CALTECH
Control & Dynamical Systems

Manifolds, Mappings, Vector Fields

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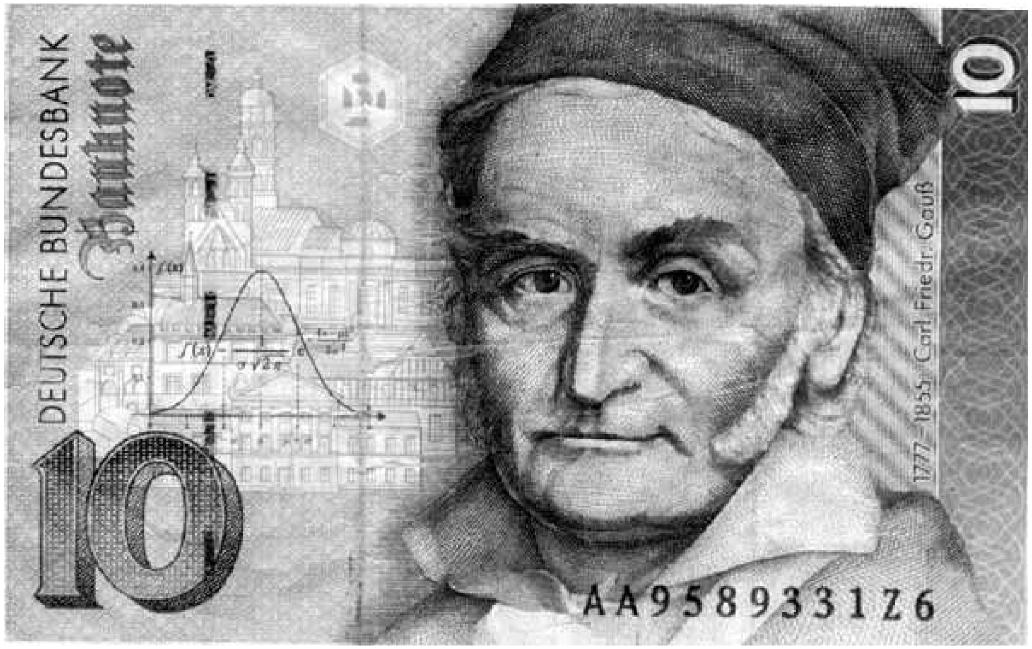
Some History

Carl Friedrich Gauss

- **Born:** 30 April 1777 in Brunswick (Germany)
- **Died:** 23 Feb 1855 in Göttingen (Germany)
- PhD, 1799, University of Helmstedt, advisor was Pfaff. Dissertation was on *the fundamental theorem of algebra*.
- 1801, predicted position of asteroid Ceres; used averaging methods; famous as an astronomer; 1807, director of the Göttingen observatory
- 1801–1830, Continued to work on algebra, and increasingly in geometry of surfaces, motivated by non-Euclidean geometry.
- 1818, Fame as a geologist; used geometry to do a geodesic survey.
- 1831, work on potential theory (Gauss' theorem), terrestrial magnetism, etc.
- 1845 to 1851, worked in finance and got rich.

Some History



Some History

□ Carl Gustav Jacob Jacobi

- **Born:** 10 Dec 1804 in Potsdam
- **Died:** 18 Feb 1851 in Berlin
- 1821, entered the University of Berlin
- 1825, PhD *Disquisitiones Analyticae de Fractionibus Simplicibus*
- 1825-26 Humboldt-Universität, Berlin
- 1826–1844 University of Königsberg
- 1827, Fundamental work on elliptic functions
- 1844–1851 University of Berlin
- *Lectures on analytical mechanics* (Berlin, 1847 - 1848); gave a detailed and critical discussion of Lagrange's mechanics.

Some History

Some History

Georg Friedrich Bernhard Riemann

- **Born:** 17 Sept 1826 in Breselenz (Germany)
- **Died:** 20 July 1866 in Selasca, Italy
- PhD, 1851, Göttingen, advisor was Gauss; thesis in complex analysis on Riemann surfaces
- 1854, Habilitation lecture: *Über die Hypothesen welche der Geometrie zu Grunde liegen (On the hypotheses that lie at the foundations of geometry)*, laid the foundations of manifold theory and Riemannian geometry
- 1857, Professor at Göttingen
- 1859, Reported on the Riemann hypothesis as part of his induction into the Berlin academy
- 1866, Died at the age of 40 in Italy,

Some History

Some History

Felix Klein

- **Born:** 25 April 1849 in Dsseldorf, Germany
- **Died:** 22 June 1925 in Göttingen, Germany
- 1871; University of Christiania (the city which became Kristiania, then Oslo in 1925)
- PhD, 1868, University of Bonn under Plücker, on applications of geometry to mechanics.
- 1872, professor at Erlangen, in Bavaria; laid foundations of geometry and how it connected to group theory (Erlangen Programm); collaborations with Lie.
- 1880 to 1886, chair of geometry at Leipzig
- 1886, moved to Göttingen to join Hilbert's group
- 1890–1900+, More on mathematical physics, mechanics; corresponded with Poincaré

Some History

Some History

Marius Sophus Lie

- **Born:** 17 Dec 1842 in Nordfjordeide, Norway
- **Died:** 18 Feb 1899 in Kristiania (now Oslo), Norway
- 1871; University of Christiania (the city which became Kristiania, then Oslo in 1925)
- PhD, 1872, University of Christiania *On a class of geometric transformations*
- 1886–1898 University of Leipzig (got the chair of Klein); where he wrote his famous 3 volume work *Theorie der Transformationsgruppen*
- 1894, Kummer was his student at Leipzig
- 1898–1899, Returned to the University of Christiania; dies shortly after his return.

Some History

Some History

Henri Poincaré

- **Born:** 29 April 1854 in Nancy, Lorraine, France
- **Died:** 17 July 1912 in Paris, France
- 1879, Ph.D. University of Paris, Advisor: Charles Hermite
- 1879, University of Caen
- 1881, Faculty of Science in Paris in 1881
- 1886, Sorbonne and École Polytechnique until his death at age 58
- 1887, elected to the Acadmie des Sciences; 1906, President.
- 1895, *Analysis situs* (algebraic topology)
- 1892–1899, *Les Méthodes nouvelles de la méchanique céleste* (3 volumes) and, in 1905, *Leçons de mecanique céleste*
- His cousin, Raymond Poincaré, was President of France during World War I.

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- Usually, we denote $\varphi(m)$ by $(x^1, \ldots, x^n)$ and call the x^i the ***coordinates*** of the point $m \in U \subset M$.
- Two charts (U, φ) and (U', φ') such that $U \cap U' \neq \varnothing$ are called ***compatible*** if $\varphi(U \cap U')$ and $\varphi'(U' \cap U)$ are open subsets of $\mathbb{R}^n$ and the maps

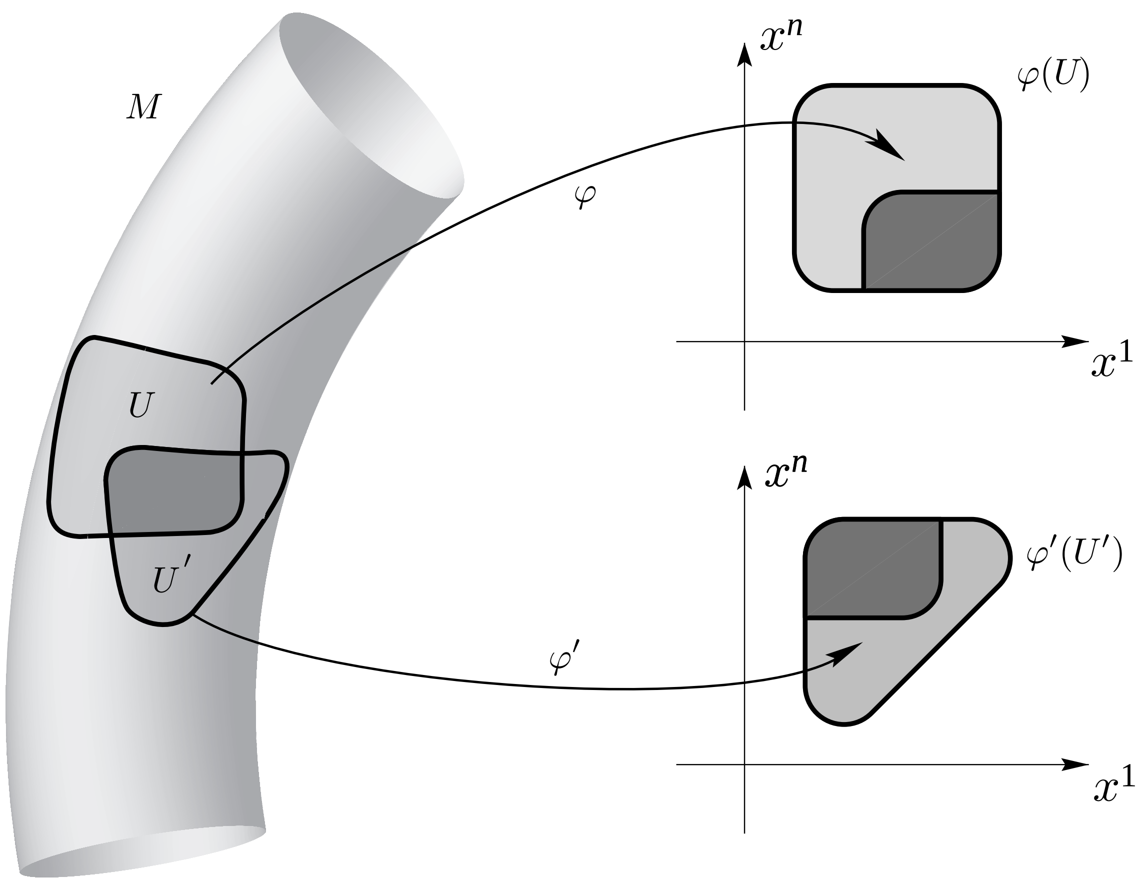
$$\varphi' \circ \varphi^{-1} | \varphi(U \cap U') : \varphi(U \cap U') \longrightarrow \varphi'(U \cap U')$$

and

$$\varphi \circ (\varphi')^{-1} | \varphi'(U \cap U') : \varphi'(U \cap U') \longrightarrow \varphi(U \cap U')$$

Manifolds

are C^∞. Here, $\varphi' \circ \varphi^{-1}|\varphi(U \cap U')$ denotes the restriction of the map $\varphi' \circ \varphi^{-1}$ to the set $\varphi(U \cap U')$.



Manifolds

- We call M a ***differentiable n-manifold*** when:

  M1. *The set M is covered by a collection of charts, that is, every point is represented in at least one chart.*

  M2. *M has an **atlas**; that is, M can be written as a union of compatible charts.*

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- **Example:** Start with $\mathbb{R}^3$ as a manifold with simply one (identity) chart. We then allow other charts such as those defined by spherical coordinates—this is then a differentiable structure on $\mathbb{R}^3$. This will be understood to have been done when we say we have a manifold.

Manifolds

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- **Topology.** Every manifold has a topology obtained by declaring open neighborhoods in charts to be open neighborhoods when mapped to M by the chart.
- **Tangent Vectors.** Two curves $t \mapsto c_1(t)$ and $t \mapsto c_2(t)$ in an n-manifold M are called ***equivalent*** at the point m if

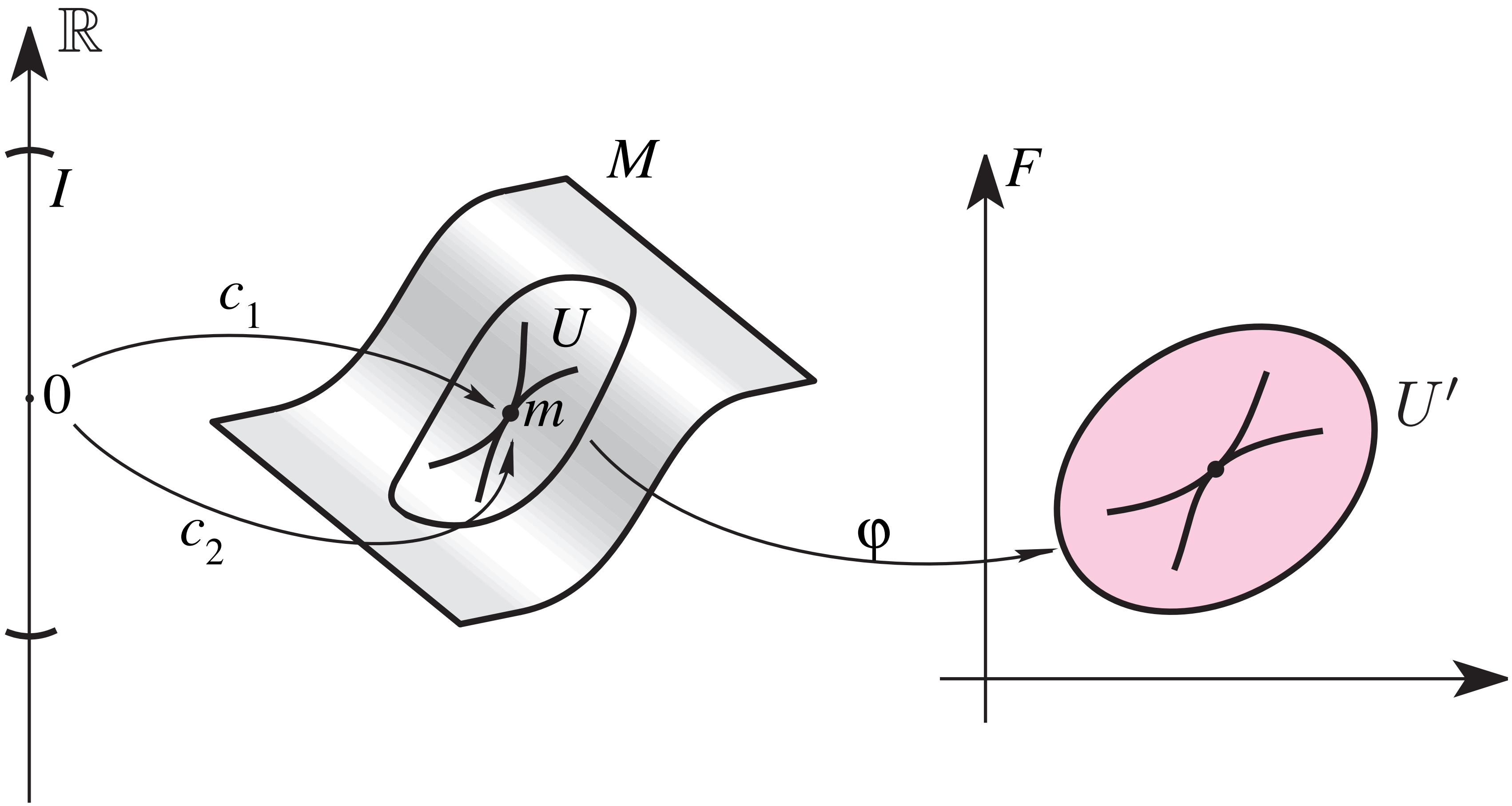
$$c_1(0) = c_2(0) = m$$

and

$$\frac{d}{dt}(\varphi \circ c_1)\bigg|_{t=0} = \frac{d}{dt}(\varphi \circ c_2)\bigg|_{t=0}$$

in some chart φ.

Manifolds



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- Set of tangent vectors to M at m is a vector space.
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- We think of $\mathbf{v} \in T_m M$ as tangent to a curve in M.

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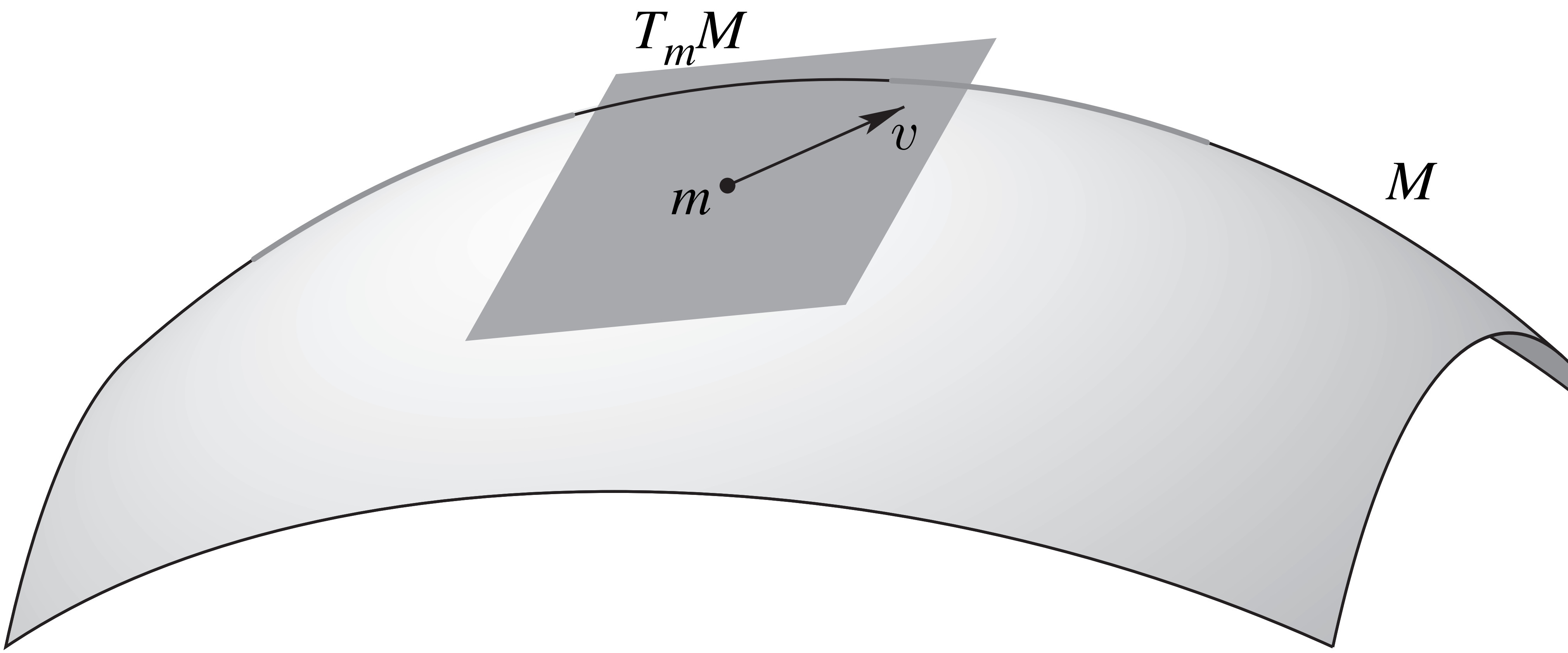

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- c a curve representative of the equivalence class v.
- The ***components*** of v are the numbers $v^1, \ldots, v^n$ defined by taking the derivatives of the components of the curve $\varphi \circ c$:

$$v^i = \frac{d}{dt}(\varphi \circ c)^i \bigg|_{t=0},$$

where $i = 1, \ldots, n$.

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  where $i = 1, \ldots, n$.
- Components are independent of the representative curve chosen, but they do depend on the chart chosen.

Tangent Bundles

- **Tangent bundle of M**, denoted by TM, is the disjoint union of the tangent spaces to M at the points $m \in M$, that is,

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- The inverse image $\tau_M^{-1}(m)$ of $m \in M$ is the tangent space $T_m M$—the ***fiber*** of TM over the point $m \in M$.

Differentiable Maps

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  - For $v \in T_m M$, choose a curve c in M with $c(0) = m$, and velocity vector $dc/dt\,|_{t=0} = v$.
  - $T_m f \cdot v$ is the velocity vector at $t = 0$ of the curve $f \circ c : \mathbb{R} \to N$, that is,

$$T_m f \cdot v = \frac{d}{dt} f(c(t))\bigg|_{t=0}.$$

Chain Rule

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- If $f : M \to N$ is of class C^k, then $Tf : TM \to TN$ is a mapping of class C^{k-1}.
- If $f : M \to N$ and $g : N \to P$ are differentiable maps (or maps of class C^k), then $g \circ f : M \to P$ is differentiable (or of class C^k), and the ***chain rule*** holds:

$$T(g \circ f) = Tg \circ Tf.$$

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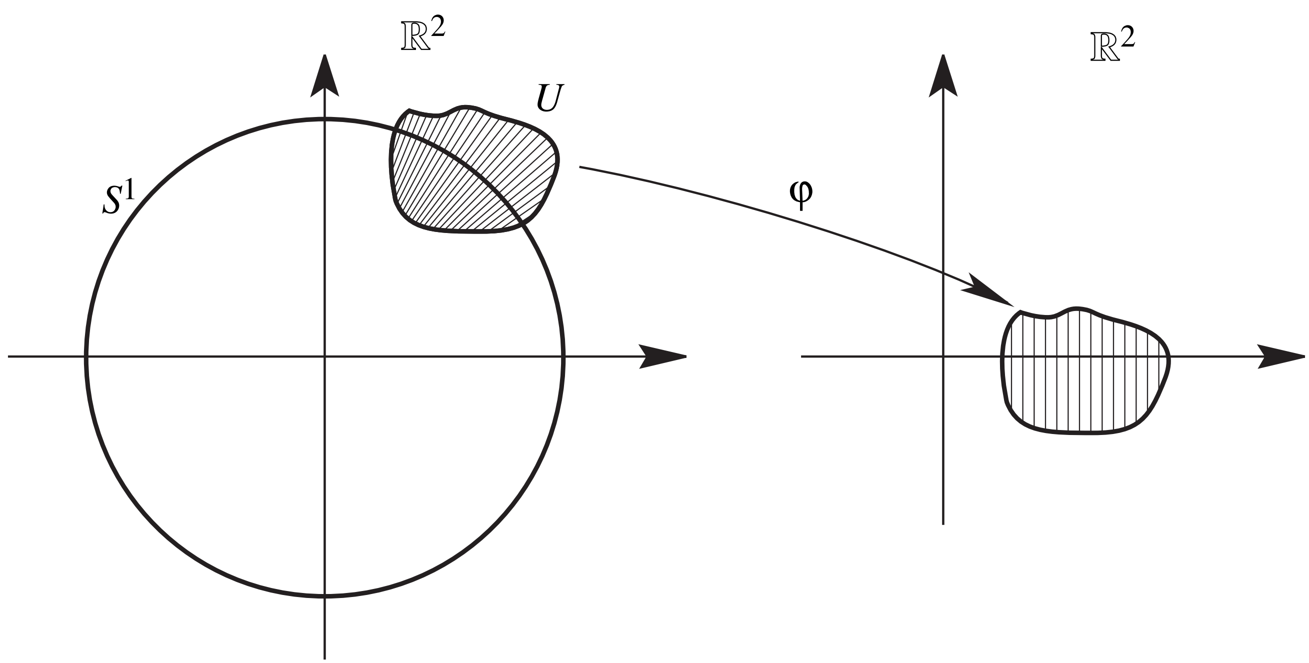
- The set of all diffeomorphisms $f : M \to M$ forms a group under composition, and the chain rule shows that $T(f^{-1}) = (Tf)^{-1}$.

Submanifolds

- **Submanifold (dim k):** $S \subset M$; for $s \in S$ there is a chart (U, φ) in M with the ***submanifold property:***

  SM. $\varphi : U \to \mathbb{R}^k \times \mathbb{R}^{n-k}$ and $\varphi(U \cap S) = \varphi(U) \cap (\mathbb{R}^k \times \{\mathbf{0}\})$.

- S is a manifold in its own right.



Submanifolds

- If $f : M \to N$ is a smooth map, a point $m \in M$ is a ***regular point*** if $T_m f$ is surjective; otherwise, m is a ***critical point*** of f. A value $n \in N$ is ***regular*** if all points mapping to n are regular.

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- **Submersion theorem:** If $f : M \to N$ is a smooth map and n is a regular value of f, then $f^{-1}(n)$ is a smooth submanifold of M of dimension $\dim M - \dim N$ and

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$$T_m\left(f^{-1}(n)\right) = \ker T_m f.$$
- This is an excellent technique for showing various sets are manifolds, such as $O(n)$, the set of orthogonal $n \times n$ matrices.

Vector Fields and Flows

- ***Vector field*** X on M: a map $X : M \to TM$ that assigns a vector $X(m)$ at the point $m \in M$; that is, $\tau_M \circ X = \text{identity}$.

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- An ***integral curve*** of X with initial condition m_0 at $t = 0$ is a (differentiable) map $c :]a, b[\to M$ such that $]a, b[$ is an open interval containing 0, $c(0) = m_0$, and

$$c'(t) = X(c(t))$$

for all $t \in]a, b[$; i.e., a ***solution curve*** of this ODE.

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  for all $t \in]a, b[$; i.e., a ***solution curve*** of this ODE.
- ***Flow*** of X: The collection of maps $\varphi_t : M \to M$ such that $t \mapsto \varphi_t(m)$ is the integral curve of X with initial condition m.

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  along with the initial condition $\varphi_0 =$ identity.
- This generalizes the situation where $M = V$ is a *linear* space, $X(m) = Am$ for a (bounded) *linear* operator A, and where

  $$\varphi_t(m) = e^{tA}m$$

  to the *nonlinear* case.

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- An important estimate that is the key to proving things like this is the ***Gronwall inequality***: if on an interval $[a, b]$, we have

$$f(t) \leq A + \int_a^t f(s)g(s)ds$$

where $A \geq 0$, and $f, g \geq 0$ are continuous, then

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- This is used on the integral equation defining a flow (using a chart).

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- Strategy for proving that the flow is C^1:
- Formally linearize the equations and find the equation that $\mathbf{D}\varphi_t$ should satisfy. This is the ***first variation equation***.
- Use the basic existence theory to see that the first variation equation has a solution.
- Show that this solution actually satisfies the definition of the derivative of φ_t by making use of Gronwall.

Vector Fields and Flows

- ***Time-dependent vector field:*** $X : M \times \mathbb{R} \rightarrow TM$ such that $X(m,t) \in T_m M$ for $m \in M$, $t \in \mathbb{R}$.

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such that $t \mapsto \varphi_{t,s}(m)$ is the integral curve $c(t)$ with initial condition $c(s) = m$ at $t = s$.
- Existence and uniqueness ⇒ ***time-dependent flow property***

$$\varphi_{t,s} \circ \varphi_{s,r} = \varphi_{t,r}.$$

If X happens to be time independent, the two notions of flows are related by $\varphi_{t,s} = \varphi_{t-s}$.

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- Notion of ***completeness of vector fields***; integral curves are defined for all time.
- Criteria discussed in book; an important case is that if, for example, by making a priori estimates, one can show that any integral curve defined on a finite open time interval (a, b) remains in a compact set, then the vector field is complete.
- For instance sometimes such an estimate can be obtained as an ***energy estimate***; eg, the flow of $\ddot{x} + x^3 = 0$ is complete since it has the preserved energy $H(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + \frac{1}{4}x^4$, which confines integral curves to balls for finite time intervals.

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- If we replace each vector space $T_m M$ with its dual $T_m^* M$, we obtain a new $2n$-manifold called the ***cotangent bundle*** and denoted by $T^* M$.
- We call $\mathbf{d}f$ the ***differential*** of f. For $v \in T_m M$, we call $\mathbf{d}f(m) \cdot v$ the ***directional derivative*** of f in the direction v.

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- Use the ***summation convention***: drop the summation sign when there are repeated indices.
- For short, we write

$$\mathbf{d}f(m) \cdot v = \frac{\partial f}{\partial x^i} v^i.$$

Vector Fields as Differential Operators

- Specifying the directional derivatives completely determines a vector, and so we can identify a basis of $T_m M$ using the operators $\partial/\partial x^i$. We write

$$\{e_1, \ldots, e_n\} = \left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right\}$$

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- There is a one to one correspondence between vector fields X on M and the differential operators

$$X[f](x) = \mathbf{d}f(x) \cdot X(x)$$

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- We also have
$$X[f] = X^i \frac{\partial f}{\partial x^i}$$
which is why we write
$$X = X^i \frac{\partial}{\partial x^i}$$

Jacobi-Lie Bracket

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- Easiest to see what is going on by computing the commutator in coordinates:

Jacobi-Lie Bracket

- We get

$$
\begin{aligned}
[X, Y] &= \left[X^i \frac{\partial}{\partial x^i}, Y^j \frac{\partial}{\partial x^j}\right] \\
&= \left[X^i \frac{\partial}{\partial x^i} Y^j \frac{\partial}{\partial x^j} - Y^j \frac{\partial}{\partial x^j} X^i \frac{\partial}{\partial x^i}\right] \\
&= X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial}{\partial x^i} \\
&= \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}\right) \frac{\partial}{\partial x^j}
\end{aligned}
$$

- Note that the second derivative terms $\partial^2 / \partial x^i \partial x^j$ canceled out. This is the ***secret*** why the bracket is a differential operator and so defines a vector field.