CDS 202 Final Examination Solutions

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Attempt ***four*** *of the following six questions.*

The exam time limit is three hours; ***no aids are permitted***.

The exam has two sheets printed on both sides

Print Your Name: ______________________________

←**Note!**

The 4 questions to be graded: ______________________________

←**Note!**

You may freely use the following properties as needed. Here α and β are differential forms and X, Y, Z are vector fields on a manifold M. (In the exam, all manifolds, vector fields, and differential forms are assumed to be smooth and the manifolds are finite dimensional.)

(a) $\pounds_X(\alpha \wedge \beta) = (\pounds_X \alpha) \wedge \beta + \alpha \wedge (\pounds_X \beta)$

(b) $\pounds_{[X,Y]}\alpha = \pounds_X \pounds_Y \alpha - \pounds_Y \pounds_X \alpha$

(c) $\mathbf{i}_X(\alpha \wedge \beta) = (\mathbf{i}_X \alpha) \wedge \beta + (-1)^k \alpha \wedge (\mathbf{i}_X \beta)$, where α is a k-form.

(d) $\pounds_X \alpha = \mathbf{d}\mathbf{i}_X \alpha + \mathbf{i}_X \mathbf{d}\alpha$

(e) $\mathbf{i}_{[X,Y]}\beta = \pounds_X \mathbf{i}_Y \beta - \mathbf{i}_Y \pounds_X \beta$

(f) For γ a one-form,

$$\mathbf{d}\gamma(X,Y) = X[\gamma(Y)] - Y[\gamma(X)] - \gamma([X,Y])$$

(g) For ω a two-form,

$$\begin{aligned}\mathbf{d}\omega(X,Y,Z) = {} & X[\omega(Y,Z)] - Y[\omega(X,Z)] + Z[\omega(X,Y)] \\ & - \omega([X,Y],Z) - \omega([Z,X],Y) - \omega([Y,Z],X)\end{aligned}$$

(h) For a one form α and a vector field X,

$$(\pounds_X \alpha)_i = X^j \frac{\partial \alpha_i}{\partial x^j} + \alpha_j \frac{\partial X^j}{\partial x^i}$$

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1. Consider the following vector fields X, Y, the one form α and the three form μ on $\mathbb{R}^3$:

$$
\begin{aligned}
X &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \\
Y &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \\
\alpha &= y\,dx - x\,dy + z\,dz \\
\mu &= \,dx \wedge\,dy \wedge\,dz
\end{aligned}
$$

(a) Compute the exterior derivative $d\alpha$ and the interior product $\mathbf{i}_X\alpha$.

(b) Compute the Lie derivative $\pounds_X\alpha$

(c) Describe the flows F_t of X and G_t of Y geometrically.

(d) Compute

$$\left.\frac{d}{dt}\right|_{t=0} F_t^*\mu \text{ and } \left.\frac{d}{dt}\right|_{t=0} G_t^*\alpha$$

(e) Compute $\left.\frac{d}{dt}\right|_{t=0} F_t^*Y$.

Solution (a) The following calculations are straightforward using the given expressions for α and X in coordinates.

$$d\alpha = -2\,dx \wedge\,dy$$

$$\mathbf{i}_X\alpha = z^2 - x^2 - y^2$$

Solution (b) Using Cartan's formula,

$$\pounds_X\alpha = d(\mathbf{i}_X\alpha) + \mathbf{i}_X(d\alpha)$$

From the solution in (a),

$$\pounds_X\alpha = d(z^2 - x^2 - y^2) + \mathbf{i}_X(-2\,dx \wedge\,dy)$$

Distributing the inner product and differential gives,

$$
\begin{aligned}
\pounds_X\alpha &= 2zdz - 2xdx - 2ydy - 2\,\mathbf{i}_X(dx) \wedge\,dy + 2\,dx \wedge \mathbf{i}_X(dy) \\
&= 2zdz - 2xdx - 2ydy + 2ydy - 2xdx \\
&= d(z^2)
\end{aligned}
$$

Solution (c) The flow of F_t is a rotation in the x-y plane and exponential enlargement in z. In fact, given an initial $(x_0, y_0, z_0) \neq \mathbf{0}$ the curve $(F_t)_*(x_0, y_0, z_0)$ for $t \in [0, T]$ describes an exponentially expanding helix around the z-axis. The flow of G_t is a simple expansion of an initial $(x_0, y_0, z_0) \neq \mathbf{0}$, i.e., $(G_t)_*(x_0, y_0, z_0) = e^t(x_0, y_0, z_0)$.

Solution (d) Using the dynamical definition of the Lie-derivative and $\operatorname{div}_\mu X = 1$

$$\left.\frac{d}{dt}\right|_{t=0} F_t^* \mu = \pounds_X \mu = (\operatorname{div}_\mu X)\mu = \mu$$

Similarly, using the dynamical definition of the Lie-derivative and Cartan's magic formula one obtains,

$$\left.\frac{d}{dt}\right|_{t=0} G_t^* \alpha = \pounds_Y \alpha = d(\mathbf{i}_Y \alpha) + \mathbf{i}_Y d\alpha = 2ydx - 2xdy + 2zdz$$

Solution (e) Using the dynamical definition of the Lie-derivative,

$$\left.\frac{d}{dt}\right|_{t=0} F_t^* Y = \pounds_X Y = 0$$

This result implies that F_t and G_t commute.

2. Let M be the ellipsoidal shell in $\mathbb{R}^3$ given by $x^2 + 4y^2 + z^2 = 1$ and let S be the partial ellipsoidal shell in $\mathbb{R}^3$ defined by the conditions $(x, y, z) \in M$ and $0 \leq x \leq 1/2$.

(a) Show that M is a smooth manifold.

(b) Argue informally that S is a smooth oriented manifold with boundary; describe a specific choice of orientation.

(c) Let the one form α be defined on the open set $U = \mathbb{R}^3 \backslash x$-axis by

$$\alpha = \frac{zdy - ydz}{y^2 + z^2}$$

Compute $\mathbf{d}\alpha$.

(d) Let β be the pull-back of α to S. Is β closed? Is β exact?

(e) Compute the integral of β over ∂S.

Solution (a) The submersion theorem is used to demonstrate M is a smooth manifold. Consider the map $\varphi : \mathbb{R}^3 \to \mathbb{R}$ given by:

$$\varphi(x, y, z) = x^2 + 4y^2 + z^2$$

Note that $M = \varphi^{-1}(1)$. If 1 is a regular value of φ, then by the submersion theorem M is a manifold. To show that 1 is a regular value, its tangent map restricted to M is shown to be surjective, i.e., for every $C \in \mathbb{R}$ there exists a $B \in \mathbb{R}^3$ given in components by $B = (B_1, B_2, B_3)$ such that

$$T\varphi|_M \cdot (B_1, B_2, B_3) = 2xB_1 + 8yB_2 + 2zB_3 = C$$

This B is given by $B = C/2(x, y, z)$.

Solution (b) It is easy to show that the cylindrical shell is a manifold. Consider the diffeomorphism between the truncated ellipsoidal shell and the cylindrical shell given by,

$$(x, y, z) \mapsto (x, 2y, z)$$

This map sends the points

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + 4y^2 + z^2 = 1, 0 \leq x \leq 1/2\}$$

to the points

$$\{(u, v, w) \in \mathbb{R}^3 : u^2 + v^2 + w^2 = 1, 0 \leq u \leq 1/2\}$$

where $u = x, v = 2y, w = z$. The orientation in the interior is given by choosing an outward or inward pointing normal at every point. On the boundary ellipses (at $x = 0$ or $x = 1/2$) the orientation of the tangent space (tangents pointing in the clockwise or counterclockwise sense) is determined by the orientation in the interior. Specifically, one picks an outward pointing vector to the boundary to be the first vector of the oriented basis, and the next vector is found by ensuring that the right-hand rule gives an outward pointing normal as in the interior.

Solution (c) A direct calculations shows that $d\alpha = 0$, and hence, α is closed.

Solution (d) Let $i : S \to \mathbb{R}^3$ be the inclusion map. Then $\beta = i^*\alpha$, and since i is smooth then $d\beta = di^*\alpha = i^*d\alpha = 0$. Thus, β is closed.

However, β is not exact. One can prove this by contradiction as follows. Suppose $\beta = d\gamma$ and consider the following integral over the ellipse $C = \{(y, z) : 4y^2 + z^2 = 1\}$:

$$\int_C \beta = \int_0^{2\pi} \frac{-2d\theta}{\cos^2(\theta) + 4\sin^2(\theta)} \neq 0$$

since the integrand is strictly negative. However, by Stokes' theorem

$$\int_C \beta = \int_C d\gamma = 0$$

since $\partial C = \emptyset$.

Solution (e) By Stokes' theorem

$$\int_{\partial S} \beta = \int_S d\beta = 0$$

since β is closed.

3. Let S be the 3×3 diagonal matrix with diagonal entries $1, 1, -2$. Let G denote the set of 3×3 real matrices A that satisfy $A^T SA = S$, where A^T denotes the transpose of A.

(a) Show that, with the operation of matrix multiplication, G is a Lie group.

(b) What is its dimension? Is G compact?

(c) Show that the Lie algebra $\mathfrak{g}$ of G may be identified with the set of 3×3 matrices ξ that satisfy $\xi^T S + S\xi = 0$. What is the Lie algebra bracket?

(d) If α is a nonzero real number, show that the matrix

$$\xi = \begin{bmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

lies in the Lie algebra $\mathfrak{g}$. What is the one parameter subgroup of G that is tangent to ξ at $t = 0$?

(e) Let $\eta, \xi \in \mathfrak{g}$ be two matrices in $\mathfrak{g}$ from part (c) that commute. Let D be the distribution on G obtained by left translating the two dimensional vector space $V = \text{span}(\eta, \xi)$ around the group. Is D integrable?

Solution (a) First check the axioms of a group:

- Closure: Given $A, B \in G$, observe their product is in G, i.e.,

$$(AB)^T SAB = B^T(A^T SA)B = B^T SB = S$$

- Associativity: This property follows from the fact that matrix multiplication is associative.
- Identity: The identity matrix denoted by e is in G since,

$$e^T Se = eS = S$$

- Inverse: The inverse of any $A \in G$ is also in G, since A is invertible

$$A^T SA = S \implies \det(A)^2 = 1 \implies A \in GL(3)$$

  and

$$A^T SA = S \implies (A^T)^{-1} SA^{-1} = (A^{-1})^T SA^{-1} = S$$

G is shown to be a subgroup of $GL(3)$ by considering the properties of the map $\varphi : GL(3) \to \{\text{symmetric } 3 \times 3 \text{ matrices}\}$ given by:

$$\varphi(A) = A^T SA$$

Note that $\varphi^{-1}(S) = G$. The claim is that S is a regular value of φ, and hence, G is a submanifold of $GL(3)$ by the submersion theorem. Consider

$$D\varphi(A) \cdot B = B^T SA + A^T SB$$

where $D\varphi(A) : L(\mathbb{R}^3, \mathbb{R}^3) \to \{\text{symmetric } 3 \times 3 \text{ matrices}\}$. For any $C \in \{\text{symmetric } 3 \times 3 \text{ matrices}\}$ there is a B such that

$$D\varphi(A) \cdot B = C$$

given by

$$B = 1/2AS^{-1}C.$$

The group operation is smooth since matrix multiplication is a smooth bilinear map on $GL(3)$, and hence, its restriction to the subgroup G is also smooth. Hence, G is a Lie subgroup of G.

Solution (b) The dimension of G is given by the submersion theorem as

$$\dim G = \dim GL(3) - \dim\{\text{symmetric } 3\times 3 \text{ matrices}\} = 3$$

Moreover, G is the preimage of the closed set S under the continuous map φ. Hence, G is closed. However, G is not compact since it is not bounded. For example, for any $t \in \mathbb{R}$ consider the following $A \in G$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & t & \sqrt{2(t^2-1)} \\ 0 & \sqrt{(t^2-1)/2} & t \end{bmatrix}$$

whose matrix norm is given by:

$$\|A\| = \frac{3}{2}(-1+3t^2)$$

which cannot be bounded for all t.

Solution (c) From the submersion theorem $T_eG \sim \mathfrak{g}$ is the set of matrices in the kernal of $D\varphi(e)$, i.e., $\xi \in L(\mathbb{R}^3, \mathbb{R}^3)$ such that

$$D\varphi(e)\cdot\xi = \xi^T S + S\xi = 0$$

Since G is a Lie subgroup of $GL(3)$, the bracket is the matrix commutator restricted to $\mathfrak{g}$.

Solution (d) $\xi \in \mathfrak{g}$ since $\xi^T = -\xi$ and

$$\xi^T S + S\xi = -\xi + \xi = 0$$

The one-parameter subgroup of G generated by ξ is $SO(2)$ and given by:

$$\exp(t\xi) = \begin{bmatrix} \cos(\alpha t) & \sin(\alpha t) & 0 \\ -\sin(\alpha t) & \cos(\alpha t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and corresponds to rotations about the z-axis.

Solution (e) If $\eta, \xi \in \mathfrak{g}$ commute then the bracket of their associated left-invariant vector fields is zero, i.e.,

$$\eta\xi = \xi\eta \implies \pounds_{X_\xi} X_\eta = X_{[\xi,\eta]} = 0$$

Hence the distribution defined by left translating the vector space spanned by ξ and η is involutive and by Frobenius' theorem integrable.

4. (a) Let X and Y be the vector fields on $\mathbb{R}^3$ defined by

$$X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \quad \text{and} \quad Y = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

Show that X and Y define vector fields X_0 and Y_0 on the standard two sphere S^2 of radius one.

i. Show that, with respect to the standard volume element on S^2, $\operatorname{div} X_0 = 0$ and $\operatorname{div} Y_0 = 0$.

ii. Calculate $[X_0, Y_0]$.

(b) Let (M_1, μ_1) and (M_2, μ_2) be two compact volume manifolds without boundary and let X_1 be a smooth vector field on M_1.

i. Explain how $(M_1 \times M_2, \mu_1 \times \mu_2)$ is a volume manifold with volume element $\mu_1 \times \mu_2$ determined in a natural way from μ_1 and μ_2.

ii. Is it true that

$$\int_{M_1 \times M_2} (\operatorname{div}_{\mu_1} X_1)\ \mu_1 \times \mu_2$$

must be zero?

Solution (a) This part is straightforward. Consider the map

$$\varphi(x, y, z) = x^2 + y^2 + z^2$$

and its regular value at 1. Note that $\varphi^{-1}(1) = S^2$. Then, $X, Y \in T_(x, y, z)S^2$ since at each point $(x, y, z) \in S^2$ they are in the kernel of $T_{(x,y,z)}\varphi$.

The divergence of X and Y is zero since the flows of their vector fields correspond to pure rotations about the x and z axes respectively. Moreover,

$$[X_0, Y_0] = [X, Y]|_{S^2} = (X[Y] - Y[X])|_{S^2} = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}$$

which is a rotation about the y-axis.

Solution (b) Let $\pi_{M_i} : M_1 \times M_2 \to M_i$ denote the natural projection. Then, $M_1 \times M_2$ is a volume manifold with volume element

$$\mu_1 \times \mu_2 = (\pi_1^* \mu_1) \wedge (\pi_2^* \mu_2).$$

as described, e.g., in 7.5-10.

For the second part the answer is yes. The proof follows. Observe that

$$\begin{aligned}
\pounds_X(\mu_1 \times \mu_2) &= \operatorname{div}_{\mu_1 \times \mu_2}(X)(\mu_1 \times \mu_2) \\
&= \pounds_X \left(\pi_{M_1}^* \mu_1\right) \wedge \left(\pi_{M_2}^* \mu_2\right) + \left(\pi_{M_1}^* \mu_1\right) \wedge \pounds_X \left(\pi_{M_2}^* \mu_2\right) \\
&= \pi_{M_1}^* \left(\pounds_{(\pi_{M_1})_* X} \mu_1\right) \wedge \left(\pi_{M_2}^* \mu_2\right) + \left(\pi_{M_1}^* \mu_1\right) \wedge \pi_{M_2}^* \left(\pounds_{(\pi_{M_2})_* X} \mu_2\right) \\
&= \left(\operatorname{div}_{\mu_1}((\pi_{M_1})_* X) + \operatorname{div}_{\mu_2}((\pi_{M_2})_* X)\right) \mu_1 \times \mu_2
\end{aligned}$$

In summary,

$$\operatorname{div}_{\mu_1 \times \mu_2}(X) = \operatorname{div}_{\mu_1}((\pi_{M_1})_* X) + \operatorname{div}_{\mu_2}((\pi_{M_2})_* X)$$

Consider $X = (X_1, \mathbf{0})$ and compute

$$\begin{aligned}
\int_{M_1 \times M_2} \operatorname{div}_{\mu_1}(X_1) \mu_1 \times \mu_2 &= \int_{M_1 \times M_2} \left(\operatorname{div}_{\mu_1}(X) + \operatorname{div}_{\mu_2}(\mathbf{0})\right) \mu_1 \times \mu_2 \\
&= \int_{M_1 \times M_2} \operatorname{div}_{\mu_1 \times \mu_2}(X, \mathbf{0}) \mu_1 \times \mu_2
\end{aligned}$$

However by the divergence theorem

$$\int_{M_1 \times M_2} \operatorname{div}_{\mu_1}(X) \mu_1 \times \mu_2 = \int_{M_1 \times M_2} \operatorname{div}_{\mu_1 \times \mu_2}(X, \mathbf{0}) \mu_1 \times \mu_2 = 0$$

since $\partial M_1 = \partial M_2 = \emptyset$.

5. (a) Let S^1 be the standard two sphere of radius one in $\mathbb{R}^3$ and S^R the sphere of radius R. Let $\phi : S^1 \to S^R$ be the map that takes $\mathbf{x} \in S^1$ to $R\mathbf{x} \in S^R$. Show that ϕ is an orientation preserving diffeomorphism and state the change of variables formula for this map.

(b) Let the vector field X on $\mathbb{R}^3$ be defined by

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$$

and let F_t be its flow. Show that the flow defines, for each t, an orientation preserving diffeomorphism of S^1 to a sphere of another radius $R(t)$.

(c) Let $f(x, y, z, t)$ be a time dependent function on $\mathbb{R}^3$ and also use the notation f to denote its restriction to a sphere. Let μ_R denote the standard area form on S^R. Find an expression for

$$\frac{d}{dt}\int_{S^{R(t)}} f\mu_{R(t)}$$

where $R(t)$ is as in part (b) and check your calculation explicitly for the function f that is identically one.

Solution (a) The map ϕ is an expansion of the sphere of radius one by the factor R, and therefore, clearly preserves the outward or inward pointing normal to the sphere. For $\mu_R \in \Omega^2(S^R)$ and $m \in S^1$, the change of variables formula is given by,

$$(\phi^*\mu_R)_m(v_1, v_2) = \mu_R(T_m\phi \cdot v_1, T_m\phi \cdot v_2) = R^2\mu_1(v_1, v_2).$$

Solution (b) From problem 1c the flow is Y is simply an expansion by $R(t) = e^t$ and by part (a) defines an orientation preserving diffeo from S^1 to $S^{R(t)}$.

Solution (c) By the change of variables theorem,

$$\begin{aligned}
\frac{d}{dt}\int_{F_t(S^1)} f\mu_{R(t)} &= \frac{d}{dt}\int_{S^1} F_t^*(f\mu_{R(t)}) \\
&= \frac{d}{dt}\int_{S^1} e^{2t}(f\circ F_t)\mu_1 \\
&= \int_{S^1} \frac{d}{dt}\left(e^{2t}f(e^t x, e^t y, e^t z, t)\mu_1\right)
\end{aligned}$$

This expression can be further simplified by recalling techniques used in the proof of the transport theorem.

$$\begin{aligned}
\frac{d}{dt}\int_{F_t(S^1)} f\mu_{R(t)} &= \int_{S^1} \frac{d}{dt}\left(e^{2t}(F_t^* f)\mu_1\right) \\
&= \int_{S^1} e^{2t}F_t^*\left(2f + \pounds_X f + \frac{\partial f}{\partial t}\right)\mu_1 \\
&= \int_{S^{R(t)}} \left(2f + \pounds_X f + \frac{\partial f}{\partial t}\right)\mu_{R(t)}
\end{aligned}$$

For $f = 1$, the formula simplifies to:

$$\frac{d}{dt}\int_{S^{R(t)}} \mu_{R(t)} = 2\int_{S^{R(t)}} \mu_{R(t)} = 8\pi e^{2t}$$

6. (a) Consider the distribution on $\mathbb{R}^3 \setminus \{0\}$ that is given at the point (x, y, z) by the set of vectors $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ satisfying $6ax + 2by + 10cz = 0$. Is this distribution integrable? If so, find the corresponding integrable manifolds.

(b) Let ω be a closed two form on a manifold M and let X be a vector field with a flow F_t satisfying $F_t^*\omega = \omega$. Show that the distribution defined (at each point) to be the kernel of the one-form $\mathbf{i}_X\omega$ is integrable.

(c) Denote coordinates on $\mathbb{R}^{2n}$ by (q^i, p_i), where i ranges between 1 and n and define the two-form ω by $\omega = dq^i \wedge dp_i$ (where a sum on i is understood). Let $H(q, p)$ be a given function and let X be the vector field such that $\mathbf{i}_X\omega = \mathbf{d}H$. Show that the conditions of part (b) hold and determine the foliation in this case.

Solution (a) For a vector field $X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}$ one can write the condition as:

$$df(X) = \mathbf{i}_X df = 0$$

where $f = 3x^2 + y^2 + 5z^2$ and $df = 6xdx + 2ydy + 10zdz$. Suppose that X and Y are in the distribution, i.e., $\mathbf{i}_X df = 0$ and $\mathbf{i}_Y df = 0$. Then by formula (e)

$$\mathbf{i}_{[X,Y]}df = \pounds_X(\mathbf{i}_Y df) - \mathbf{i}_Y(\pounds_X df) = -\mathbf{i}_Y(\pounds_X df)$$

Using Cartan's formula to expand the Lie derivative gives,

$$\mathbf{i}_{[X,Y]}df = -\mathbf{i}_Y(\mathbf{i}_X ddf + d\mathbf{i}_X df) = 0$$

Hence, the distribution defined by $\mathbf{i}_X df = 0$ is involutive, and by the Frobenius theorem, integrable. The corresponding integrable manifolds are the level sets of f.

Solution (b) We again use Frobenius. First observe that,

$$F_t^*\omega = \omega \implies \pounds_X\omega = 0 \implies d\mathbf{i}_X\omega = 0$$

since ω is closed. This property is used to show that if X_1 and X_2 are in the distribution then so is their bracket. Similar to (a),

$$\begin{aligned} \mathbf{i}_{[X_1,X_2]}\mathbf{i}_X\omega &= \pounds_{X_1}(\mathbf{i}_{X_2}\mathbf{i}_X\omega) - \mathbf{i}_{X_2}(\pounds_{X_1}\mathbf{i}_X\omega) \\ &= -\mathbf{i}_{X_2}(\pounds_{X_1}\mathbf{i}_X\omega) \\ &= -\mathbf{i}_{X_2}(d\mathbf{i}_{X_1}\mathbf{i}_X\omega + \mathbf{i}_{X_1}d\mathbf{i}_X\omega) = -\mathbf{i}_{X_2}\mathbf{i}_{X_1}d\mathbf{i}_X\omega = 0 \end{aligned}$$

Thus, the distribution is involutive, and hence, integrable.

Solution (c) A direct calculation can be used to show that ω is in fact closed. By the dynamical definition of the Lie derivative,

$$\frac{d}{dt}F_t^*\omega = F_t^*\pounds_X\omega = F_t^*(d\mathbf{i}_X\omega + \mathbf{i}_X d\omega) = F_t^*(ddH + \mathbf{i}_X d\omega) = F_t^*(\mathbf{i}_X d\omega) = 0$$

since ω is closed. Hence, the distribution defined by the kernel of $\mathbf{i}_X\omega$ is integrable. The integrable manifolds in this case are the level sets of constant H or energy.