_\beta \boldsymbol{P}_\beta \cdot \left(\dot{\boldsymbol{Q}}_\beta - \sum_k \boldsymbol{u}_k \psi_k(\boldsymbol{Q}_\beta) \right) \right) \mathrm{d}t + \frac{1}{\sigma^2} \hat{f}(\boldsymbol{Q}(1)) \right)$$

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Semi-discrete equations are

$$\begin{aligned} \dot{\boldsymbol{Q}}_\beta &= \textstyle\sum_k \boldsymbol{u}_k \psi_k(\boldsymbol{Q}_\beta) \\ \dot{\boldsymbol{P}}_\beta &= -\boldsymbol{P}_\beta \cdot \textstyle\sum_k \boldsymbol{u}_k \nabla \psi_k(\boldsymbol{Q}_\beta) \\ \frac{\partial}{\partial \boldsymbol{u}_k} \|\boldsymbol{u}\|_g^2 &= \textstyle\sum_\beta \boldsymbol{P}_\beta \psi_k(\boldsymbol{Q}_\beta) \end{aligned}$$

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$$\begin{aligned} \frac{\partial}{\partial t} \boldsymbol{Q}_t(s) &= \boldsymbol{u}_t(\boldsymbol{Q}_t(s)) \\ \frac{\partial}{\partial t} \boldsymbol{P}_t(s) &= -(\nabla \boldsymbol{u}(\boldsymbol{Q}_t(s)))^T \cdot \boldsymbol{P}_t(s) \\ \boldsymbol{m}_t &= \textstyle\int_S \boldsymbol{P}_t(s) \delta(\boldsymbol{x} - \boldsymbol{P}_t(s)) \mathrm{d}s \end{aligned}$$

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Discrete matching condition

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Use a particle-mesh approach

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$$\mu_k^1 = \sum_\beta \hat{\mu}_\beta^A \psi_k(\boldsymbol{Q}_\beta(1))$$

$$\mu_k^B = \sum_\beta \hat{\mu}_\beta^B \psi_k(\boldsymbol{Q}_\beta^B)$$

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$$\hat{f}(\boldsymbol{Q}(1)) = \sum_{kl} K_{kl}\nu_k\nu_l, \quad \nu_k = \mu_k^1 - \mu_k^B$$

Time discretisation

Discretise in time in the functional

$$\min_{\boldsymbol{u},\boldsymbol{Q},\boldsymbol{P}} \left(\Delta t \sum_{n=1}^{N} \left(\|\boldsymbol{u}^{n-1}\|_g^2 + \sum_\beta \boldsymbol{P}_\beta^{n-1} \cdot \left(\frac{\boldsymbol{Q}_\beta^n - \boldsymbol{Q}_\beta^{n-1}}{\Delta t} - \sum_k \boldsymbol{u}_k^{n-1} \psi_k(\boldsymbol{Q}_\beta^{n-1}) \right) \right) \mathrm{d}t + \frac{1}{\sigma^2} \hat{f}(\boldsymbol{Q}^N) \right)$$

Get symplectic Euler discretisation

$$\begin{aligned}
\boldsymbol{Q}_\beta^n &= \boldsymbol{Q}_\beta^{n-1} + \Delta t \sum_k \boldsymbol{u}_k \psi_k(\boldsymbol{Q}_\beta^{n-1}) \\
\boldsymbol{P}_\beta^n &= \boldsymbol{P}_\beta^{n-1} - \Delta t \boldsymbol{P}_\beta^n \cdot \sum_k \boldsymbol{u}_k \nabla \psi_k(\boldsymbol{Q}_\beta^{n-1}) \\
\frac{\partial}{\partial \boldsymbol{u}_k} \|\boldsymbol{u}^{n-1}\|_g^2 &= \sum_\beta \boldsymbol{P}_\beta^{n-1} \psi_k(\boldsymbol{Q}_\beta^{n-1})
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Discretise in time in the functional

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Can get higher-order methods by discretising the dynamical equation with a RK method; get a symplectic PRK method

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$$\frac{\partial}{\partial t}\boldsymbol{Q}(s,t) = \sum_k \boldsymbol{u}_k \psi_k(\boldsymbol{Q}(s,t))$$

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$$\frac{\partial}{\partial t}\frac{\partial \boldsymbol{g}(\boldsymbol{x},t)}{\partial \boldsymbol{x}} = \sum_k \boldsymbol{u}_k \nabla\psi_k(\boldsymbol{g}(\boldsymbol{x},t)) \cdot \frac{\partial \boldsymbol{g}(\boldsymbol{x},t)}{\partial \boldsymbol{x}}$$

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$$\frac{\partial}{\partial t} J_\beta = \sum_k \boldsymbol{u}_k \nabla \psi_k(\boldsymbol{Q}_\beta) \cdot J_\beta, \quad J_\beta = \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}(\boldsymbol{Q}_\beta(t), t)$$

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This gives a simple way of computing Jacobi information

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This property is preserved by any variational integrator

This means that, if the momentum starts normal to the shape it will stay normal to the shape in the numerical solution

Solution methods

Three different approaches

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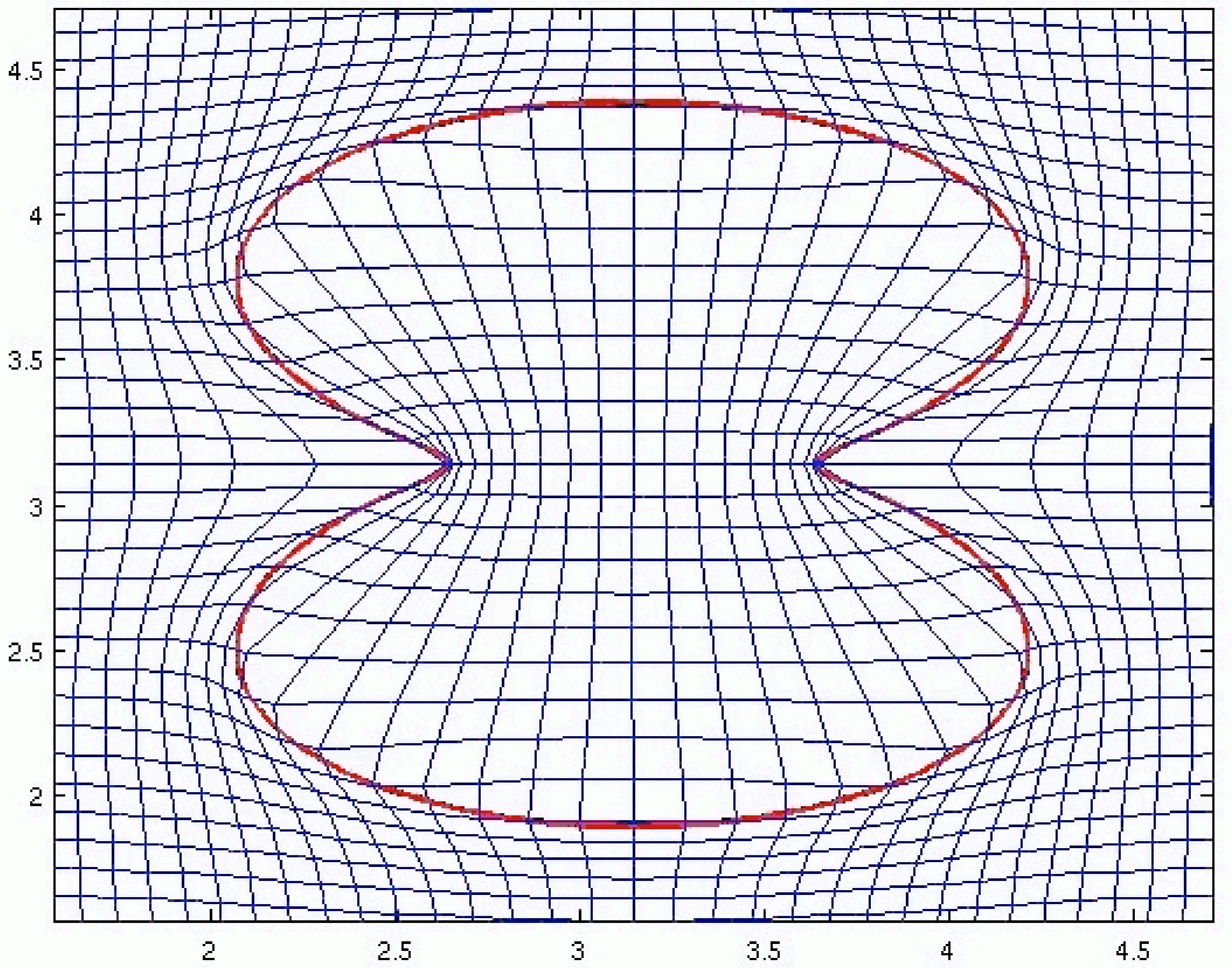
Extremise the discrete functional directly

Some movies

What does the shortest path between two shapes look like?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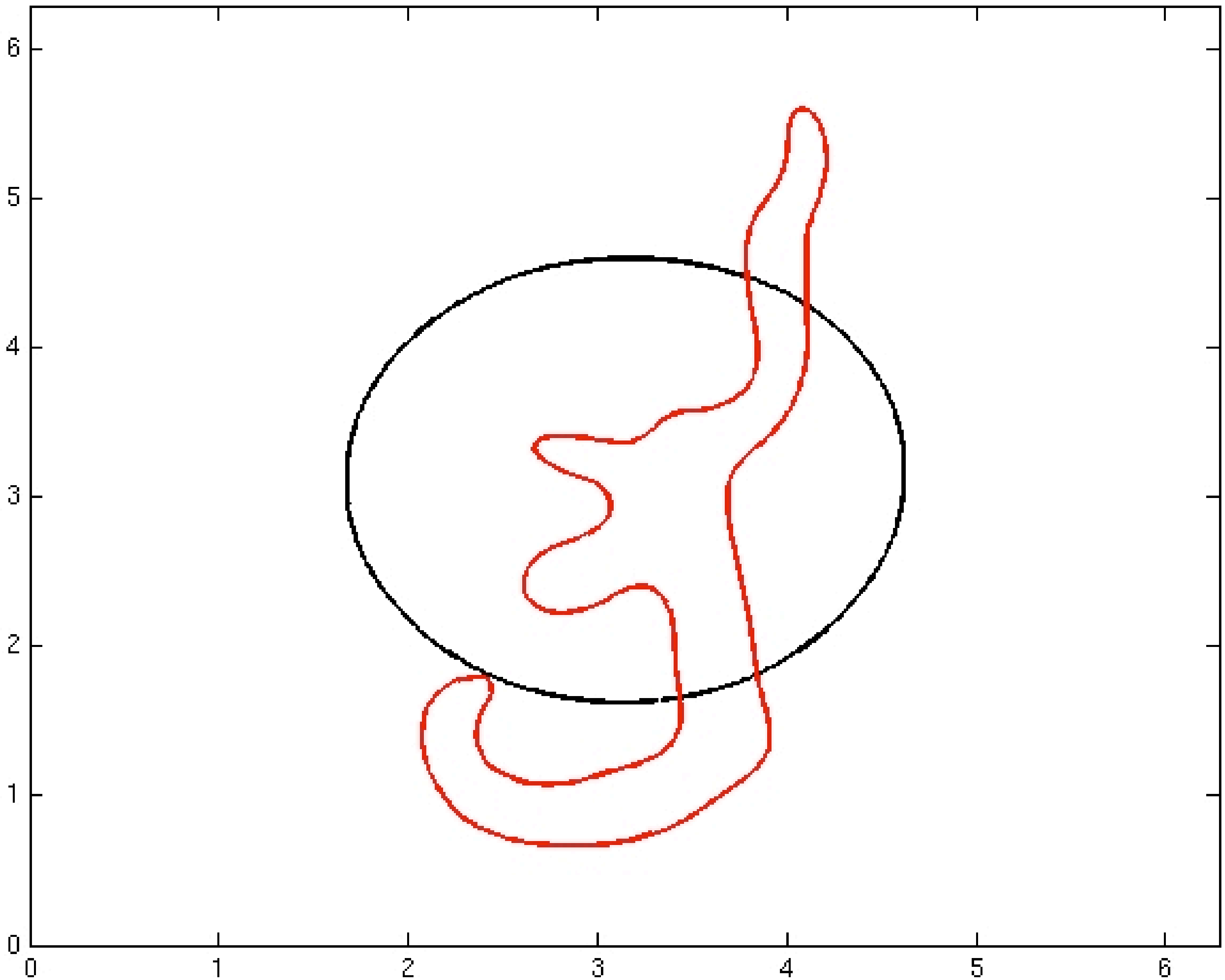

6
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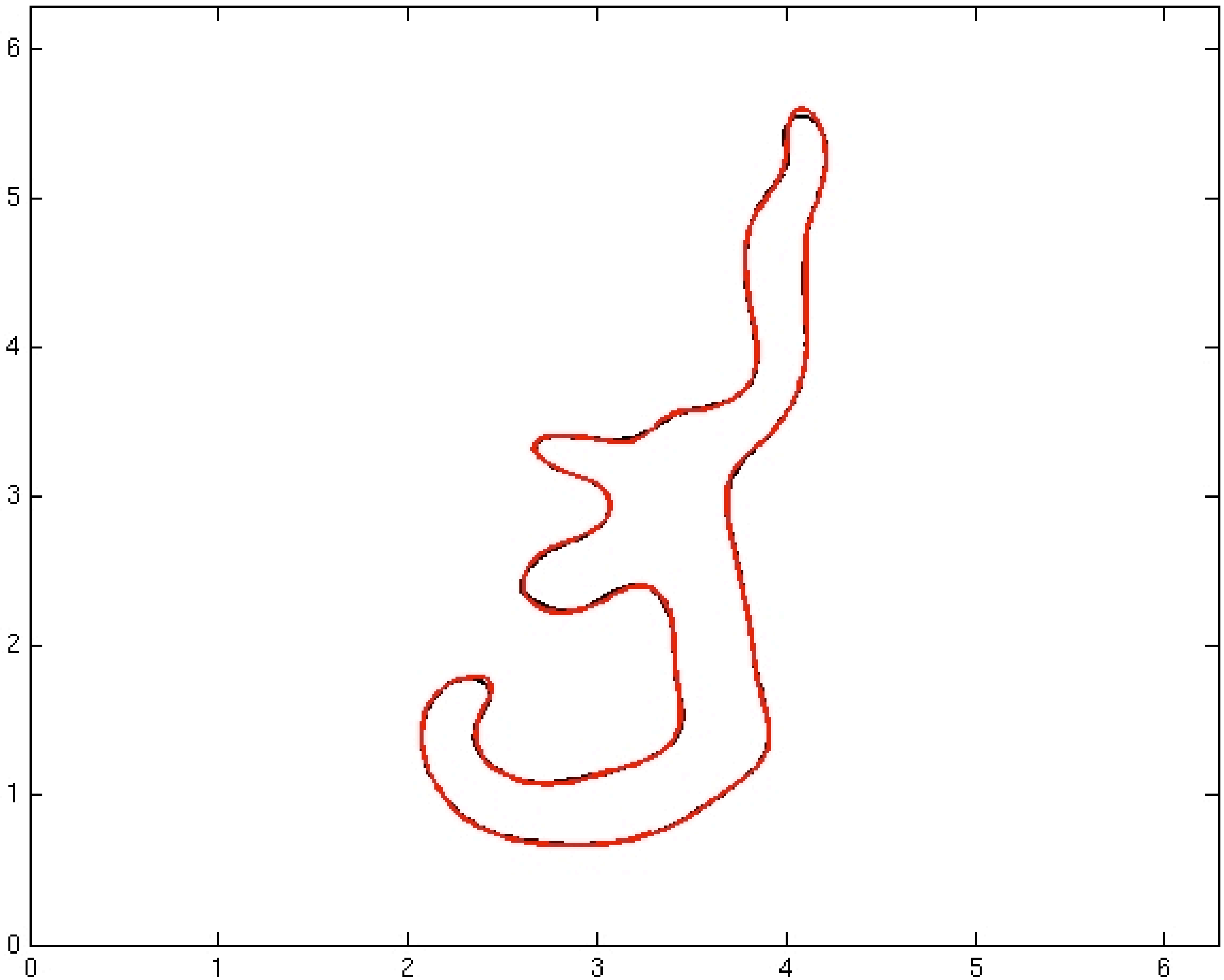

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Summary

- Variational shape matching is useful in a wide variety of applications: analysis of shape datasets and shape optimisation
- Necessary when large deformations are needed
- Shapes are embedded in a flow which follows geodesics in diffeomorphism group
- Particle-mesh method gives simple discretisation with geometric properties for matching curves/surfaces

Outlook

- Collaborations with engineers at ICL and elsewhere
- Parallel algorithm for matching complex shapes
- Developing statistical analysis of shapes with Sofia Olhede (Maths, ICL→UCL)

The End