CDS 202 Winter 2005 Final Exam

Problem 1 (Scribe: Nawaf Bou-Rabee)

(a) Let M be an n-manifold and X and Y be vector fields on M. Suppose that α is a closed k-form on M and that $\mathbf{i}_Y\alpha = 0$. Is it true that

$$\mathbf{i}_{[X,Y]} = -\mathbf{i}_Y \mathbf{d}\mathbf{i}_X \alpha?$$

(b) Let the vector fields X and Y on $\mathbb{R}^3$ be given by

$$\begin{aligned} X &= \frac{\partial}{\partial x} - z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} \\ Y &= x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}. \end{aligned}$$

Compute the flows of F_t of X and G_t of Y.

(c) Compute $[X, Y]$.

(d) Let $\alpha = xdy + ydx$ adn compute

$$\left.\frac{d}{dt}\right|_{t=0} F_t^*\alpha \quad \text{and} \quad \left.\frac{d}{dt}\right|_{t=0} G_t^*\alpha$$

(e) Is the identity in part (a) valid for Y in (b) and α in (d)?

Solution for (a) Yes this result follows from identity (e), $\mathbf{i}_Y\alpha = 0$ and α being closed.

$$\begin{aligned} \mathbf{i}_{[X,Y]} &= \pounds_X \mathbf{i}_Y \alpha - \mathbf{i}_Y \pounds_X \alpha \\ &= -\mathbf{i}_Y \pounds_X \alpha \quad \text{since } \mathbf{i}_Y\alpha = 0 \\ &= -\mathbf{i}_Y(\mathbf{d}\mathbf{i}_X\alpha + \mathbf{i}_X\mathbf{d}\alpha) \\ &= -\mathbf{i}_Y\mathbf{d}\mathbf{i}_X\alpha \quad \text{since } \alpha \text{ is closed} \end{aligned}$$

Solution for (b) The ode associated with X follows:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

with solution:

$$F_t(x_0, y_0, z_0) = (t + x_0, \cos t y_0 - \sin t z_0, \sin t y_0 + \cos t z_0)$$

Similarly the ode associated with Y follows:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

with solution:

$$G_t(x_0, y_0, z_0) = (e^t x_0, e^{-t} y_0, e^t z_0)$$

Solution for (c) This is straightforward.

$$\begin{aligned}
[X, Y] =& X[Y] - Y[X] \\
=& \left(\frac{\partial}{\partial x} - z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}\right)\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right) \\
& - \left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} - z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}\right) \\
=& \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial z} + 2z\frac{\partial}{\partial y}
\end{aligned}$$

Solution for (d) This follows from the Lie derivative formula and Cartan's magic formula.

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} F_t^* \alpha &= \pounds_X \alpha \\
&= \mathbf{d}\mathbf{i}_X \alpha + \mathbf{i}_X \mathbf{d}\alpha \\
&= \mathbf{d}\mathbf{i}_X \alpha \quad \text{since } \alpha \text{ is closed} \\
&= \mathbf{d}(y - zx) \\
&= \mathbf{d}y - x\mathbf{d}z - z\mathbf{d}x
\end{aligned}$$

Similarly

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} G_t^* \alpha &= \pounds_Y \alpha \\
&= \mathbf{d}\mathbf{i}_Y \alpha \\
&= \mathbf{d}(xy - yx) \\
&= 0
\end{aligned}$$

Solution for (e) Yes since α is closed and $\mathbf{i}_Y \alpha = 0$.

Problem 2 (Scribe: Nawaf Bou-Rabee)

Let M be the "partial ellipsoid" in $\mathbb{R}^3$ defined by the conditions

$$4x^2 + y^2 + z^2 = 4 \quad \text{and} \quad -1 \le x \le 0.$$

(a) Show that the full ellipsoid $4x^2 + y^2 + z^2 = 4$ is a smooth manifold.

(b) Argue informally that M is a smooth oriented manifold with boundary; describe a specific choice of orientation.

(c) Let the one form α be defined on the open set $U = \mathbb{R}^3 \backslash$x-axis by

$$\alpha = \frac{ydz - zdy}{y^2 + z^2}.$$

Compute $\mathbf{d}\alpha$ on U.

(d) Let β be the pull-back of α to the set V consisting of M minus the point $(-1, 0, 0)$. Is β closed? exact? Compute the integral $\int_{\partial M} \beta$. Is Stokes' theorem violated? Explain.

(e) Let the vector field on $\mathbb{R}^3$ defined by

$$X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}.$$

Show that X defines a vector field on M. Show that, with respect to the standard volume element on M, $\text{div}Y = 0$.

(f) Compute the Lie derivate $\pounds_Y \beta$ on V.

Solution for (a) To see this directly apply the submersion theorem as follows. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by:

$$f(x, y, z) = 4x^2 + y^2 + z^2$$

Note that

$$Df(x, y, z) \cdot (v_1, v_2, v_3) = (8x, 2y, 2z) \cdot (v_1, v_2, v_3)$$

$Df(x, y, z)$ is onto since,

$$(8x, 2y, 2z) \cdot (v_1, v_2, v_3) = c$$

has a solution given by $(2x, y, z)/(2(4x^2 + y^2 + z^2))c$ provided $x, y, z \neq 0$. Therefore f is a submersion and by submersion theorem $f^{-1}(4)$ is a submanifold of $\mathbb{R}^3$.

Or use the following diffeomorphism from S^2 to the ellipsoid given by $f(x, y, z) = (2x, y, z)$. Therefore, the ellipsoid is a manifold because S^2 is a manifold.

Solution for (b) One way to orient the manfiold is by using the outwarding point normal which is continuous on the manifold and induces a counterclockwise rotation on the boundary about the x-axis.

Solution for (c) We did this direct calculation in class (cf. lecture 14): $d\alpha = 0$.

Solution for (d) i^* commutes with $\mathbf{d}$ because i is smooth and therefore $\beta = i^*\alpha$ is closed since

$$\mathbf{d}\beta = \mathbf{d}i^*\alpha = i^*\mathbf{d}\alpha = 0$$

because α is closed from part (c).

However, β is not exact since on the one hand if it was exact $\beta = d\gamma$ and therefore $\int_C \beta = \int_C d\gamma = 0$ for any closed curve C; on the other hand, as shown below $\int_{\partial M} \beta = -2\pi$. This contradiction arises because the region which β is defined is not contractible.

The integral is computed by transforming to polar coordinates to obtain

$$\int_{\partial M} \beta = -\int_0^{2\pi} d\theta = -2\pi.$$

Stokes' theorem is not violated since the integrand is not defined on all of M.

Solution for (e) X is a vector field on M since it is parallel to M, i.e., $X[f] = 0$.

Moreover, the flow of X is a pure rotation about the x-axis which is volume-preserving on $\mathbb{R}^3$, and hence, Y (its restriction to M) is area-preserving. Therefore $\text{div}Y = 0$.

Solution for (f)

$$\begin{aligned}
\pounds_Y \beta &= \mathbf{d}\mathbf{i}_Y\beta + \mathbf{i}_Y\mathbf{d}\beta \quad \text{by Cartan} \\
&= \mathbf{d}\mathbf{i}_Y\beta \quad \text{since } \beta \text{ is closed} \\
&= d\left(\frac{y^2+z^2}{y^2+z^2}\right) \\
&= d(1) = 0
\end{aligned}$$

Problem 3 (Scribe: Nawaf Bou-Rabee)

Let G denote the set of 4×4 real matrices that have the block form

$$\begin{bmatrix} A & a \\ 0 & 1 \end{bmatrix}$$

where A is a 3×3 orthogonal matrix of determinant 1 and where $a \in \mathbb{R}^3$.

(a) Show that, with the operation of matrix multiplication, G is a Lie group that is isomorphic to the group of transformations of $\mathbb{R}^3$ to itself of the form $T_{A,a} : \mathbb{R}^3 \to \mathbb{R}^3$; $x \mapsto Ax + a$.

(b) Show that the Lie algebra of G, $\mathfrak{g}$ may be identified with the set of 4×4 matrix of the form $\begin{bmatrix} \hat{x} & y \\ 0 & 0 \end{bmatrix}$, where $\hat{x}$ is the 3×3 skew matrix associated with $x \in \mathbb{R}^3$ (satisfying $\hat{x} \cdot u = x \times u$ for all vectors $u \in \mathbb{R}^3$). Describe the Lie bracket operation.

(c) Let $\xi \in \mathfrak{g}$ and $\eta_1, \eta_2 \in \mathfrak{g}$ be defined by choosing for ξ, $x = (0,0,1)$, $a = 0$ and for η_1, $x = 0$, $a = (1,0,0)$ and for η_2, $x = 0$, $a = (0,1,0)$. Let $V \subset \mathfrak{g}$ be the vector subspace spanned by ξ, η_1, η_2. Let D be the distribution on G obtained by left translating V around the group. Is D integrable?

(d) Let $f : G \to \mathbb{R}$ be defined by $f(K) = \text{trace}(K^T K)$ for $K \in G$. Show that f is a smooth function and calculate its derivative at the identity element of G.

Solution for (a) G is a manifold since $G \sim SO(3) \times \mathbb{R}^3$ and $SO(3)$, $\mathbb{R}^3$ are manifolds. G is also a group since 1) G is closed under matrix multiplication

$$\begin{bmatrix} A_1 & a_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A_2 & a_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A_1 A_2 & A_1 a_2 + a_1 \\ 0 & 1 \end{bmatrix} \in G \quad \text{since} \quad A_1 A_2 \in SO(3) \quad \text{and} \quad A_1 a_2 + a_1 \in \mathbb{R}^3$$

2) G contains an identity element given explicitly by:

$$e = \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix}$$

and 3) G contains an inverse given explicity by:

$$\begin{bmatrix} A^{-1} & -A^{-1}a \\ 0 & 1 \end{bmatrix}$$

Moreover, G is isomorphic to $T_{A,a}$ since the map $f : G \mapsto T_{A,a}$ is clearly bijective and preserves the group structure because

$$\begin{aligned} f(A_1, a_1) f(A_2, a_2) &= T_{A_1,a_1} T_{A_2,a_2} \\ &= T_{A_1 A_2, A_1 a_2 + a_1} \\ &= f((A_1 A_2, A_1 a_2 + 1_1)) \\ &= f((A_1, a_1) \cdot (A_2, a_2)) \end{aligned}$$

and the same is true for the inverse of f.

Solution for (b) The tangent space at the identity of G is simply

$$\mathfrak{g} = T_e G = T_e(SO(3) \times \mathbb{R}^3) = so(3) \times \mathbb{R}^3$$

which may be identified with 4×4 matrices of the desired form. Since $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(4)$ the Lie bracket operation is the matrix commutator given explicitly by

$$\begin{aligned}[(\hat{x}, u), (\hat{y}, v)] &= \begin{bmatrix} \hat{x} & u \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{y} & v \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \hat{y} & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} & u \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \hat{x}\hat{y} - \hat{y}\hat{x} & \hat{x}v - \hat{y}u \\ 0 & 0 \end{bmatrix}\end{aligned}$$

Solution for (c) D is integrable since it is involutive as the following reveals. Observe that

$$X_\xi, X_{\eta_1}, \text{ and } X_{\eta_2} \in D$$

by definition. Also note that

$$[\xi, \eta_1] = \eta_2, \quad [\xi, \eta_2] = -\eta_1, \quad [\eta_1, \eta_2] = 0$$

Therefore,

$$X_{[\xi,\eta_1]} = X_{\eta_2}, \; X_{[\xi,\eta_2]} = X_{-\eta_1}, \text{ and } X[\eta_1, \eta_2] = 0 \in D$$

Hence D is involutive and by Frobenius' theorem D is integrable.

Solution for (d) f is smooth since it is the composition of the trace and matrix multiplication on G which are each smooth operations. Its derivative is given by:

$$Df(K) \cdot \delta = 2\text{trace}(\delta^T K)$$

which vanishes at the identity

$$Df(e) \cdot \delta = 2\text{trace}(\delta) = 0$$

since $Df : TG \to \mathbb{R}$ and elements of T_eG have zero trace.

Problem 4 (Scribe: Nawaf Bou-Rabee)

Let $M = S^2$ be the standard two sphere in $\mathbb{R}^3$ defined by $x^2 + y^2 + z^1 = 1$. Let X be the vector field on $\mathbb{R}^3$ defined by:

$$X = zy\frac{\partial}{\partial x} + zx\frac{\partial}{\partial y} - 2xy\frac{\partial}{\partial z}.$$

(a) Show that X defines a vector field, that we shall call Y, on M.

(b) Show that the flows of both Y and X are complete.

(c) Let C be the circle defined by $x^2 + y^2 = 1$ and $z = 0$. Let $C_t = F_t(C)$, where F_t is the flow of Y. Show that C_t is a smooth one dimensional submanifold of M.

(d) Let ω be the one form on $\mathbb{R}^3$ defined by $\omega = ydx - xdy + xydz$. Let γ be the pull-back of ω from $\mathbb{R}^3$ to M. Express $\mathbf{d}\gamma$ as the pull-back of a two form on $\mathbb{R}^3$ to S^2.

(e) Compute

$$\left.\frac{d}{dt}\right|_{t=0} \int_{C_t} \omega.$$

Solution for (a) Y is parallel to M since $X[f] = 0$.

Solution for (b) Y and X are C^∞ vector fields and evolve on spheres of fixed radius for all time. Since the flows generated by Y and X evolve on compact level-sets, Y and X are complete by a basic theorem from the class (Proposition 4.1.19).

Solution for (c) The flow of Y is a diffeomorphism on M since Y is complete, and therefore C and C_t are diffeomorphic. It follows that C_t is a smooth one-dimensional submanifold since C is.

Solution for (d)

$$\mathbf{d}\gamma = i^*\mathbf{d}\omega = \mathbf{d}i^*\omega = i^*(2dy \wedge dx + ydx \wedge dz + xdy \wedge dz)$$

Solution for (e)

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} \int_{C_t} \omega &= \left.\frac{d}{dt}\right|_{t=0} \int_{C} F_t^*\omega \\
&= \int_C \pounds_Y \omega \\
&= \int_C \mathbf{d}\mathbf{i}_Y\omega + \mathbf{i}_Y d\omega \\
&= \int_C \mathbf{d}(\mathbf{i}_Y\omega) \quad \text{since } d\omega = 0 \text{ because } \omega \text{ is a one-form on a one-manifold } C \\
&= 0 \quad \text{since } C \text{ is a closed curve}
\end{aligned}$$

Problem 5 (Scribe: Ling Shi)

(a) Let the tensor h on $\mathbb{R}^3$ be defined by

$$h = xdx \otimes dx - yzdx \otimes dz$$

Let the vectors u and v be based at the point $(1, 1, 1)$ and have components given by $u = (1, -1, 0)$ and $v = (0, -1, 2)$. Compute $h(u, v)$.

(b) Let the vector field X be defined by

$$X = x\frac{\partial}{\partial x} - z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}.$$

Show that the flow F_t of X is defined for all time.

(c) Compute

$$\left.\frac{d}{dt}\right|_{t=0} F_t^*(h \otimes dx)$$

Solution for (a) As

$$h(u,v) = \cdot\langle dx, u\rangle \cdot \langle dx, v\rangle - \langle dx, u\rangle \cdot \langle dz, v\rangle,$$

and we have

$$\langle dx, u\rangle = [1\ \ 0\ \ 0] \cdot [1\ \ -1\ \ 0]' = 1, \langle dx, v\rangle = 0, \langle dz, v\rangle = 2,$$

we obtain

$$h(u,v) = 0 - 2 = -2.$$

Solution for (b) We can write the system in ODE form as

$$\begin{aligned} \dot{x} &= x \\ \dot{y} &= -z \\ \dot{z} &= y, \end{aligned}$$

and hence we can easily find the flow to be:

$$F_t(x_0, y_0, z_0) = (x_0 e^t, y_0 \cos t - z_0 \sin t, y_0 \sin t + z_0 \cos t),$$

which is clearly defined for all t.

Solution for (c) By the Lie-derivative formula,

$$\left.\frac{d}{dt}\right|_{t=0} F_t^*(h \otimes dx) = \mathfrak{L}_X(h \otimes dx).$$

It is easy to compute the following:

$$\mathfrak{L}_X x = x, \mathfrak{L}_X dx = dx, \mathfrak{L}_X dz = dy, \mathfrak{L}_X yz = -z^2 + y^2.$$

Hence we obtain

$$\mathfrak{L}_X(h \otimes dx) = 4x dx \otimes dx \otimes dx + (z^2 - 2yz - y^2) dx \otimes dz \otimes dx - yz dx \otimes dy \otimes dx.$$

Problem 6 (Scribe: Ling Shi)

(a) (i) Let M be a smooth connected, oriented n-manifold with boundary and $f : M \to N$ a smooth map to a k-manifold N, $k \geq n$. Suppose that α is a closed $n-1$-form on N and let $\beta = f^*\alpha$. Show that $\int_{\partial M} \beta = 0$.

(ii) For β as in (a), show that $\mathbf{d}\beta$ must vanish somewhere in M.

(b) (i) Let S^3 denote the three sphere, the subset of $\mathbb{R}^4$ (with coordinates denoted (w, x, y, z)) defined by $w^2 + x^2 + y^2 + z^2 = 1$. Show that S^3 is a smooth manifold and describe its tangent space at a point (w, x, y, z).

(ii) Let β be the oneform on $\mathbb{R}^4$ defined by

$$\beta = xdw + wdx + wdz + zdw + ydz + zdy$$

and let γ be the pull-back of β to S^3 by the inclusion map $i : S^3 \to \mathbb{R}^4$. Show that γ is closed.

(iii) Let D be defined as the set of all tangent vectors u to S^3 that satisfy $\langle \gamma, u \rangle = 0$. Identity an open set U of S^3 on which D is a distribution (3-dimensional at each point of U). Is D integrable on U?

Solution for (a) (i)

$$\int_{\partial M} \beta = \int_M d\beta = \int_M df^*\alpha = \int_M f^* d\alpha = 0.$$

Note that the first equality is from Stoke's Theorem and the last one follows from the fact that α is closed and hence $d\alpha = 0$.

Solution for (a) (ii) As $\beta = f^*\alpha$ and f is smooth, α is a form and hence β is smooth. From (i),

$$\int_M d\beta = 0,$$

hence either $d\beta = 0$ all over M or $d\beta$ has both positive and negative parts, hence by the continuity of $d\beta$, we conclude that $d\beta$ must vanish somewhere.

Solution for (b) (i) Let $f : \mathbb{R}^4 \to \mathbb{R}$ be defined by $f(x) = ||x||^2$. Then it's clear that $S^3 = f^{-1}(1)$. The derivative of f is surjective onto $\mathbb{R}$ as $Df(x) \cdot v = 2x^T v$, and for any $a \in \mathbb{R}$, let $v = \frac{ax}{2||x||^2}$, we have $2x^T v = a$. Therefore by the submersion theorem, S^3 is a smooth manifold and the tangent space at a point $x \in S^3$ is given by

$$T_x S^3 = \ker Df(x) = \{v \in \mathbb{R}^4 : x^T v = 0\},$$

which is a three dimensional hyperplane in $\mathbb{R}^4$.

Solution for (b) (ii) It's easy to compute that $d\beta = 0$ and hence $d\gamma = di^*\beta = i^*d\beta = 0$, hence γ is closed.

Solution for (b) (iii) D is the set of all tangent vectors u to S^3 such that $\langle \gamma, u \rangle = 0$ which means that for any vector field X and Y in D, $i_X\gamma = i_Y\gamma = 0$. One needs to check that the kernel of γ is two dimensional at each point of an open set U in S^3, to ensure that D is a distribution on that set.

Thus we have

$$\begin{aligned} i_{[X,Y]}\gamma &= \mathfrak{L}_X i_Y \gamma - i_Y \mathfrak{L}_X \gamma \\ &= -i_Y \mathfrak{L}_X \gamma \\ &= -i_Y (d i_X \gamma + i_X d\gamma) \\ &= 0. \end{aligned}$$

Hence $[X, Y] \in D$, i.e. D is involutive, and by Frobenious' Theorem, D is integrable.