CDS 202 Final Examination

J. Marsden, March, 2003

Attempt ***five*** *of the following six questions.*

This exam has ***four*** *pages including this cover page.*

The exam time limit is three hours; ***no aids are permitted.***

The exam must be turned in by noon on Friday, March 21.

Print Your Name: ______________________	←**Note!**
The 5 questions to be graded: ______________________	←**Note!**

You may freely use the following properties as needed. Here α and β are differential forms and X, Y, Z are vector fields on a manifold M. (In the exam, all manifolds, vector fields, and differential forms are assumed to be smooth and the manifolds are finite dimensional.)

(a) $\pounds_X(\alpha \wedge \beta) = (\pounds_X \alpha) \wedge \beta + \alpha \wedge (\pounds_X \beta)$

(b) $\pounds_{[X,Y]}\alpha = \pounds_X \pounds_Y \alpha - \pounds_Y \pounds_X \alpha$

(c) $\mathbf{i}_X(\alpha \wedge \beta) = (\mathbf{i}_X \alpha) \wedge \beta + (-1)^k \alpha \wedge (\mathbf{i}_X \beta)$, where α is a k-form.

(d) $\pounds_X \alpha = \mathbf{d}\mathbf{i}_X \alpha + \mathbf{i}_X \mathbf{d}\alpha$

(e) $\mathbf{i}_{[X,Y]}\beta = \pounds_X \mathbf{i}_Y \beta - \mathbf{i}_Y \pounds_X \beta$

(f) For γ a one-form,

$$\mathbf{d}\gamma(X, Y) = X[\gamma(Y)] - Y[\gamma(X)] - \gamma([X, Y])$$

(g) For ω a two-form,

$$\begin{aligned}\mathbf{d}\omega(X, Y, Z) = X[\omega(Y, Z)] - Y[\omega(X, Z)] + Z[\omega(X, Y) \\ - \omega([X, Y], Z) - \omega([Z, X], Y) - \omega([Y, Z], X)\end{aligned}$$

(h) For a one form α and a vector field X,

$$(\pounds_X \alpha)_i = X^j \frac{\partial \alpha_i}{\partial x^j} + \alpha_j \frac{\partial X^j}{\partial x^i}$$

1. (a) Let M be an n-manifold and suppose that X is a vector field on M and μ is a volume form on M. Show that

$$\pounds_X \mathbf{i}_X \mu = \mathbf{i}_X \pounds_X \mu$$

(b) Let the vector field X on $\mathbb{R}^3$ be given by

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

Compute the flow F_t of X.

(c) Letting $\mu = dx \wedge dy \wedge dz$, and letting F_t be the flow found in (b), compute $F_t^* \mu$.

(d) Consider the two form ω on $\mathbb{R}^3$ given by $\omega = \mathbf{i}_X(dx \wedge dy \wedge dz)$, where X is the vector field given in (b). Compute $\pounds_X \omega$.

(e) Show that the standard volume form μ on the unit two sphere in $\mathbb{R}^3$ is closed but is not exact.

2. Let $M = C^2$ be the cylinder with boundary in $\mathbb{R}^3$ defined by the conditions $x^2 + y^2 = 1$ and $0 \leq z \leq 1$. Let α be the one form on $\mathbb{R}^3$ minus the z-axis defined by

$$\alpha = \frac{xdy - ydx}{x^2 + y^2} + df$$

where $f(x, y, z) = \exp(x^2 + y^2)$. Also, let the vector field X on $\mathbb{R}^3$ be given by

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

(a) Compute $\mathbf{d}\alpha$ and relate your result to the computation of the curl of a certain vector field in $\mathbb{R}^3$.

(b) Let $i : M \to \mathbb{R}^3$ be the inclusion and let $\beta = i^*\alpha$. Is β closed? exact? Compute the integral $\int_{\partial M} \beta$.

(c) In what sense does X define a vector field Y on M?

(d) Compute the function $g = \mathbf{i}_X \alpha$.

(e) Compute $\pounds_Y \beta$.

3. Let G denote the set of 3×3 real matrices A satisfying the condition $AKA^T = K$, where K is the 3×3 diagonal matrix with diagonal entries $1, 2, -2$.

(a) Show that G is a Lie group; determine its Lie algebra $\mathfrak{g}$, and find the dimension of G.

(b) Find, explicitly, two elements ξ and η in the Lie algebra $\mathfrak{g}$ such that the Lie bracket $[\xi, \eta]$ is not in the span of ξ and η.

(c) Let X_ξ and X_η be the left invariant vector fields on G equaling ξ and η at the identity and let D be the distribution that is spanned by X_ξ and X_η. Is D and integrable distribution?

(d) Let $f : G \to \mathbb{R}$ be defined by $f(A) = \operatorname{trace} A$. Show that f is a smooth function and compute the derivative of f at the identity element of G.

4. Let M be the hyperboloid with boundary in $\mathbb{R}^3$ defined by the conditions $x^2 + y^2 - z^2 = 1$ and $-1 \le z \le 1$. Let $S = \partial M$ be its boundary.

(a) Give M and S consistent orientations; illustrate with a figure.

(b) Let α be a smooth closed one form defined on $\mathbb{R}^3$ minus the z-axis and write α as

$$\alpha = P\,dx + Q\,dy + R\,dz$$

Write out what it means for α to be closed in terms of P, Q, R. Let β be the pull back of α to M. Show that β is closed. Must it be exact?

(c) Let $f : M \to \mathbb{R}$ be a given smooth real valued function. Show that for β as above,

$$\int_M \mathbf{d}f \wedge \beta = \int_S f\beta$$

(d) Let γ be a smooth closed two-form on $\mathbb{R}^3$ minus the origin. Show that the integral of γ over the unit two sphere S^2 defined by $x^2 + y^2 + z^2 = 1$ equals the integral of γ over the two sphere of radius 2 defined by $x^2 + y^2 + z^2 = 4$.

5. (a) Let X be a complete smooth vector field on $\mathbb{R}^3$ with flow F_t. Let S^2 be the unit sphere in $\mathbb{R}^3$ and let $S_t^2 = F_t(S^2)$. Let β be a smooth two form on $\mathbb{R}^3$. Show that

$$\int_{S^2} \mathbf{i}_X \mathbf{d}\beta = \left.\frac{d}{dt}\right|_{t=0} \int_{S_t^2} \beta.$$

(b) Let X be the vector field on $\mathbb{R}^3$ defined by

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + \frac{\partial}{\partial z},$$

and let g denote the tensor field

$$g = dx \otimes dx + (x^2 + y^2)\, dx \otimes dy + dy \otimes dy.$$

i. Compute the interior product $\mathbf{i}_X g$.

ii. Compute the Lie derivative $\pounds_X g$.

6. Let the one form β on $\mathbb{R}^3$ be written as $\beta = P\,dx + Q\,dy + R\,dz$ and suppose that β is nonzero at every point. Let D be the pointwise two-dimensional distribution defined at the point (x, y, z) to be the set of vectors with components (u, v, w) such that

$$Pu + Qv + Rw = 0$$

(a) Find necessary and sufficient conditions involving the two-form $\mathbf{d}\beta$ for D to be integrable. Give an explicit example of such a β.

(b) Let X be a vector field such that $\pounds_X \beta = 0$. Show that, in an appropriate sense, the distribution D is invariant under the flow F_t of X.

(c) Let Y and Z be two (complete) linearly independent vector fields taking values in D and suppose that D is integrable. Let F_t be the flow of Y and let $c(s), 0 < s < 1$ be an integral curve of Z. Let S be the the collection of points of the form $F_t(c(s))$ for $0 < t < 1$ and $0 < s < 1$. Show that S is an integral manifold of D.