CDS 202 - Geometry of Nonlinear Systems

Winter 2003

Solution for Final Exam

April 2, 2003

This contains scanned solutions to this year's final exam. Solutions for 1 – 5 are from Asa Hopkins and for 6 from Yongqiang Liang.

Scanned solutions start on next page.

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CDS 202
Final Exam
3/16/2003

1)

a) $\mathcal{L}_X i_X \mu = d i_X i_X \mu + i_X d i_X \mu$ (Cartan's MF)

μ is a differential form, so $i_X i_X \mu = 0$

$\Rightarrow \mathcal{L}_X i_X \mu = i_X d i_X \mu$

$i_X \mathcal{L}_X \mu = i_X d i_X \mu + i_X i_X d\mu$ (Cartan's MF)

again, $i_X i_X d\mu = 0$

$\Rightarrow i_X \mathcal{L}_X \mu = i_X d i_X \mu = \mathcal{L}_X i_X \mu$ ✓

b)

$\dot{x} = y \qquad x(t) = x_0 \cos t + y_0 \sin t$

$\dot{y} = -x \qquad y(t) = -x_0 \sin t + y_0 \cos t$

$\dot{z} = 1 \qquad z(t) = z_0 + t$

$F_t(x_0, y_0, z_0) = (x_0 \cos t + y_0 \sin t, -x_0 \sin t + y_0 \cos t, z_0 + t)$ ✓

c) $\mu = dx \wedge dy \wedge dz$

$F_t^* \mu = (\cos t \, dx + \sin t \, dy) \wedge (-\sin t \, dx + \cos t \, dy) \wedge (dz)$

$= -\cos t \sin t \, \cancel{dx \wedge dx}^{\,0} \wedge dz + \sin t \cos t \, \cancel{dy \wedge dy}^{\,0} \wedge dz$
$+ \cos^2 t \, dx \wedge dy \wedge dz - \sin^2 t \, dy \wedge dx \wedge dz$

$= (\sin^2 t + \cos^2 t) \, dx \wedge dy \wedge dz = dx \wedge dy \wedge dz = \mu$

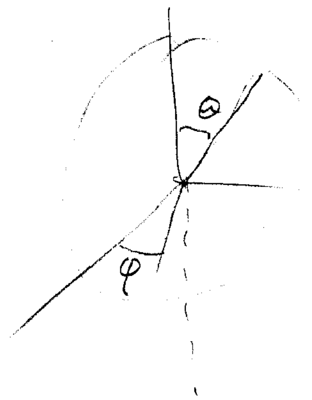
(as expected, since X is divergence-free) ✓

d) $\mathcal{L}_X \omega = \mathcal{L}_X i_X \mu = i_X \mathcal{L}_X \mu$ (by part a)

$$\mathcal{L}_X \mu = \frac{d}{dt}\Big|_{t=0} F_t^* \mu = \frac{d}{dt}\Big|_{t=0} \mu = 0$$

$\Rightarrow \mathcal{L}_X \omega = 0$ ✓

e) $\mu = \sin\theta \, d\theta \wedge d\varphi$



$\varphi : 0$ to 2π
$\theta : 0$ to π

- μ is a 2-form on a 2-manifold, $d\mu$ would be a 3-form, which is not allowed, so $d\mu = 0 \Rightarrow \mu$ is closed ✓

- if μ were exact, $\mu = d\omega$, then

$$0 = \int_{\partial S^2} \omega = \int_{S^2} \mu = 4\pi$$ ✓

(since ∂S^2 is nonexistent) (by Stokes's) (area of the sphere)

$\Rightarrow \mu$ is not exact

2) $\alpha = \dfrac{x\,dy - y\,dx}{x^2+y^2} + df$

$= \dfrac{x\,dy - y\,dx}{x^2+y^2} + 2x\left(\exp(x^2+y^2)\right)dx + 2y\left(\exp(x^2+y^2)\right)dy$

a) $d\alpha = \left(\dfrac{1}{x^2+y^2} - \dfrac{x(2x)}{(x^2+y^2)^2}\right)dx \wedge dy - \left(\dfrac{1}{x^2+y^2} - \dfrac{y(2y)}{(x^2+y^2)^2}\right)dy \wedge dx$

$+ 4xy\left(\exp(x^2+y^2)\right)dy \wedge dx + \cancel{4xy\left(\exp(x^2+y^2)\right)dx \wedge dy}$

✓ $= \dfrac{1}{(x^2+y^2)^2}\left[(x^2+y^2-2x^2)\,dx \wedge dy + \left[-(x^2+y^2)+2y^2\right]dy \wedge dx\right]$

$= \dfrac{1}{(x^2+y^2)^2}\left[(y^2-x^2)\,dx \wedge dy + (x^2-y^2)\,dx \wedge dy\right]$

$= 0$

$\Rightarrow$ the curl of the vector field

✓ $\left(\dfrac{x}{x^2+y^2} + 2y\,e^{x^2+y^2}\right)\dfrac{\partial}{\partial y} + \left(\dfrac{-y}{x^2+y^2} + 2x\,e^{x^2+y^2}\right)\dfrac{\partial}{\partial x}$

is zero

b) $\underline{d}\beta = \underline{d}\,i^*\alpha = i^*\,\underline{d}\alpha = 0 \Rightarrow \beta$ is closed

(-1) $\displaystyle\int_{\partial M}\beta = \int_M d\beta = 0$ (if we were to calculate $\int_{\partial M}\beta$ explicitly, the top and bottom edges would cancel)

give an example!

(β is not exact, since M is not contractable, and α blows up for precisely the points that can't be removed from being encircled, the z axis ($\mathbb{R}^3$ minus the z-axis isn't contractable)

($i: M \to \mathbb{R}^3$ is inclusion map)

c) $X = i^* Y$ where Y is the velocity vector field of curves on M, Y in M-only coordinates (θ, z)

(-1) would be $-\frac{\partial}{\partial \theta} + \frac{\partial}{\partial z}$

(X contains only vectors which lie in $T_{\underline{x}} M$ for $\underline{x} \in M$.)

d) $g = i_X \alpha = \alpha(X)$

✓ $= \left(\frac{-y}{x^2+y^2} + 2x\, e^{x^2+y^2}\right) y + \left(\frac{x}{x^2+y^2} + 2y\, e^{x^2+y^2}\right)(-x)$

$= \underline{-1}$

e) ✓ $\mathcal{L}_Y \beta = \mathcal{L}_{i^* X}\, i^* \alpha = i^* \mathcal{L}_X \alpha = i^*(d\, i_X \alpha + i_X\, d\alpha)$

$= i^*(d(-1) + i_X 0) = 0$

3) a) - G is a group:

group operation: matrix multiplication

$A \in G$, $B \in G$

$$ABK(AB)^T = ABKB^TA^T = AKA^T = K$$

$$\Rightarrow AB \in G$$

$I \in G$, so we have our identity element

if $A \in G$, then $A^{-1} \in G$

- G is a manifold:

I'll prove G is a submanifold of $GL(3,\mathbb{R})$

Let $f: GL(3,\mathbb{R}) \to$ symmetric matrices

$$f(A) = AKA^T$$

$$Df(A)\cdot B = BKA^T + AKB^T$$

is this onto?

for any $C \in$ symmetric matrices,

$$BKA^T + AKB^T = C \quad \text{if } B = \frac{CK^{-1}A}{2} \ \checkmark$$

$$\left(\frac{CK^{-1}AKA^T}{2} + \frac{AKA^TK^{-1}C^T}{2} = \frac{CK^{-1}K}{2} + \frac{KK^{-1}C^T}{2} = C\right)$$

(for $f(A)=K$, the level set we're interested in)

$\Rightarrow f^{-1}(K) = G$ is a submanifold of $GL(3,\mathbb{R})$

- Matrix multiplication is a linear map in GL, and is therefore C^∞, so the group op. is a C^∞ map and so its restriction on G is C^∞

$\Rightarrow G$ is a Lie Group

$\mathfrak{g}$, the Lie algebra, is the kernel of Df

$$KA^T + AK = 0$$

these are matrices of the form:

(G is a submanifold of $GL(n,\mathbb{R})$, so it inherits the commutator bracket for $\mathfrak{g}$)

$$\begin{bmatrix} 0 & -2a & 2b \\ a & 0 & c \\ b & c & 0 \end{bmatrix} \checkmark$$

G is 3 dimensional, since $\mathfrak{g}$ is 3 dimensional $\checkmark$

b) $\xi = \begin{bmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\eta = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$\xi\eta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ $\eta\xi = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix}$

$[\xi\ \eta] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \in \mathfrak{g}$, but not in span$(\xi\ \eta)$

($[\xi, \eta]$ is linearly independent of ξ and η,
$[\xi,\eta] \neq a\xi + b\eta$, $a, b \in \mathbb{R}$)

a) b)
-1

why is commutator the Lie bracket ?

c) D is not integrable. By Frobenius's thearem:

D is integrable iff $[X_\xi, X_\eta] \in D$ for $X_\xi, X_\eta \in D$. However, at the identity, $[X_\xi, X_\eta] = [\xi, \eta]$ which is not in the span of ξ and η (using ξ and η from part b).

$\Rightarrow D$ is not integrable

d) $f: G \to \mathbb{R}$, $f(A) = \text{trace } A$

- the trace is smooth on $GL(n,\mathbb{R})$, so it is smooth on a smooth submanifold of $GL(n,\mathbb{R})$, that is, G ✓
- $Df(I) = 0$ because

  $Df(I) = f(T_e G) = f(a)$, $a \in \mathfrak{g}$

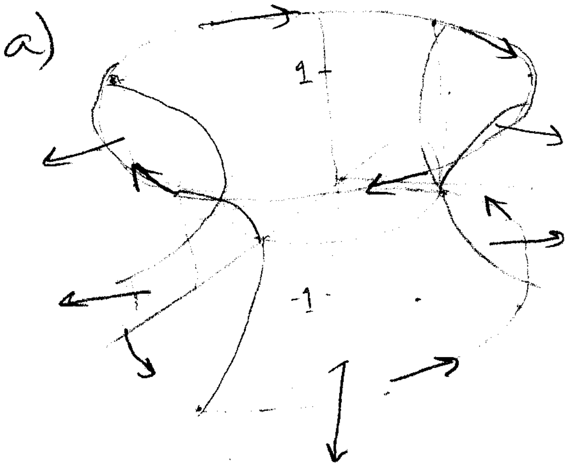
  The elements of $\mathfrak{g}$ are traceless, so $Df(I) = 0$.

✓

4) M: $x^2+y^2-z^2=1$, $-1\le z\le 1$

$S=\partial M$

a)



✓

orient M so the outward direction is towards increasing x^2+y^2

S then orients so as to be positive going in the $+\theta$ direction for $z=-1$ and $-\theta$ direction for $z=1$

b) $\alpha = P\,dx + Q\,dy + R\,dz$

$\underline{d}\alpha = 0$

$$\Rightarrow \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 0 \quad \text{(coefficient of } dx\wedge dy)$$

✓ $$\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} = 0 \quad (\text{'' '' } dx\wedge dz)$$

$$\text{and } \frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} = 0 \quad (\text{'' '' } dy\wedge dz)$$

$i: M \to \mathbb{R}^3$ inclusion

✓ $\beta = i^*\alpha$

$\underline{d}\beta = \underline{d}(i^*\alpha) = i^*\underline{d}\alpha = 0 \Rightarrow \beta$ is closed

no, β doesn't have to be exact (α could be ✓ the one from problem #2, for example)

c) $f: M \to \mathbb{R}$ Show $\int_M \underline{d} f \wedge \beta = \int_S f\beta$

$$\int_S f\beta = \int_M \underline{d}(f\beta) = \int_M \underline{d}f \wedge \beta + (-1)^0 f \wedge \cancel{\underline{d}\beta}^{\to 0}$$

✓

$$= \int_M \underline{d}f \wedge \beta$$

d) γ: smooth, closed ($\underline{d}\gamma = 0$), 2-form on $\mathbb{R}^3$ minus origin

$$\int_{S^2(r=2)} \gamma - \int_{S^2} \gamma = \int_{\substack{B(r=2) \\ (r=2 \text{ ball})}} \underline{d}\gamma - \int_{\substack{B \\ (\text{unit Ball})}} \underline{d}\gamma = \int_{r=1}^{r=2} \underline{d}\gamma = 0$$

✓ The difference between the 2 surface integrals is equal to the integral over the volume between the surfaces of $\underline{d}\gamma$, which is zero

5) b) i) $X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$

$g = dx\otimes dx + (x^2+y^2)\, dx\otimes dy + dy\otimes dy$

$i_X g(Y) = g(X,Y)$

$\Rightarrow i_X g = y\, dx + y(x^2+y^2)\, dy - x\, dy$

$= y\, dx + [y(x^2+y^2) - x]\, dy$ ✓

ii) $(\mathcal{L}_X g)_{11} = y\frac{\partial}{\partial x}(1) - \cancel{x\frac{\partial}{\partial y}(1)} + \frac{\partial}{\partial z}(1)$

$+ 1\cancel{\frac{\partial}{\partial x}(y)} + (x^2+y^2)\frac{\partial}{\partial x}(-x) + \cancel{1\frac{\partial}{\partial x}(y)}$

$+ 0 \quad = -(x^2+y^2)$

[I'm using: $(\mathcal{L}_X g)_{ij} = X^k\frac{\partial g_{ij}}{\partial x^k} + g_{ik}\frac{\partial X^k}{\partial x^j} + g_{kj}\frac{\partial X^k}{\partial x^i}$]

$(\mathcal{L}_X g)_{12} = y\frac{\partial}{\partial x}(x^2+y^2) - x\frac{\partial}{\partial y}(x^2+y^2)$

$+1\frac{\partial}{\partial y}y + (x^2+y^2)\frac{\partial}{\partial y}(-x) + (x^2+y^2)\frac{\partial}{\partial x}(y)$

$+1\frac{\partial}{\partial x}(-x)$

$= 2xy - 2xy + 1 - 1 = 0$

$(\mathcal{L}_X g)_{21} = 1\frac{\partial}{\partial x}(-x) + 1\frac{\partial}{\partial y}(y) = 0$

$(\mathcal{L}_X g)_{22} = y\frac{\partial}{\partial x}(1) - x\cancel{\frac{\partial}{\partial y}}(1) + 1\frac{\partial}{\partial y}(-x) + (x^2+y^2)\frac{\partial}{\partial y}(y)$

$+1\frac{\partial}{\partial y}(-x) = x^2+y^2$

rest $= 0$ $(g_{13}=g_{31}=g_{23}=g_{32}=g_{33}=0)$

$$\Rightarrow \mathcal{L}_X g = -(x^2+y^2)\, dx \otimes dx + (x^2+y^2)\, dy \otimes dy \checkmark$$

a) show that

$$\int_{S^2} i_X d\beta = \frac{d}{dt}\bigg|_{t=0} \int_{S_t^2} \beta$$

$$\int_{S^2} i_X d\beta = \int_{S^2} \mathcal{L}_X \beta - d i_X \beta = \int_{S^2} \mathcal{L}_X \beta - \int_{\partial S^2} i_X \beta$$

$$= \int_{S^2} \mathcal{L}_X \beta$$ since S^2 doesn't have a boundary

$$= \int_{S^2} \frac{d}{dt}\bigg|_{t=0} F_t^* \beta = \frac{d}{dt}\bigg|_{t=0} \int_{S^2} F_t^* \beta$$

$$= \frac{d}{dt}\bigg|_{t=0} \int_{F_t(S^2)} \beta = \frac{d}{dt}\bigg|_{t=0} \int_{S_t^2} \beta \quad \checkmark$$

$\checkmark$

6. a). $Pu + Qv + Rw = 0$ implies

$i_{(u,v,w)}\beta = 0$ for $(u,v,w) \in D$.

if $X, Y \in D$. then by Frobenius' theorem,

D integrable $\Leftrightarrow [X, Y] \in D$

$\Leftrightarrow i_{[X,Y]}\beta = 0$

$\Leftrightarrow i_{[X,Y]}\beta = \mathcal{L}_X \overset{\to 0}{i_Y\beta} - i_Y \mathcal{L}_X\beta = i_Y \mathcal{L}_X \beta = 0$

$\Leftrightarrow i_Y(\cancel{d i_X \beta}^{\to 0} + i_X d\beta) = 0$

$\Leftrightarrow i_Y i_X d\beta = 0 \quad \forall X, Y \in D$

For example, such a β could be $\beta = d\gamma$.

$i_Y i_X d\beta = 0$ since $d\beta = 0$

b)

if $Y \in D$. then $i_Y \beta = 0$

$F_t^* i_Y \beta = i_{F_t^* Y} F_t^* \beta$

since $\frac{d}{dt}(F_t^*\beta) = F_t^* \mathcal{L}_X \beta = 0$

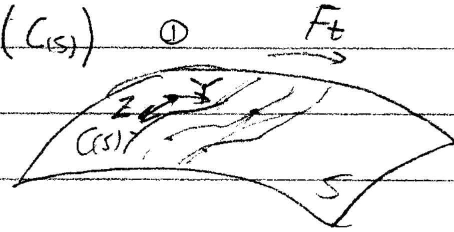
$\therefore F_t^* \beta = \beta$

i.e. $i_{F_t^* Y}\beta = F_t^* i_Y \beta = 0$

$\Rightarrow F_t^* Y \in D$

i.e. D is invariant under F_t.

a) $\frac{d}{dt} F_t(c(s)) = Y(F_t(c(s))) = F_{t*} Y(c(s))$ ①



$\frac{d}{ds} F_t(c(s)) = F_{t*}(Z(c(s)))$ ②

not terribly clear

-2

by definition of flow & integral curve of v.f.

So at each point $F_t(c(s))$, TS is spanned by

$F_{t*}(Y_{c(s)})$ & $F_{t*}(Z(c(s)))$, which are l.i.

$\Rightarrow$ TS is a section of D.

$\therefore$ S is an integral manifold of D