On a microscopic model of viscous friction

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Abstract. We study a microscopic model of a heavy object, subject to an external force E, and immersed in an infinitely extended perfect gas. We assume the gas to be described by the mean-field approximation and interacting elastically with the body. In this set up, denoting by V_0 the initial velocity of the body and being V_∞ its asymptotic velocity, if $|V_0 - V_\infty|$ is small enough, then $|V(t) - V_\infty| \approx C\,t^{-d-2}$ for t large, where d is the dimension of the space.

The reason for the power law approach to the limit velocity instead of the exponential one (typical in viscous friction problems), is due to the memory effects induced by re-collisions.

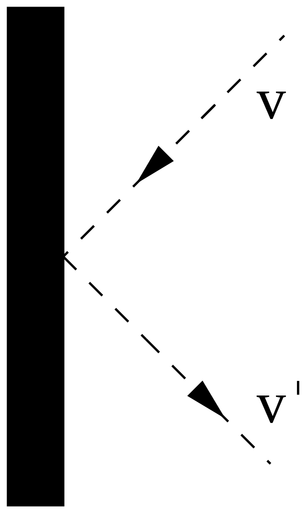
Three-dimensional case with null external force

The evolution equation for the gas outside the body is:

$$(\partial_t + v \cdot \nabla_x) f(x, v; t) = 0.$$

Let us take for simplicity the body to be a disk orthogonal to the x-axis and moving along the same axis (but the same considerations hold for a general convex body).

On the right and left face of the disk we have reflecting boundary conditions:



$$
\begin{aligned}
v'_x &= 2V(t) - v_x \\
v'_\perp &= v_\perp,
\end{aligned}
$$

where v_x, $v_\perp$ are the components of the velocity of a gas particle along the x-axis and the orthogonal plane respectively.

We impose also the continuity of $f(x, v; t)$ along the trajectories of gas particles (in particular, before and after a collision with the disk), and we take as initial distribution:

$$
f(x, v; 0) = \rho \left(\frac{\beta}{\pi}\right)^{3/2} e^{-\beta v^2}. \tag{1}
$$

The evolution equation for the disk is:

$$\dot{X}(t) = V(t), \qquad \dot{V}(t) = -F(t), \qquad (2)$$

$$X(0) = 0, \qquad V(0) = V_0,$$

where

$$\begin{aligned} F(t) &= 2\int_{D(t)} dx \int_{v_x < V(t)} dv\,(V(t) - v_x)^2 f(x, v; t) \\ &- 2\int_{D(t)} dx \int_{v_x \geq V(t)} dv\,(V(t) - v_x)^2 f(x, v; t) \end{aligned}$$

is the action of the gas on the disk.

Neglecting recollisions between particles and disk, the friction term assumes the simplified form (using (1)):

$$\begin{aligned} F_0(V) &= 2\rho \left(\frac{\beta}{\pi}\right)^{3/2} \sigma \left[\int_{v_x < V} dv\,(V - v_x)^2 e^{-\beta v^2} \right. \\ &\left. - \int_{v_x \geq V} dv\,(V - v_x)^2 e^{-\beta v^2}\right]. \end{aligned}$$

Such $F_0(V)$ is an increasing odd function, positive and convex in the interval $(0, +\infty)$. Substituting F_0 in place of F in Eq. (2), we obtain exponential decay:

$$V_0\, e^{-C_1 t} \leq V(t) \leq V_0\, e^{-C_2 t}.$$

In the full problem, where we include recollisions, we rewrite the full friction term F as:

$$F(t) = F_0(V(t)) + r^+(t) + r^-(t)$$

where

$$r^+(t) = 2\int_{D(t)} dx \int_{v_x < V(t)} dv\, (v_x - V(t))^2 \times \left(f(x,v;t) - \rho \left(\frac{\beta}{\pi}\right)^{3/2} e^{-\beta v^2} \right) \quad (3)$$

and a similar expression for r^- ($r^\pm$ contain contributions from right and left recollisions). The result is the following:

Theorem. There exists $\gamma_0 > 0$ sufficiently small such that, for any initial velocity $V_0 = \gamma \in (0, \gamma_0)$ there exists at least one solution $(V(t), f(t))$. Moreover, any solution $(V(t), f(t))$ satisfies the following properties:

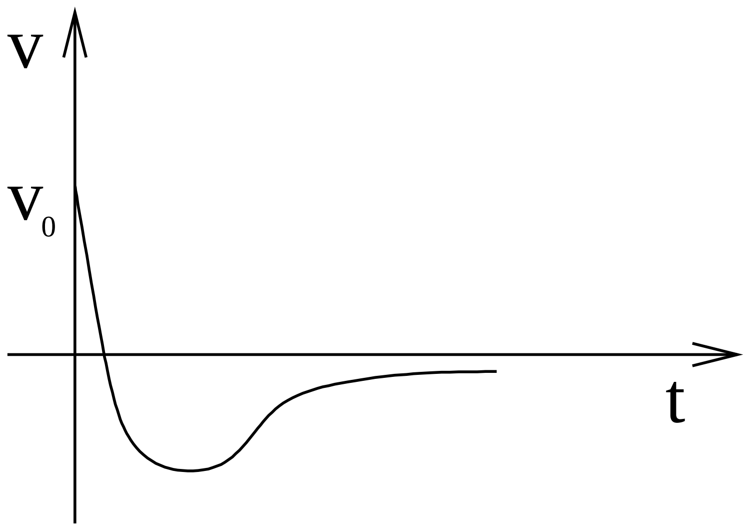
(i) for any $t \geq 0$ it is:

$$V(t) \geq \gamma\, e^{-C_1 t} - \gamma^3 \frac{A_1}{(1+t)^5}, \quad (4)$$

(*ii*) there exists a sufficiently large $\bar{t}$, depending on γ, such that for any $t \geq 0$:

$$V(t) \leq \gamma\, e^{-C_2 t} - \gamma^5 \frac{A_2}{t^5} \chi(\{t \geq \bar{t}\}) \qquad (5)$$

where $\chi(\{\ldots\})$ is the characteristic function of the set $\{\ldots\}$.



The power-law approach to the equilibrium velocity and the change of sign of $V(t)$ are due to the memory effect of recollisions.

Idea of the proof

We introduce an a.e. differentiable function with bounded derivative, $t \to W(t)$, with $W(0) = V_0$, decreasing up to a fixed time t_0 (depending on γ), and satisfying the estimates (4)-(5).

We consider then the modified problem

$$\dot{V}_W(t) = -\frac{F_0(W(t))}{W(t)} V_W(t) - r_W^+(t) - r_W^-(t),$$

where $r_W^{\pm}$ are computed according to (3), for a disk moving with assigned velocity $W(t)$. Thus we find a new velocity, V_W. The solution to our problem is the fixed point of the map $W \to V_W$.

Hence we have to prove that V_W enjoys the same properties of W.

The main efforts are the estimates of the recollision terms $r_W^{\pm}$, in particular the lower bound of Lemma 3:

Lemma 1. For any $t \geq 0$ and γ sufficiently small,

$$|r_W^{+}(t)| \leq C\frac{\gamma^{(5+\frac{1}{4})}A_1^3}{(1+t)^5}\,\chi(\{t > t_0\}).$$

Lemma 2. For any $t \geq 0$ and γ sufficiently small,

$$r_W^{-}(t) \leq C\frac{(\gamma + A_1\gamma^3)^3}{(1+t)^5}.$$

Lemma 3. Suppose γ sufficiently small. Then, for $t \geq t_0$ we have:

$$r_W^{-}(t) \geq C\frac{\gamma^5}{t^5}.$$

Along the same lines it can be treated the case in which the disk is also subject to an external constant force, reaching the asymptotic velocity with the same power-law, and the case in which the external force is of elastic type, arriving at the power-law decay for the position of the disk.

A generalization of the shape of the body has been also performed, including any body having a convex shape.

Finally, it has been analysed the case of a different interaction between the body and the gas particles, considering diffusive boundary conditions.

References

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