Weak Solutions of Hydrodynamic Equations

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Outline:

1. Hydrodynamic Equations

2. Weak solutions

2. Onsager Conjecture

3. Outlook

Euler Eqns

Eulerian

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = 0, \\ -\Delta p = \nabla \cdot (u \cdot \nabla u) \end{cases}$$

$\nabla \cdot u = 0$ = invariant constraint of incompressibility

Lagrangian

$$\begin{cases} a \mapsto X(a,t), \quad X(a,0) = a, \\ \partial_t^2 X + (\nabla_x p)(X,t) = 0, \\ -\Delta_x p = \\ \nabla_x \cdot \left((\partial_t X \circ X^{-1}) \cdot \nabla_x (\partial_t X \circ X^{-1}) \right) \end{cases}$$

$\det(\nabla_a X) = 1$ invariant constraint of incompressibility

Back-to-Labels $\begin{cases} \partial_t A + u \cdot \nabla A = 0, \quad A(x,0) = x, \\ u = \mathbb{P}\left((\nabla A)^* u_0(A)\right) \end{cases}$

Theorem 1 $u_0 \in C^s$, $s > 1$, $\nabla \cdot u_0 = 0$, $\nabla \times u_0 \in L^p$, $1 < p < \infty$. $\exists T > 0$, $A, u \in L^\infty([0,T], C^s)$.

Filtered $\begin{cases} \partial_t A + u \cdot \nabla A = 0, \quad A(x,0) = x, \\ u = J_\alpha \mathbb{P}\left((\nabla A)^* u_0(A)\right) \end{cases}$

$$\omega = \nabla \times u.$$

Vorticity evolution

$$\partial_t \omega + u \cdot \nabla \omega = \omega \cdot \nabla u$$

Regularity = smooth solution on time interval $[0, T]$.

BKM: Sufficient for regularity:

$$\int_0^T \|\omega\|_{L^\infty(dx)} dt < \infty$$

Amplification factor of arbitrary tracers

$$\int_0^T \|\nabla u\|_{L^\infty(dx)} dt < \infty$$

Surface Quasi-Geostrophic equation = QG

QGκs

$$\begin{cases} \partial_t\theta + u\cdot\nabla\theta + \kappa\Lambda^s\theta = 0, \\ u = R^{\perp}\theta \end{cases}$$

with $\Lambda = (-\Delta)^{1/2}$ the Zygmund operator and $R = (\nabla)\Lambda^{-1}$ Riesz operators, $\kappa \geq 0$, $s > 0$, $n = 2$.

Instead of vortex lines: iso-θ lines.

"Vorticity" evolution

$$\partial_t\left(\nabla^{\perp}\theta\right) + u\cdot\nabla\left(\nabla^{\perp}\theta\right) + \kappa\Lambda^s\left(\nabla^{\perp}\theta\right) = \left(\nabla^{\perp}\theta\right)\cdot\nabla u$$

Same BKM.

Weak Solutions for Nonlinear Eqns

No such thing, in general. Typical methodology: good approximation, integration by parts, weak continuity.

Continuity does not imply weak continuity. Simple example: $u \mapsto \|u\|^2$ with $u \in L^2([0, 2\pi])$. Take $u_n(x) = \sin(nx)$ for $x \in [0, 2\pi]$: converges weakly to zero, norms bounded away from zero.

Weak solutions: solve the equation in a very large space. (Not enough to have equation solved almost everywhere. Not enough to have the nonlinear term well-defined.)

Weak Solutions

(E)

$$u \in C_w[0, T; L^2_{loc,u}]$$

$$\int u(t) \cdot \varphi dx - \int u_0 \cdot \varphi dx = \int_0^t \int \text{Trace}\left[(u \otimes u)(\nabla \varphi)\right] dx ds$$

(QGκs)

$$\theta \in C_w[0, T; L^2_{loc,u}] \cap L^2[0, T; H^{\frac{s}{2}}]$$

$$u = R^{\perp}\theta,$$

$$\int \theta(t) \cdot \varphi dx - \int \theta_0 \cdot \varphi dx = \int_0^t \int \left[(\theta u) \cdot \nabla \varphi - \kappa \theta \Lambda^s \varphi\right] dx ds$$

Desirable: locally square integrable, evolutionary weak solutions obtained as limits of good approximate solutions u^ϵ. Needed: weak continuity of approximations in L^2. (Weak continuity is stronger than strong continuity).

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^3} \text{Trace}\left[(u^\epsilon \otimes u^\epsilon)(\nabla\varphi)\right] dx = \int_{\mathbb{R}^3} \text{Trace}\left[(u \otimes u)(\nabla\varphi)\right] dx$$

for all smooth divergence–free φ, when

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^3} (u^\epsilon \cdot \varphi)\, dx = \int_{\mathbb{R}^3} (u \cdot \varphi)\, dx$$

holds for all φ. Known for surface QG (Resnick, '95), not for Euler. The reason is structural not dimensional.

Weak solutions for QG

$$\partial_t \theta + u \cdot \nabla \theta = 0, \quad u = R^{\perp}\theta.$$

For periodic $\theta = \sum_{j \in \mathbb{Z}^2} \widehat{\theta}(j) e^{i(j \cdot x)}$ infinite ODE:

$$\frac{d}{dt}\widehat{\theta}(l) = \sum_{j+k=l} \left(j^{\perp} \cdot k\right) |j|^{-1} \widehat{\theta}(j)\widehat{\theta}(k)$$

Using the antisymmetry:

$$\frac{d}{dt}\widehat{\theta}(l) = \sum_{j+k=l} \gamma^{l}_{j,k} \widehat{\theta}(j)\widehat{\theta}(k)$$
$$\gamma^{l}_{j,k} = \frac{1}{2}(j^{\perp} \cdot k) \left(\frac{1}{|j|} - \frac{1}{|k|}\right)$$

$$\left|\gamma^{l}_{j,k}\right| \le \frac{|l|^2}{\max\{|j|\,,|k|\}}$$

Consequently

$$\|(-\Delta)^{-1}\left[B(\theta_1,\theta_1) - B(\theta_2,\theta_2)\right]\|_w \le$$
$$C\left\{\|\theta_1-\theta_2\|_w\left(1+\log_+\|\theta_1-\theta_2\|_w\right)\right\}\left(\|\theta_1\|_{L^2}+\|\theta_2\|_{L^2}\right)$$

with $\|f\|_w = \sup_{j\in\mathbb{Z}^2}\left|\widehat{f}(j)\right|$. Quasi-Lipschitz, with loss of two derivatives. Loss of derivatives does not impede existence theory, but prevents a proof of uniqueness.

Dissipative QG

Regularity and uniqueness: with critical dissipation ($s = 1$): Cordoba-Wu-C (small data), Kiselev-Nazarov-Volberg and Caffarelli-Vasseur, all data. For Burgers, there is blow up at $s = 1$! (Kiselev, Nazarov, Shterenberg).

For supercritical dissipation ($0 < s < 1$) there is a gap in passing from L^∞ to C^δ, no gap in passing from L^2 to L^∞, nor from C^δ to C^∞ if $\delta > 1 - s$ (Wu-C).

(QGκs)

$$\partial_t \theta + u \cdot \nabla \theta + \kappa \Lambda^s \theta = 0$$

with $u = R^\perp \theta$, $\kappa > 0$, $0 < s$.

The theorems:

Theorem 2 *Let* $\theta_0 \in L^2(\mathbb{R}^n)$ *and let* θ *be a corresponding Leray-Hopf weak solution of QG*κs*:*

$$\theta \in L^\infty([0,\infty), L^2(\mathbb{R}^n)) \cap L^2([0,\infty); H^{s/2}(\mathbb{R}^n)).$$

Then, for any $t > 0$*,*

$$\sup_{\mathbb{R}^n} |\theta(x,t)| \le C \, \frac{\|\theta_0\|_{L^2}}{t^{\frac{n}{2s}}}.$$

As a consequence,

$$\|u(\cdot,t)\|_{BMO(\mathbb{R}^n)} \le C \, \frac{\|\theta_0\|_{L^2}}{t^{\frac{n}{2s}}}$$

for any $t > 0$*.*

Idea of proof: (De Giorgi)

$$\theta_k = (\theta - C_k)_+, \quad C_k = M(1 - 2^{-k}),$$

M to be determined. Fix any $t_0 > 0$. Let $t_k = t_0(1 - 2^{-k})$. Consider the quantity U_k,

$$U_k = \sup_{t \geq t_k} \int \theta_k^2(x,t)\,dx + 2\int_{t_k}^{\infty} \int |\Lambda^{\frac{s}{2}} \theta_k|^2 dx\,dt,$$

$$V_k = \frac{2^{\gamma k}\,U_k}{t_0^{2/(q-2)}\,M^2\,2^{(-\gamma q-2)/(q-2)}} \quad \text{with} \quad \gamma = \frac{2(q-1)}{q-2} > 0$$

with $q = q(s) > 2$. Using Gagliardo-Nirenberg and the Cordoba-Cordoba-Wu-C dissipativity

$$V_k \leq V_{k-1}^{\frac{q}{2}}$$

Choosing M large enough, $V_0 < 1$ and then $V_k \to 0$. This implies boundedness in terms of the initial L^2 norm, and scaling invariance implies the bound.

Theorem 3 *Let θ be a solution of QG$_{\kappa s}$ satisfying*

$$\theta \in L^{\infty}([0,\infty), L^2(\mathbb{R}^n)) \cap L^2([0,\infty); H^{s/2}(\mathbb{R}^n)).$$

Let $t_0 > 0$. Assume that

$$\theta \in L^{\infty}(\mathbb{R}^n \times [t_0,\infty))$$

and

$$u \in L^{\infty}([t_0,\infty); C^{1-s}(\mathbb{R}^n)).$$

Then θ is in $C^{\delta}(\mathbb{R}^n \times [t_0,\infty))$ for some $\delta > 0$.

Proof highly technical:

Ideas: 1) Sylvestre-Caffarelli harmonic extension of fractional Laplacian. 2) De Giorgi quantitative isoperimetric inequality. 3) Caffarelli-Vasseur diminishing oscillation lemma. 4) De Giorgi blowup methodology.

Theorem 4 *Let θ be a Leray-Hopf weak solution of QGκs,*

$$\theta \in L^\infty([0,\infty); L^2(\mathbb{R}^2)) \cap L^2([0,\infty); H^{s/2}(\mathbb{R}^2)).$$

Let $\delta > 1 - s$ and let $0 < t_0 < t < \infty$. If

$$\theta \in L^\infty([t_0, t]; C^\delta(\mathbb{R}^2)),$$

then

$$\theta \in C^\infty((t_0, t] \times \mathbb{R}^2).$$

Idea of proof: Littlewood-Paley energy estimates

Littlewood-Paley decomposition.

$$u = \sum_{j=-1}^{\infty} \Delta_j u$$

$$\text{supp}\, \mathcal{F}(\Delta_j(u)) \subset 2^j[\tfrac{1}{2}, \tfrac{5}{4}]$$
$$\Delta_j \Delta_k \neq 0 \Rightarrow |j-k| \leq 1,$$
$$\left(\Delta_j + \Delta_{j+1} + \Delta_{j+2}\right) \Delta_{j+1} = \Delta_{j+1}$$
$$\Delta_j \left(S_{k-2}(u)\Delta_k(v)\right) \neq 0 \Rightarrow k \in [j-2, j+2]$$
$$S_k(u) = \textstyle\sum_{q=-1}^{k} \Delta_q.$$

$$\Delta_j = \Psi_j(D) = \Psi_0(2^{-j}D), \quad \Delta_{-1}u = \Phi_{-1}(D)u.$$

Φ_{-1}: radial, nonincreasing, C^∞

$$\begin{cases} \Phi_{-1} = 1,\ 0 \le r \le a \\ \Phi_{-1} = 0,\ r \ge b \\ 0 < a < b < 1 \end{cases}$$

$$\Psi_0(r) = \Phi_{-1}(r/2) - \Phi_{-1}(r), \quad \Psi_j(r) = \Psi_0(2^{-j}r).$$

$$(\Psi(D)u)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x\cdot\xi)}\Psi(\xi)\widehat{u}(\xi)d\xi$$

$\widehat{u}(\xi) = \int_{\mathbb{R}^n} e^{-i(x\cdot\xi)}u(x)dx.$ $a < b < \frac{4}{3}a$ (e.g. $a = 1/2,\ b = 5/8$)

Inhomogeneous Besov space

$$\|u\|_{B^s_{p,q}} = \left\| \left\{ 2^{sj} \|u_j\|_{L^p} \right\}_j \right\|_{\ell^q(\mathbb{N})}.$$

The space $B^s_{p,c(\mathbb{N})}$ is the subspace of $B^s_{p,\infty}$ formed with functions such that

$$\lim_{j\to\infty} 2^{sj} \|u_j\|_{L^p} = 0.$$

Bernstein Inequalities

$$\|\Delta_j u\|_{L^b} \leq 2^{jd(\frac{1}{a}-\frac{1}{b})} \|\Delta_j u\|_{L^a} \quad \text{for } b \geq a \geq 1.$$

Besov Space Embeddings

If $b \geq a \geq 1$

$$B^s_{a,r} \subset B_{b,r}^{s-d\left(\frac{1}{a}-\frac{1}{b}\right)},$$

$$B^0_{a,2} \subset L^a, \text{ for } a \geq 2.$$

Euler weak solutions: main difficulty

$$B(u,v) = \mathbb{P}(u\cdot\nabla v) = \Lambda\mathbb{H}(u\otimes v)$$

where

$$[\mathbb{H}(u\otimes v)]_i = R_j(u_j v_i) + R_i(R_k R_l(u_k v_l)),$$

$\mathbb{P}$ is the Leray-Hodge projector, $\Lambda = (-\Delta)^{\frac{1}{2}}$ is the Zygmund operator and $R_k = \partial_k \Lambda^{-1}$ are Riesz transforms.

$$\Delta_q(B(u,v)) = C_q(u,v) + I_q(u,v)$$

$$C_q(u,v) = \sum_{p\ge q-2,\ |p-p'|\le 2} \Delta_q(\Lambda\mathbb{H}(\Delta_p u, \Delta_{p'} v))$$

$$I_q(u,v) = \sum_{j=-2}^{2} \Big[\Delta_q \Lambda\mathbb{H}(S_{q+j-2}u, \Delta_{q+j}v) + \Delta_q \Lambda\mathbb{H}(S_{q+j-2}v, \Delta_{q+j}u)\,\Big]$$

For L^2 weak solutions it would be desirable to have a bound of the type

$$\|\Lambda^{-M}\left(B(u_1,u_1) - B(u_2,u_2)\right)\|_w \le C\|u_1 - u_2\|_w^a \left[\|u_1\|_{L^2} + \|u_2\|_{L^2}\|\right]^{2-a}$$

with $a > 0$ and $\|f\|_w$ a weak enough norm so that weak convergence in L^2 implies convergence in the w norm, (e.g $B^{-s}_{\infty,\infty},\ s > 3/2$) and M as large as needed. This is true for $I(u,v)$ but not for $C(u,v)$. For weak solutions in $B^{\frac{1}{3}}_{3,q}$, $C(u,v)$ is good and $I(u,v)$ is bad.

Littlewood-Paley Energy Flux

$$\Pi_N := \int_{\mathbb{R}^3} \text{Trace } [S_N(u \otimes u) \nabla S_N(u)]\, dx.$$

This is the (formal) time derivative

$$\Pi_N = \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |S_N(u(t))|^2\, dx$$

of the energy contained in $S_N(u)$ when u solves the Euler equation.

Onsager Conjecture

$$u \in C^s,\ s > \frac{1}{3} \Leftrightarrow \frac{dE}{dt} = 0$$

Eyink, C-E-Titi, Duchon-Robert $'' \Rightarrow ''$.

Theorem 5 *(Cheskidov-C-Friedlander-Shvydkoy) Weak solutions*

$$u \in L^3([0,T], B^{1/3}_{3,c(\mathbb{N})})$$

of the Euler equations conserve energy. There exist functions in $B^{1/3}_{3,\infty}$ *that are divergence-free and obey* $\liminf_{N\to\infty} |\Pi_N| > 0$.

Similar results for helicity. See also work of Chae. In two dimensions, infinite time, damped and driven NS: absence of anomalous dissipation of enstrophy. (Ramos-C.)

Idea of Proof

Let

$$K(j) = \begin{cases} 2^{\frac{2j}{3}}, & j \leq 0; \\ 2^{-\frac{4j}{3}}, & j > 0, \end{cases}$$

and

$$d_j = 2^{j/3} \|\Delta_j u\|_3, \text{ for } j \geq -1, \quad d_j = 0 \text{ for } j < -1$$
$$d^2 = \{d_j^2\}_j$$

Proposition 6 *If $u \in L^2$ then*

$$|\Pi_N| \leq C(K * d^2)^{3/2}(N).$$

Consequently

$$\limsup_{N\to\infty} |\Pi_N| \leq \limsup_{N\to\infty} d_N^3.$$

Indeed, following C-E-T:

$$S_N(u \otimes u) - S_N u \otimes S_N u = r_N(u,u) - (u - S_N) \otimes (u - S_N),$$

with

$$r_N(u,u) = \int_{\mathbb{R}^3} h_N(y)(u(x-y) - u(x)) \otimes (u(x-y) - u(x))dy,$$

and $\widehat{h_N}(\xi) = \Phi_{-1}(2^{-N}\xi)$. Substituting in definition of Π_N:

$$\Pi_N = \int_{\mathbb{R}^3} \text{Trace}[r_N(u,u) \cdot \nabla S_N u]dx \\ - \int_{\mathbb{R}^3} \text{Trace}[(u - S_N) \otimes (u - S_N) \cdot \nabla S_N u]dx.$$

We bound the first term by

$$\|r_N(u,u)\|_{3/2} \|\nabla S_N u\|_3,$$

and use

$$\|r_N(u,u)\|_{3/2} \le \int_{\mathbb{R}^3} |h_N(y)|\, \|u(\cdot - y) - u(\cdot)\|_3^2 dy$$

Besov embedding and Bernstein inequalities:

$$\begin{aligned}\|u(\cdot - y) - u(\cdot)\|_3^2 &\le C \sum_{j\le N} |y|^2 2^{2j} \|\Delta_j u\|_3^2 + C \sum_{j>N} \|\Delta_j u\|_3^2 \\ &= C 2^{4N/3} |y|^2 \sum_{j\le N} 2^{-4(N-j)/3} d_j^2 \\ &+ C 2^{-2N/3} \sum_{j>N} 2^{2(N-j)/3} d_j^2 \\ &\le C(2^{4N/3}|y|^2 + 2^{-2N/3})(K * d^2)(N).\end{aligned}$$

Collecting and integrating, we find

$$\left|\int_{\mathbb{R}^3} \text{Trace}[r_N(u,u)\cdot\nabla S_N u]dx\right| \le$$

$$C(K*d^2)(N)\left(\int_{\mathbb{R}^3} |h_N(y)|\, 2^{4N/3}|y|^2 dy + 2^{-2N/3}\right)\left[\sum_{j\le N} 2^{2j}\|\Delta_j u\|_3^2\right]^{1/2}$$

$$\le C(K*d^2)(N)2^{-2N/3}\left[\sum_{j\le N} 2^{4j/3} d_j^2\right]^{1/2}$$

$$\le C(K*d^2)^{3/2}(N)$$

Similarly,

$$\int_{\mathbb{R}^3} \text{Trace}[(u - S_N) \otimes (u - S_N) \cdot \nabla S_N u] dx$$
$$\leq \|u - S_N u\|_3^2 \|\nabla S_N u\|_3$$
$$\leq C \left(\sum_{j>N} \|\Delta_j u\|_3^2 \right) \left(\sum_{j \leq N} 2^{2j} \|\Delta_j u\|_3^2 \right)^{1/2}$$
$$\leq C (K * d^2)^{3/2} (N).$$

Outlook

- Weak solutions in right spaces require special nonlinear structure.

- Supercritical QG problem open.

- Anomalous dissipation: close to optimal spaces.