Degenerate relative equilibria

— and the concept of criticality in hydrodynamics —

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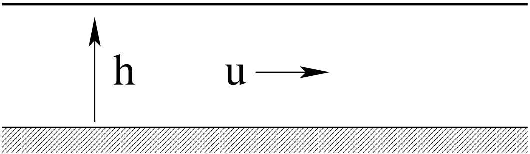
Criticality in fluid mechanics

In shallow water, a uniform flow with velocity u and depth h,



is said to be critical when $u^2 = gh$ (Froude number unity).

Another characterization: the flow is critical when the speed of a plane wave in the linearization about uniform flow equals speed of uniform flow:

$$\text{speed of plane waves} \;=\; \pm\sqrt{gh}$$

Criticality in shallow water hydrodynamics

Shallow water equations

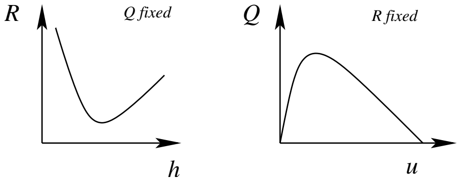
$$u_t + uu_x + gh_x = 0 \quad \text{and} \quad h_t + uh_x + hu_x = 0\,.$$

Steady solutions realize constant values of Q and R

$$R = gh + \tfrac{1}{2}u^2 \quad \text{and} \quad Q = hu\,.$$

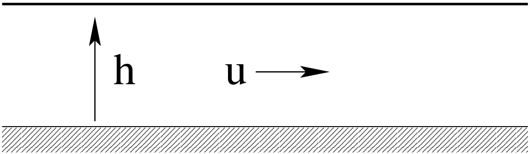
Given Q and R find values of h and u. Criticality:

- the state at which R is a minimum for fixed $Q \neq 0$
- the state at which Q is a maximum for fixed $R > 0$ ($u > 0$)



Criticality and solitary waves

In shallow water, a uniform flow with velocity u and depth h,



is said to be critical when $u^2 = gh$ (Froude number unity).

Criticality is a bifurcation point for solitary waves.

In this case, Froude number unity is a bifurcation point for the KdV solitary wave (add dispersion to SWE to see this).

Generalize criticality to non-trivial states?

Use quasi-static approximation, or consider the flow to be slowly-varying in the $x-$direction and use WKB theory

- GILL (1977) *J. Fluid Mech.*
- KILLWORTH (1992) *J. Fluid Mech.*
- JOHNSON & CLARKE (2001) *Ann. Rev. Fluid Mech.*

Define criticality to be when eigenvalues of the linearization pass through zero.

- BENJAMIN (1971) A unified theory of conjugate flows, *Phil. Trans. Royal Soc. London A*

But restricted to parallel flows (independent of x), and criticality is treated as a one-parameter problem.

Observation: uniform flows are relative equilibria

Criticality is an $n-$parameter problem with $n = \dim(\mathfrak{g})$

- (h, u) are coordinates for a Lie algebra
- (R, Q) are coordinates for a momentum map

Criticality of uniform flows corresponds to degeneracy of RE
Degenerate RE generate zero eigenvalues: a saddle-center bifurcation transverse to the group orbit

Saddle-center leads to homoclinic bifurcation (SW)

- Role of curvature of the momentum map
- Geometric phase along group
- Thom-Boardman classification of singularities

Generalize criticality: can define criticality for any flow which can be characterized as a RE !

$G-$equivariant Hamiltonian systems

Consider a Hamiltonian system with symmetry. For example, take

$$\mathbb{J}\mathbf{u}_t = \nabla H(\mathbf{u})\,, \quad \mathbf{u} \in M = \mathbb{R}^{2n+2}\,,$$

and suppose that it is equivariant with respect to an $n-$dimensional abelian Lie group G (subgroup of the Euclidean group) with Lie algebra $\mathfrak{g}$, action $\Phi_g(\mathbf{u})$ and generator

$$\xi_M(\mathbf{u}) := \frac{d}{ds}\bigg|_{s=0} \Phi_{\exp(t\xi)}(\mathbf{u})\,, \quad \xi \in \mathfrak{g}\,.$$

Suppose G is symplectic and the Hamiltonian function is G$-$invariant, etc., and momentum map

$$\mathbf{J} \;:\; M \to \mathfrak{g}^*\,.$$

Symplectic relative equilibria

Relative equilibria are solutions which travel along a group orbit at constant speed. An RE is of the form

$$\mathbf{u}(t) = \Phi_{\exp(t\xi)}(\varphi) \quad \text{for some} \quad \xi \in \mathfrak{g}\,,$$

where $\varphi : \mathfrak{g} \to M$ is a critical point of the augmented Hamiltonian

$$H_\xi(\mathbf{u}) := H(\mathbf{u}) - \langle \mathbf{J}(\mathbf{u}) - \mu, \xi \rangle\,.$$

A critical point, φ, of H_ξ is a mapping from $\mathfrak{g}$ into M. Substitution into the momentum gives

$$\mu = \mathbf{J} \circ \varphi(\xi)\,.$$

a mapping from $\mathfrak{g}$ into $\mathfrak{g}^*$.
The equation $DH_\xi = 0$ can also be interpreted as the Lagrange necessary condition for a constrained variational principle: find critical points of H restricted to level sets of the momenta.

(cf. MARSDEN (1992), MARSDEN & RATIU (1994))

Degenerate relative equilibria

A RE is non-degenerate when the second variation of H_ξ at a critical point is a non-degenerate quadratic form on the subspace consisting of vectors tangent to $\mathbf{J}^{-1}(\mu)$ and transverse to the group orbit.

Four types of degeneracy

• Singularity of the momentum map.

Assume throughout that μ is a regular value of the momentum map.

• Failure of G-Morse: the dimension of the kernel of the second variation of H_ξ is greater than the dimension of the group. Related to $\mathrm{D}\mathbf{c}(\mu)$ singular.

• Failure of G-Morse: μ fixed, but external parameters varied; example: PALMORE, Measure of degenerate relative equilibria I, *Ann. Math.* (1976).

• $\det[\mathrm{D}\mathbf{P}(\mathbf{c})] = 0$, $\mathbf{P}(\mathbf{c}) := \mathbf{J} \circ \varphi$ and $\mathbf{c}$ are coordinates for $\mathfrak{g}$.

Degenerate RE and the Jacobian

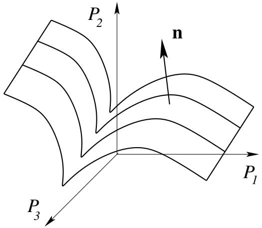
The key to the study of the nonlinear behaviour transverse to the group orbit near degenerate RE is the geometry of

$$\mathbf{P} \ : \ \mathfrak{g} \to \mathfrak{g}^* \, .$$

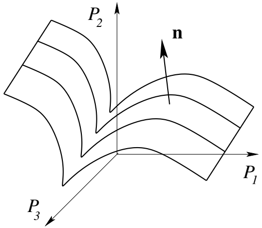
The condition

$$\det[\mathrm{D}\mathbf{P}(\mathbf{c})] = 0 \, ,$$

defines a hypersurface in $\mathfrak{g}$ with image in $\mathfrak{g}^*$



Geometry of $\det[\mathrm{D}\mathbf{P}(\mathbf{c})] = 0$



When $\mathrm{D}\mathbf{P}(\mathbf{c})$ has rank $n-1$ there exists $\mathbf{n} \in T_{\mathbf{c}}\,\mathfrak{g}$

$$[\mathrm{D}\mathbf{P}(\mathbf{c})]\mathbf{n} = 0\,.$$

The image of the hypersurface in $\mathfrak{g}^*$ can have singularities. By introducing a metric, $\mathbf{n}$ can be interpreted as a normal vector to the surface in $\mathfrak{g}^*$, at regular points on the surface.

The surface in $\mathfrak{g}^*$ is locally a barrier to the existence of RE.

Thom-Boardman singularity theory

For a mapping $P : \mathbb{X} \to \mathbb{Y}$, with $\mathbb{X}, \mathbb{Y}$ $n-$dimensional vector spaces, the subsets

$$\Sigma^k(\mathbf{P}) = \{\mathbf{c} \in \mathbb{X} \ : \ \mathrm{rank}(\mathrm{Jac}(\mathbf{c})) = n - k\,\}$$

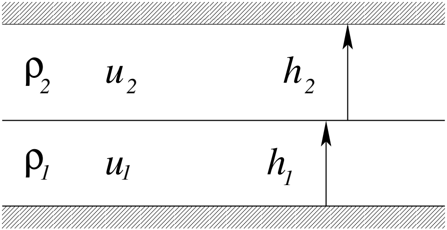
are known in singularity theory as the Thom-Boardman singularities. Restrict to the case $k = 1$. There is a hierarchy of singular sets, for example

$$\Sigma^{11}(\mathbf{P}) = \Sigma^1\left(\mathbf{P}\Big|_{\Sigma^1(\mathbf{P})}\right)$$

is the set where Jacobian of the kernel of **P** restricted to $\Sigma^1(\mathbf{P})$ drops in rank by one. The classification continues until the dimension is exhausted. The connection with degenerate RE:

- Momentum map $\mathbf{P}(\mathbf{c}) \in \Sigma^1(P) \Rightarrow$ saddle-center bifurcation
- Nonlinearity: $\mathbf{P}(\mathbf{c}) \notin \Sigma^{11}(P) \Rightarrow$ homoclinic bifurcation

Bifurcation of internal solitary waves



Taking a Boussinesq model for internal waves (e.g. CHOI & CAMASSA (1999) *J.Fluid Mech.*), can formulate the steady part as a Hamiltonian system on $\mathbb{R}^8$ with a three-dimensional group of affine translations.

The Lie algebra can be coordinatized by the parameters associated with the uniform flow (h_1, u_1, u_2), and the momentum map can be coordinatized by (R, Q_1, Q_2) where R is the Bernoulli energy and Q_j are the mass flux in each layer. Rigid lid implies $h_1 + h_2 = d$.

Criticality and geometry of $\mathbf{P} : \mathfrak{g} \longrightarrow \mathfrak{g}^*$

$$\mathbf{P}(\mathbf{c}) := \mathbf{J} \circ \varphi = R(\mathbf{c})\xi_1^* + Q_1(\mathbf{c})\xi_2^* + Q_2(\mathbf{c})\xi_3^* \,,$$

with $\mathbf{c} = (h_1, u_1, u_2)$ and

$$\begin{aligned} R(\mathbf{c}) &= \tfrac{1}{2}\rho_1 u_1^2 - \tfrac{1}{2}\rho_2 u_2^2 + (\rho_1 - \rho_2) g h_1 \\ Q_1(\mathbf{c}) &= \rho_1 h_1 u_1 \\ Q_2(\mathbf{c}) &= \rho_2 (d - h_1) u_2 \,. \end{aligned}$$

$$\mathrm{D}\mathbf{P}(\mathbf{c}) = \begin{bmatrix} (\rho_1 - \rho_2) g & \rho_1 u_1 & -\rho_2 u_2 \\ \rho_1 u_1 & \rho_1 h_1 & 0 \\ -\rho_2 u_2 & 0 & \rho_2 (d - h_1) \end{bmatrix} ,$$

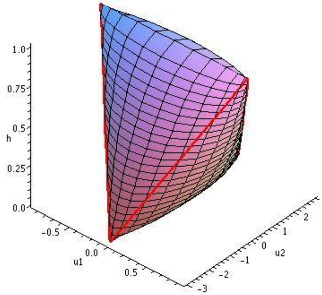
and there exists $\mathbf{n}$ satisfying $\mathrm{Jac}(\mathbf{c})\mathbf{n} = 0$ when $f(\mathbf{c}) = 0$ where

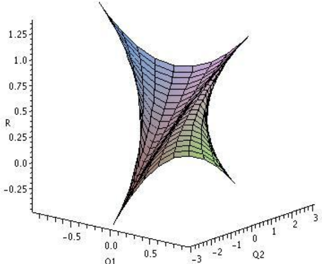
$$f(\mathbf{c}) = \det(\mathrm{Jac}(\mathbf{c})) = \rho_1 \rho_2 (\rho_1 - \rho_2) g h_1 (d - h_1) \left[1 - F_1^2 - r F_2^2 \right] ,$$

where $F_j^2 = u_j^2 / ((1 - r) g h_j)$ and $r = \rho_2 / \rho_1$.

Plot the surface $f(\mathbf{c}) = 0$ and its image in the (R, Q_1, Q_2) plane.

Criticality surfaces for two-layer flow





Criticality and $\mathrm{d}f(\mathbf{c}) \cdot \mathbf{n}$

Now

$$f(\mathbf{c}) := \det[\mathrm{Jac}(\mathbf{c})] = \mathrm{C}\left[(1-r) - \frac{u_1^2}{gh_1} - r\frac{u_2^2}{gh_2}\right], \quad \mathrm{C} = \rho_1^2\rho_2 g h_1 h_2 \,.$$

The criticality surface in (h_1, u_1, u_2) space is defined by $f^{-1}(0)$ and a vector $\mathbf{v}$ is tangent to this surface if $\mathrm{d}f \cdot \mathbf{v} = 0$. Now,

$$\mathrm{d}f = \frac{\mathrm{C}}{g}\left(\frac{u_1^2}{h_1^2} - \frac{ru_2^2}{h_2^2}, -\frac{2u_1}{h_1}, -\frac{2ru_2}{h_2}\right),$$

and so

$$\langle \mathrm{d}f, \mathbf{n}\rangle = \frac{3\mathrm{C}}{\rho_1 g}\left(\rho_1\frac{u_1^2}{h_1^2} - \rho_2\frac{u_2^2}{h_2^2}\right).$$

(cf. B & Donaldson, Phys. Fluids, 2007)

Degeneracy of $\mathrm{D}\mathbf{P}(\mathbf{c})$ and saddle-center

Linearize about a degnerate relative equilibrium

- 0 is an eigenvalue of geometric multiplicity n
- 0 is an eigenvalue of algebraic multiplicity $2n$
- 0 is an eigenvalue of (at least) algebraic multiplicity $2n+2$ if and only if $\det[\mathrm{D}\mathbf{P}(\mathbf{c})] = 0$ (invoking the $G-$Morse hypothesis).

Saddle-center bifurcation of eigenvalues in the linearization transverse to the group orbit corresponds to $\mathbf{P}(\mathbf{c}) \in \Sigma^1(\mathbf{P})$.

Transform linearization to Williamson normal form.

Leading order nonlinear normal form

For values of the momenta in a neighbourhood of a degenerate point,there exists coordinates $(\phi_1,\ldots,\phi_n,u,I_1,\ldots,I_n,v)\in\mathbb{R}^{2n+2}$ satisfying

$$\begin{aligned}
-\frac{dv}{dt} &= I_1-\tfrac{1}{2}\kappa u^2+\cdots\,, \\
\frac{du}{dt} &= s_1 v+\cdots\,, \\
-\frac{dI_j}{dt} &= 0\,, & j=1,\ldots,n \\
\frac{d\phi_1}{dt} &= u+\cdots \\
\frac{d\phi_j}{dt} &= s_j\, I_j+\cdots\,, & j=2,\ldots,n\,.
\end{aligned}$$

The coordinates $(I_1,\ldots,I_n)$ are local coordinates near a point on the criticality hypersurface in $P-$space. The coordinate I_1 is associated with the direction transverse to the hypersurface, and $I_2,\ldots,I_n$ are associated with directions tangent to the image of the hypersurface $\det[\mathrm{D}\mathbf{P}(\mathbf{c})]=0$.

Formula for κ and the symplectic signs

The coefficient of the nonlinear term in the normal form, κ, can be expressed in terms of the generalized eigenvectors,

$$\begin{aligned}\kappa &= -\langle \xi_{n+1}, D^3H(\xi_{n+1}, \xi_{n+1})\rangle - 3\langle \xi_1, D^3H(\xi_1, \xi_{2n+2})\rangle \\ &\quad +3\langle \xi_1, D^3H(\xi_{n+1}, \xi_{2n+1})\rangle\,.\end{aligned}$$

The sign $s_1 = \pm 1$ is a symplectic invariant associated with the symplectic Jordan theory.

The signs s_j for $j = 2, \ldots, n$ are the signs of the nonzero eigenvalues of $\mathrm{D}\mathbf{P}(\mathbf{c})$.

κ and the momentum map

The coefficient κ has a characterization in terms of the geometry of the momentum map $\mathbf{P}$

$$\kappa = a_0^3 \langle \mathrm{d}f(\mathbf{c}), \mathbf{n} \rangle \,, \qquad f(\mathbf{c}) := \det[\mathrm{D}\mathbf{P}(\mathbf{c})] \,,$$

where a_0 is a positive constant.

Remark: κ is the intrinsic second derivative[1] of the mapping $\mathbf{P}(\mathbf{c})$ (e.g. PORTEOUS 1971, GOLUBITSKY & GUILLEMIN 1973).

$$\langle \mathrm{d}f(\mathbf{c}), \mathbf{n} \rangle = \text{Constant} \, \langle D^2\mathbf{P}(\mathbf{c})(\mathbf{n}, \mathbf{n}), \mathbf{n} \rangle \,.$$

[1] Thanks to James Montaldi (Manchester) for this observation.

Curvature of the momentum map

Let

$$\begin{aligned} \mathfrak{g} &\cong T_{\mathbf{c}}\,\mathfrak{g} = \mathfrak{h} \oplus \mathbb{X}\,, \quad \mathfrak{h} = \mathrm{Ker}(\mathrm{D}\mathbf{P}(\mathbf{c})) \\ \mathfrak{g}^* &\cong T_{\mathbf{P}(\mathbf{c})}\,\mathfrak{g}^* = \mathbb{Y} \oplus \mathfrak{h}^*\,, \end{aligned}$$

It is the curvature of the graph of the function

$$\mathscr{K}(\mathbf{c}, s) = \langle \mathbf{n}, \mathbf{P}(\mathbf{c} + s\mathbf{n})\rangle\,,$$

on $\mathfrak{h} \times \mathfrak{h}^*$ that appears in the normal form

$$\kappa = \text{Constant}\, \frac{d^2}{ds^2}\bigg|_{s=0} \mathscr{K}(\mathbf{c}, s)\,,$$

for some positive constant.

(cf. TJB, Preprint, 2006)

Leading order nonlinear normal form

Normal form transverse to the group

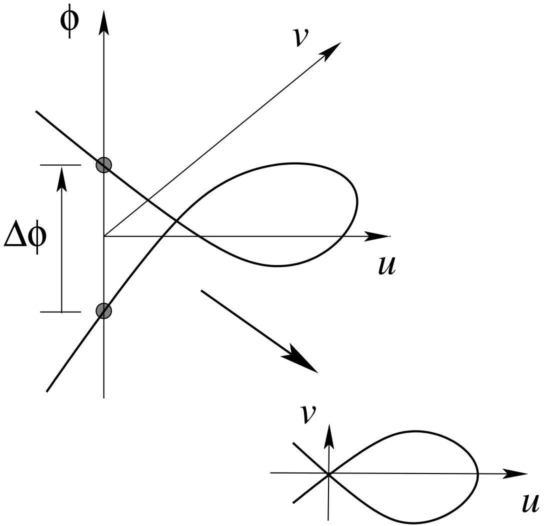
$$
\begin{aligned}
-\frac{dv}{dt} &= I_1 - \tfrac{1}{2}\kappa u^2 + \cdots \,, \\
\frac{du}{dt} &= s_1 v + \cdots \,.
\end{aligned}
$$

Normal form tangent to the group

$$
\begin{aligned}
-\frac{dI_j}{dt} &= 0\,, && j = 1, \ldots, n \\
\frac{d\phi_1}{dt} &= u + \cdots && \\
\frac{d\phi_j}{dt} &= s_j\, I_j + \cdots \,, && j = 2, \ldots, n
\end{aligned}
$$

Directional geometric phase, plus dynamic drift along the group.

Schematic of the geometric phase



Homoclinic bifurcation from invariant tori

For illustration, consider a $\mathbb{T}^2-$equivariant Hamiltonian system on $\mathbb{R}^6$,

$$\mathbb{J}u_t = \nabla H(u)\,, \quad u \in \mathbb{R}^6\,,$$

with momentum map **J**.

RE associated with this group are tori,

$$u(t) = \Phi_{g(t)}(\varphi)\,.$$

Take coordinates ω_1 and ω_2 for the Lie algebra. The family of RE is non-degenerate when

$$\det \begin{bmatrix} \frac{\partial P_1}{\partial \omega_1} & \frac{\partial P_1}{\partial \omega_2} \\ \frac{\partial P_2}{\partial \omega_1} & \frac{\partial P_2}{\partial \omega_2} \end{bmatrix} \neq 0 \quad \text{equivalently} \quad \det \begin{bmatrix} \frac{\partial \omega_1}{\partial I_1} & \frac{\partial \omega_1}{\partial I_2} \\ \frac{\partial \omega_2}{\partial I_1} & \frac{\partial \omega_2}{\partial I_2} \end{bmatrix} \neq 0\,,$$

where (P_1, P_2) are the momenta evaluated on an RE, and (I_1, I_2) can be interpreted as values of level sets.

Degenerate invariant tori and homoclinic bifurcation

Near degeneracy, there exists new coordinates $(\phi_1, \phi_2, u, I_1, I_2, v)$ satisfying

$$\begin{aligned}
-\frac{dv}{dt} &= I_1 - \tfrac{1}{2}\kappa u^2 + \cdots \\
\frac{du}{dt} &= s_1 v + \cdots \\
-\frac{dI_j}{dt} &= 0 \quad j = 1, 2 \\
\frac{d\phi_1}{dt} &= u + \cdots \\
\frac{d\phi_2}{dt} &= s_2 I_2 + \cdots
\end{aligned}$$

with

$$\kappa = a_0^3 \langle \mathrm{d}f(\boldsymbol{\omega}), \mathbf{n} \rangle , \quad f(\boldsymbol{\omega}) := \det \begin{bmatrix} \frac{\partial P_1}{\partial \omega_1} & \frac{\partial P_1}{\partial \omega_2} \\ \frac{\partial P_2}{\partial \omega_1} & \frac{\partial P_2}{\partial \omega_2} \end{bmatrix} .$$

– There is a geometric phase shift on the invariant torus.
– A new mechanism for saddle-center bifurcation of tori?
– Even the case $n = 1$ is new!
(B & DONALDSON (2005) *Phys. Rev. Lett.*)

HANSSMANN (1998) takes a saddle-center bifurcation in the plane, and adds an integrable $n-$torus.

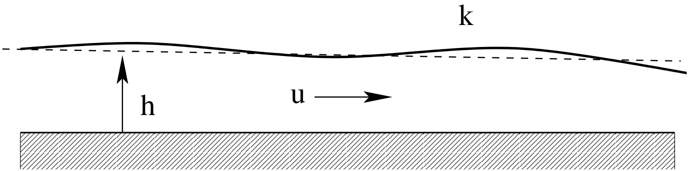
$$\begin{aligned} -\frac{dv}{dt} &= \lambda + b(\omega)u^2 \\ \frac{du}{dt} &= a(\omega)v \\ -\frac{dI_j}{dt} &= 0 \\ \frac{d\theta_j}{dt} &= \omega_j\,, \quad j = 1, \ldots, n\,. \end{aligned}$$

Then perturbation terms are added which break the symmetry (integrability) and persistence of the bifurcation on Cantor subsets of parameter space is proved.

See also BROER, HANSSMANN & YOU (2005).

Shallow water hydrodynamics

–degenerate RE, mean flow, secondary criticality and dark solitary waves



Stokes waves in shallow water coupled to a mean flow are RE associated with $G = \mathbb{R}^2 \times S^1$ with $\mathbb{R}^2$ associated with mean flow, and S^1 associated with the periodic wave (the Stokes wave):

$$(h, u, k) \to (R, Q, B)$$

When these RE are degenerate,

$$\det \left[\frac{\partial(R, Q, B)}{\partial(h, u, k)} \right] = 0 \,,$$

the flow is critical and a class of solitary waves is generated: steady "dark solitary waves".

(cf. B & DONALDSON, J. Fluid Mech. 2006)

Model Hamiltonian system with $\mathbb{S}^1 \times \mathbb{R}^2$ symmetry

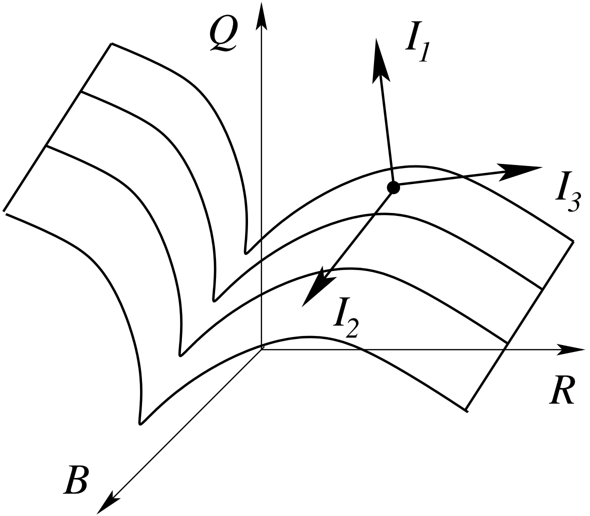
$$\begin{aligned} a\,A_{xx} + 2\mathrm{i}b\,A_x + \beta|A|^2 A &= -2\,(\ell h_x + m u_x)A \\ r\,h_{xx} + c\,u_{xx} &= \ell\,(|A|^2)_x \\ c\,h_{xx} + s\,u_{xx} &= m\,(|A|^2)_x\,, \end{aligned}$$

where $a, b, \beta, \ell, m, r, s$ and c are given (in general nonzero) real parameters with $rs - c^2 \neq 0$. (For water waves $gh_0 - c_g^2 \neq 0$.)

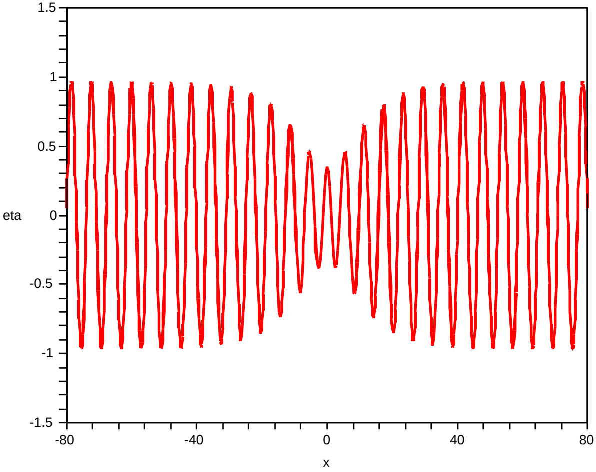
$$\mathbb{J}u_x = \nabla H(u)\,, \quad u \in \mathbb{R}^8\,.$$

When RE associated with the group $\mathbb{S}^1 \times \mathbb{R}^2$ are degenerate, a homoclinic bifurcation occurs which corresponds to a form of steady dark solitary wave. Found also in full water wave problem (cf. B & Donaldson J. Fluid Mech. 2006).

Schematic of the image of $\Sigma^1(P)$ for degenerate Stokes waves



Schematic of steady dark solitary waves



Degenerate RE and internal solitary waves

- Two-layer flow with a rigid lid
  - uniform flows = 3D RE, critical surface is 2D
  - $\langle \mathrm{d}f, \mathbf{n} \rangle = 0$ separates solitary waves of elevation from solitary waves of depression.
  - 3D mean flow (uniform flow) coupled to a periodic wave = 4D RE, 3D critical surface, bif. to internal steady DSWs
- Two-layer flow with a free surface
  - uniform flow = 4D RE, critical surface is 3D
  - $\langle \mathrm{d}f, \mathbf{n} \rangle = 0$ is a 2D manifold
  - uniform flow (mean flow) coupled to a periodic wave = 5D RE, 4D critical surface, bif. to internal steady DSWs

Theory predicts manifold of bifurcating solitary waves from each family of degenerate RE. The bifurcating SWs may have exponentially small tails in the case of two layers with free surface.

(cf. B & DONALDSON, Phys Fluids (2007) + in preparation)

Remarks on dimension and group action

When the group has dimension n, $2n + 2$ is the lowest dimension phase space in which the phenomena can occur.

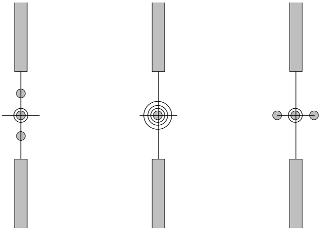
— Dimension N with $N > 2n + 2$: complementary dimensions hyperbolic, can use center-manifold reduction.

– Dimension N with $N > 2n + 2$: complementary dimensions elliptic, will get persistence issues and exponentially-small tails, as in the case without symmetry (e.g. Iooss & Lombardi, J. Diff. Eqns 2006)

– When the group is non-abelian need to bring in more theory to do the tangent/transverse splitting of the vectorfield (e.g. ROBERTS, WULFF & LAMB J. Diff. Eqns 2002), but one expects the basic idea to persist (geometry of momentum map on RE determining the nonlinear normal form transverse to group).

Infinite dimensions – with continuous spectrum

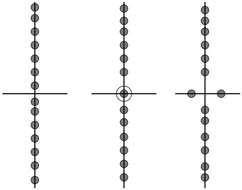
Suppose there is a continuous spectrum on the imaginary axis and a saddle-center bifurcation



Normal form theory goes through to leading order, but the continuous spectrum will be an obstacle to persistence of the homoclinic orbit.

This example arises in nonlinear Schrödinger equation with non-Kerr nonlinearity where the RE is a solitary wave.

Saddle-center with infinite dimensional center



Formal normal form theory goes through to leading order for the saddle-center coupled to an infinite number of elliptic modes. But the elliptic modes will be an obstacle to persistence.

This example arises in the time-dependent water-wave problem. There is a sequence (possibly infinite) of saddle-center bifurcations, and the attendant homoclinic bifurcations – have been found to be associated with a form of wave breaking – *micro-breakers*.

– LONGUET-HIGGINS & DOMMERMUTH, J. Fluid Mech. (1997)
– TJB, J. Fluid Mech. (2004)

Summary

– Generalization of criticality in fluid mechanics –

- Hamiltonian formulation
- Any flow that can be characterized as a RE has a concept of criticality: degeneracy of the RE
- Criticality generates solitary waves
- Properties of the bifurcating solitary wave (homoclinic orbit) encoded in the geometry of the momentum map evaluated on a family of RE
- Used to find new families of solitary waves in shallow water hydrodynamics

- New observations in dynamical systems: e.g. mechanism for homoclinic bifurcation from invariant tori.

- TJB & N.M. Donaldson.
  Degenerate periodic orbits and homoclinic torus bifurcation.
  Phys. Rev. Lett. **95** 104301 (2005).

- TJB & N.M. Donaldson.
  Secondary criticality of water waves. Part I. Definition, bifurcation and solitary waves.
  J. Fluid Mech. **565** 381–417 (2006).

- TJB
  Degenerate relative equilibria, curvature of the momentum map, and homoclinic bifurcation.
  Preprint, University of Surrey (2006).

- TJB & N.M. Donaldson.
  Reappraisal of criticality of two-layer flows and its role in the generation of internal solitary waves.
  Physics of Fluids (in press, 2007).