CDS 202 - Geometry of Nonlinear Systems
Winter 2003

Solution for Sample Final Exam (Winter 2002 Final)

March 14, 2003 ; Updated March 14, 4 p.m

This contains solutions to most of the 2002 Final. Solutions for 1, 2(a), 3 and 4 are from this year's scribe. In addition there are scanned solutions for all problems except Problem 2. These are from a student's solution from last year.

Problem 1 *(Solution scribe : Ather Gattami, Winter 2003)*

(i) Let's first calculate the Lie-derivative for each component:

$$\begin{aligned}
\pounds_X(g_{11}) &= x\frac{\partial}{\partial}g_{11} + y\frac{\partial}{\partial y}g_{11} = 4x^2 + 0 = 4x^2 \\
\pounds_X(g_{12}) &= x\frac{\partial}{\partial x}x + 0 = x \\
\pounds_X(g_{21}) &= x \\
\pounds_X(g_{22}) &= 0 + y\frac{\partial}{\partial y}(1+y^2) = 2y^2 \\
\pounds_X(dx) &= d\pounds_X(x) = dx \\
\pounds_X(dy) &= d\pounds_X(y) = d(0 + y\frac{\partial}{\partial y}y) = dy
\end{aligned}$$

Hence,

$$g = (8x^2+2)dx \otimes dx + 3xdx \otimes dy + 3xdy \otimes dx + (4y^2+2)dy \otimes dy$$

(ii) The vector field is $(X^1, X^2) = (x, y)$. The flow F_t is the solution to the system of equations

$$\left\{ \begin{array}{rcl} \dot{x} &=& x \\ \dot{y} &=& y \end{array} \right. .$$

Thus, $F_t = (xe^t, ye^t)$.

(iii) According to the Lie-derivative theorem, the expression to be calculated is simply $\pounds_X(g)$.

Problem 2 - Part a *(Solution scribe : Tosin Otitoju, Winter 2003)*

Solution for (i)

$$\alpha = -\tfrac{y}{x^2+y^2}dx + \tfrac{x}{x^2+y^2}dy + dz$$

Let

$$F_1(x,y,z) = -\tfrac{y}{x^2+y^2};\ F_2(x,y,z) = \tfrac{x}{x^2+y^2};\ F_3(x,y,z) = 1$$

Then

$$\alpha = F_1 \ dx + F_2 \ dy + F_3 \ dz$$

Taking the exterior derivative:

$$\begin{aligned}
\mathbf{d}\alpha &= \mathbf{d}(F_1 \ dx) + \mathbf{d}(F_2 \ dy) + \mathbf{d}(F_3 \ dz) \\
&= \mathbf{d}F_1 \wedge dx + \mathbf{d}F_2 \wedge dy + \mathbf{d}F_3 \wedge \ dz \\
&= (\tfrac{\partial F_2}{\partial x} - \tfrac{\partial F_1}{\partial y})dx \wedge dy + (\tfrac{\partial F_3}{\partial x} - \tfrac{\partial F_1}{\partial z})dx \wedge dz + (\tfrac{\partial F_3}{\partial y} - \tfrac{\partial F_2}{\partial z})dy \wedge dz \\
&= 0
\end{aligned}$$

Since the Curl $F = (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}) \ \mathbf{i} + (\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}) \ \mathbf{j} + (\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}) \ \mathbf{k}$

Then it is clear that Curl $F = 0$.

Solution for (ii) $\beta = i^*\alpha$

$$\mathbf{d}\beta = \mathbf{d}(i^*\alpha) = i^*(\mathbf{d}\alpha) = 0$$

so β is closed.

Problem 3 *(Solution Scribe Jonathan Pritchard, Winter 2003)*
Let O(4) denote the set of 4×4 real orthogonal matrices.

(a) Show that O(4) is a manifold and a Lie group; what is its dimension? Is it connected?

(b) Show that its Lie algebra consists of 4×4 skew matrices

(c) Let ξ and η be the Lie algebra elements

$$\xi = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \eta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Compute the Lie algebra bracket $[\xi, \eta]$.

(d) Let X_ξ and X_η be the left invariant fields on O(4) equaling ξ and η at the identity and let D be the distribution that is spanned by X_ξ and X_η. Is D integrable?

Solution for (a) Consider $GL(n, \mathbb{R})$, which we know is a Lie group. We seek to show that O(4) is a closed subgroup of $GL(n, \mathbb{R})$. If successful we know that O(4) is then a regular Lie subgroup and so a manifold and a Lie group in its own right.

First recall the definition of O(4) in term of the standard inner product of matrices. For $A, B \in O(4)$,

$$< Ax, Ay > = < x, y >, \forall x, y \in \mathbb{R}^4$$

We note that the standard matrix product AB lies within $O(4)$

$$< ABx, ABy > = < Bx, By > = < x, y >$$

and so conclude that $O(4)$ forms a group under standard matrix multiplication.

To show that $O(4)$ is closed consider the smoothly convergent sequence

$$A_n \rightarrow A.$$

If $A_n \in O(4)$ satisfies

$$< A_n x, A_n y > = < x, y >,$$

then by continuity of $< \cdot, \cdot >$ we must have

$$< Ax, Ay > = < x, y > .$$

From this we conclude that $A \in O(4)$ and so $O(4)$ is closed.

Invoking the deep property that a closed subgroup of a Lie group is a Lie subgroup we have that $O(4)$ is a Lie Group and so a manifold.

Alternative solution to (a) Work to show that $O(4)$ is a Lie group directly from the definition. We must show

(i) $O(4)$ is a group

(ii) $O(4)$ is a manifold

(iii) The group operation is smooth

We have already shown that $O(4)$ is a group. In order to show (ii) we seek to invoke the submersion theorem to show that $O(4)$ is a submanifold and so a manifold. First we show that $O(4)$ is the level set of some function.

Define $O(4) = \{A \in 4 \times 4 \text{ matrices } | AA^T = I\}$. Consider then the function

$$f : GL(4, \mathbb{R}) \to Sym(4, \mathbb{R}), f(A) = AA^T - I,$$

where $Sym(n, \mathbb{R})$ is the set of real, symmetric 4×4 matrices. We then have that $O(4) = f^{-1}(0)$, a level set. Next we must show that $f(A)$ is a submersion. The tangent map is

$$\mathbf{D}f(A) \cdot B = AB^T + BA^T.$$

To show that this is surjective $\forall A \in O(4)$ we need $C \in Sym(4, \mathbb{R})$ s.t. $AB^T + BA^T = C$. Notice that a solution is

$$B = \frac{CA}{2}.$$

So $T_A f$ is surjective, f is a submersion and so by the Submersion Theorem $O(4)$ is a manifold.

Finally we recall that standard matrix multiplication is a bilinear mapping

$$GL(n, \mathbb{R}) \times GL(n, \mathbb{R}) \to GL; A, B \mapsto AB.$$

This is a smooth mapping so the restriction of this mapping to $O(4)$ is also smooth.

The Lie algebra of $O(4)$ consists of skew-symmetric 4×4 matrices. The dimension is the number of independent components of these matrices.

$$dim = \frac{4(4-1)}{2} = 6$$

$O(4)$ is not connected. Considering the determinant map we see that $det(A) = \pm 1$. Examples of A for these two cases are the identity and the identity with the sign of one diagonal element flipped. The determinant map is continuous so if it takes only two discrete values by invocation of the intermediate value theorem we see that $O(4)$ must contain two disconnected parts.

Solution for (b) Consider the tangent space at the identity. For a curve, A(t) through the identity (A(0)=I) we have by the definition of $O(4)$

$$A(t)A(t)^T = I.$$

We take the derivative of this along the curve

$$\frac{dA}{dt}A^T + A\frac{dA^T}{dt} = 0.$$

Now evaluate at t=0

$$\frac{dA}{dt}I + I\frac{dA^T}{dt} = 0.$$

So at the identity

$$\frac{dA}{dt} = -\frac{dA^T}{dt}.$$

Hence $dA/dt|_{t=0}$ must be skew-symmetric. As $O(4)$ is a manifold these curves passing through the identity must represent the tangent space and so we conclude that the Lie Algebra of $O(4)$ is skew-symmetric.

Alternative solution for (b) Using the function $f(A)$ from (a) and the submersion theorem we have that the tangent space at the identity is given by

$$\begin{aligned} T_I O(4) &= T_I f^{-1}(0) \\ &= ker T_I f. \end{aligned}$$

Evaluating this we see that

$$ker(\mathbf{D}f) = ker(AB^T + BA^T).$$

Setting $A = I$ the right hand side becomes

$$ker(B + B^T).$$

So B and thus the Lie Algebra must be skew-symmetric.

Solution for (c) The Lie bracket in $GL(n, \mathbb{R})$ is the matrix commutator, this is preserved under restriction to $O(4)$. So the Lie bracket is

$$[\xi, \eta] = \xi\eta - \eta\xi.$$

Evaluating this yields

$$[\xi, \eta] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Solution for (d) First note that at each point dim D=2, ie the span of two vectors. To get a good distribution we require that X_ξ and X_η are linearly independent at each point. This is clearly true at the identity, elsewhere we may apply a left translation to obtain the vectors at a new point. Left translation is a diffeomorphism mapping basis to basis. It must be the case then that the two vectors are linearly independent everywhere in $O(4)$.

Now we seek to apply Frobenius' theorem which states that the distribution will be integrable if it is involutive. D is involutive if

$$\forall X, Y \in D, [X, Y] \in D.$$

Check the two vector fields we're given which span D and note that it will be enough to check at the identity by the above argument. At the identity

$$[X_\xi, X_\eta] = [\xi, \eta],$$

but from the result of (c) this is not expressible as a linear combination of ξ and η. Hence we conclude that D is neither involutive nor integrable.

Problem 4 *(Solution Scribe Melvin E. Flores, Winter 2003)*

Let H denote the upper hemisphere in $\mathbb{R}^3$ defined by

$$H = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \text{ and } z \geq 0\}$$

and let $S = \partial H$ be its boundary.

a) *Give H and S consistent orientations; illustrate with a figure.*

For this recall the picture drawn in class the boundary going counterclockwise and the normal vector point outward from the surface.

b) *Let α and β be one forms on H and let X be a vector field on H. Is it true that*

$$\int_H (\pounds_X \alpha) \wedge \beta - \int_{\partial H} (\mathbf{i}_X \alpha) \wedge \beta = \int_H (\pounds_X \beta) \wedge \alpha - \int_{\partial H} (\mathbf{i}_X \beta) \wedge \alpha?$$

Combining terms under common integrals:

$$\int_H (\pounds_X \alpha) \wedge \beta - \int_H (\pounds_X \beta) \wedge \alpha \stackrel{?}{=} \int_{\partial H} (\mathbf{i}_X \alpha) \wedge \beta - \int_{\partial H} (\mathbf{i}_X \beta) \wedge \alpha$$

$$\int_H (\pounds_X \alpha) \wedge \beta - (\pounds_X \beta) \wedge \alpha \stackrel{?}{=} \int_{\partial H} (\mathbf{i}_X \alpha) \wedge \beta - (\mathbf{i}_X \beta) \wedge \alpha$$

Using $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$ on the terms $(\pounds_X \beta) \wedge \alpha$ and $(\mathbf{i}_X \beta) \wedge \alpha$ we can re-write the integral as follows:

$$\int_H (\pounds_X \alpha) \wedge \beta + \alpha \wedge (\pounds_X \beta) \stackrel{?}{=} \int_{\partial H} (\mathbf{i}_X \alpha) \wedge \beta - \alpha \wedge (\mathbf{i}_X \beta)$$

Note that

$$(\pounds_X \alpha) \wedge \beta + \alpha \wedge (\pounds_X \beta) = \pounds_X (\alpha \wedge \beta) \text{ and } (\mathbf{i}_X \alpha) \wedge \beta - \alpha \wedge (\mathbf{i}_X \beta) = \mathbf{i}_X (\alpha \wedge \beta)$$

Therefore, we can re-write the integrals more compactly as follows:

$$\int_H \pounds_X (\alpha \wedge \beta) \stackrel{?}{=} \int_{\partial H} \mathbf{i}_X (\alpha \wedge \beta)$$

Using Cartan's magic formula $\pounds_X \gamma = \mathbf{d}\mathbf{i}_X \gamma + \mathbf{i}_X \mathbf{d}\gamma$ we can write the integral on the left as follows:

$$\int_H \pounds_X (\alpha \wedge \beta) = \int_H \mathbf{d}\mathbf{i}_X (\alpha \wedge \beta) + \int_H \mathbf{i}_X \mathbf{d}(\alpha \wedge \beta)$$

Note that the last term on the right is zero because the manifold in consideration here is two-dimensional, $(\alpha \wedge \beta)$ is a two form and $\mathbf{d}(\alpha \wedge \beta)$ is a three-form. Consequently we have that

$$\int_H \pounds_X(\alpha\wedge\beta) = \int_H \mathbf{d}\mathbf{i}_X(\alpha\wedge\beta)$$

Using Stokes' theorem

$$\int_H \pounds_X(\alpha\wedge\beta) = \int_H \mathbf{d}\mathbf{i}_X(\alpha\wedge\beta) = \int_{\partial H} \mathbf{i}_X(\alpha\wedge\beta)$$

therefore, the relation

$$\int_H \pounds_X(\alpha\wedge\beta) = \int_{\partial H} \mathbf{i}_X(\alpha\wedge\beta)$$

is true.

c) *Let α be the one form on $\mathbb{R}^3$ defined by*

$$\alpha = \tfrac{x\ dy - y\ dx}{x^2+y^2} + dz$$

a let β be α pulled back to H. Compute

$$\int_H \mathbf{d}\beta$$

The one form α is not continuously differentiable in the region of interest so we may question its integrability. One solution

Let $\beta = i^*\alpha$ then

$$\int_H \mathbf{d}\beta = \int_H \mathbf{d}(i^*\alpha)$$

Since $i^* \circ \mathbf{d} = \mathbf{d} \circ i^*$

$$\int_H \mathbf{d}\beta = \int_H \mathbf{d}(i^*\alpha) = \int_H i^*(\mathbf{d}\alpha)$$

Since $\mathbf{d}\alpha = 0$ from Problem 2 - Part a then

$$\int_H \mathbf{d}\beta = \int_H \mathbf{d}(i^*\alpha) = \int_H i^*(\mathbf{d}\alpha) = 0$$

The other interpretation of $\int_H \mathbf{d}\beta$ gives the answer 2π :

$$\int_H \mathbf{d}\beta = \int_{\partial H} \beta = \int_C \frac{xdy - ydx}{x^2+y^2} = 2\pi$$

where C is the unit circle in the x-y plane. This inconsistency is a sign that something is wrong. Indeed what is wrong is the singularity at origin.

10/10

1. ⓐ $\mathcal{L}_{x\frac{\partial}{\partial x}} g = \mathcal{L}_{x\frac{\partial}{\partial x}} g_{11} dx\otimes dx + \mathcal{L}_{x\frac{\partial}{\partial x}} g_{12}\, dx\otimes dy + \mathcal{L}_{x\frac{\partial}{\partial x}} g_{21} dy\otimes dx + \mathcal{L}_{x\frac{\partial}{\partial x}} g_{22}\, dy\otimes dy$

$= (4x^2 dx\otimes dx + (1+2x^2) dx\otimes dx + (1+2x^2) dx\otimes dx)$

$+ (x dx\otimes dy + x dx\otimes dy) + (x dy\otimes dx + x dy\otimes dx) + 0$

$= (2+8x^2) dx\otimes dx + 2x\, dx\otimes dy + 2x\, dy\otimes dx$

$\mathcal{L}_{y\frac{\partial}{\partial y}} g = \mathcal{L}_{y\frac{\partial}{\partial y}} g_{11} dx\otimes dx + \mathcal{L}_{y\frac{\partial}{\partial y}} g_{12} dx\otimes dy + \mathcal{L}_{y\frac{\partial}{\partial y}} g_{21} dy\otimes dx + \mathcal{L}_{y\frac{\partial}{\partial y}} g_{22}\, dy\otimes dy$

$= 0 + x\, dx\otimes dy + x dy\otimes dx$

$+ (2y^2 dy\otimes dy + (1+y^2) dy\otimes dy + (1+y^2) dy\otimes dy)$

$= x dx\otimes dy + x\, dy\otimes dx + (2+4y^2) dy\otimes dy$

$\Rightarrow \mathcal{L}_X g = \mathcal{L}_{x\frac{\partial}{\partial x}} g + \mathcal{L}_{y\frac{\partial}{\partial y}} g$

$= (2+8x^2) dx\otimes dx + 3x dx\otimes dy + 3x dy\otimes dx + (2+4y^2) dy\otimes dy$ ✓

ⓑ $\begin{cases} \dot{x} = x \\ \dot{y} = y \end{cases} \Rightarrow F_t(x,y) = (xe^t, ye^t)$ ✓

ⓒ By Lie Derivative Theorem,

$\frac{d}{dt} F_t^* g = F_t^* \mathcal{L}_X g$

$\Rightarrow \frac{d}{dt}\Big|_{t=0} F_t^* g = F_0^* \mathcal{L}_X g$

$= \mathcal{L}_X g$, since F_0 is identity map.

$= (2+8x^2) dx\otimes dx + 3x dx\otimes dy + 3x dy\otimes dx + (2+4y^2) dy\otimes dy$ ✓

10/10

3. ⓐ let $A, B, C \in O(4)$, define group multiplication to be matrices multiplication $\mu: O(4) \times O(4) \to O(4)$ with $(A,B) \mapsto AB$

we have
$$\begin{cases} (AB)(AB)^T = ABB^TA^T = AA^T = I \quad \Rightarrow AB \in O(4) \\ (AB)C = A(BC) \text{ by the property of matrices multiplicati} \\ A \cdot A^{-1} = I, \quad IA = A \\ I \text{ is identity element} \end{cases}$$

thus $O(4)$ is a group.

Also, let $S(4)$ be 4×4 symmetric matrix, and let

$f: GL(4,R) \to S(4)$ with $A \mapsto AA^T$, we have

$Df(A) \cdot D = AD^T + DA^T$, which is surjective since

$Df(A) \cdot D = S \in S(4)$ is solvable with $D = \frac{SA}{2}$.

Thus, by submersion theorem, $f^{-1}(I) = O(4)$ is closed submanifold of $GL(4,R)$, thus $O(4)$ is a smooth manifold

Also, we see the group multiplication μ is C^{∞} since it's matrices multiplication. So, $O(4)$ is a Lie group by definition.

Also, $GL(4,R)$ is $4^2 = 16$ dimensional, $S(4)$ is 10 dimensional,

thus $O(4)$ is $16 - 10 = 6$ dimensional by submersion theorem.

Also, $\det(AA^T) = 1 \Rightarrow \det(A) = \pm 1$.

Since $\{-1, 1\}$ is disconnected and det is a continuous map, $O(4)$ is also disconnected. Actually, it has two connected component.

ⓑ By submersion theorem, $T_I O(4) = \ker Df(I)$
$$= \{D \in L(R^4, R^4) \mid D + D^T = 0\}$$
$$= \{D \in L(R^4, R^4) \mid D \text{ is skew symmetric}\}$$

$\Rightarrow$ the Lie algebra $o(4)$ of $O(4)$ consists of 4×4 skew matrices.

(c). $I_A(B) = ABA^{-1} \Rightarrow Ad_A \eta = A \eta A^{-1}$, where $\eta \in T_I O(4)$

$\Rightarrow [\xi, \eta] = T_I(Ad_A \eta) \cdot \xi = \xi \eta - \eta \xi$, where $\xi \in T_I O(4)$

$$\Rightarrow [\xi, \eta] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \checkmark$$

(d) for left invariant vector fields, we have $[X_\xi, X_\eta] = X_{[\xi, \eta]}$

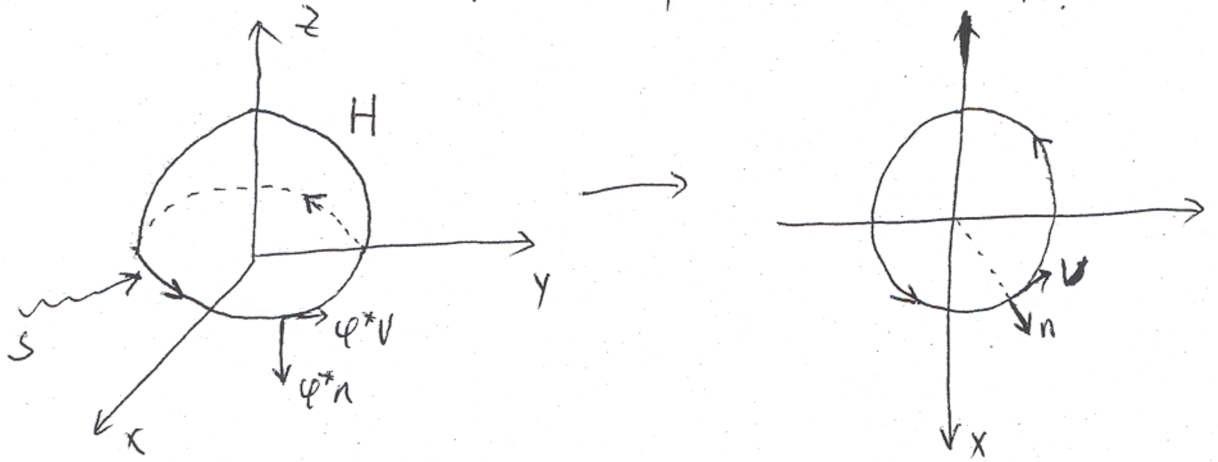
$\Rightarrow [X_\xi, X_\eta](g) = T_e L_g [\xi, \eta]$

Also, from (c) we see ξ, η and $[\xi, \eta]$ are linearly independent.

Also, $T_e L_g$ is a diffeomorphism with $(T_e L_g)^{-1} = T_g L_{g^{-1}}$,
thus $T_e L_g \xi$, $T_e L_g \eta$ and $T_e L_g [\xi, \eta]$ are linearly independent. Thus, $[X_\xi, X_\eta] \notin D$, D is not involutive
thus is not integrable by Frobenius Theorem.

4. ⓐ let $\varphi: U \subset H \to \mathbb{R}^2$ given by $(x,y,z) \longmapsto (x,y)$ be positively oriented chart. Let $\mu = dx \wedge dy$ is volume element in $\mathbb{R}^2$. thus $\varphi^*\mu$ defines an orientation on H, i.e, the basis $(\varphi^*\frac{\partial}{\partial x}, \varphi^*\frac{\partial}{\partial y})$ is positively oriented.

Also, we see the normal direction to the boundary $\partial\varphi(S)$ of $\varphi(H)$ is $n = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$, thus $\varphi^* n$ is the normal direction to the boundary S of H. So, we can choose $\varphi^* V$, where $V = \dfrac{-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}}{\sqrt{x^2+y^2}}$, as the positively oriented basis for S, since (n, V) is positively oriented basis in $\mathbb{R}^n$ thus $(\varphi^* n, \varphi^* V)$ is positively oriented in M.



ⓑ $\int_H (\mathcal{L}_X \alpha)\wedge\beta \blacksquare \int (\mathcal{L}_X \beta)\wedge\alpha = \int_H \big((\mathcal{L}_X\alpha)\wedge\beta + \alpha\wedge(\mathcal{L}_X\beta)\big) = \int_H \mathcal{L}_X(\alpha\wedge\beta)$

$= \int_H \big(d\, i_X(\alpha\wedge\beta) + i_X d(\alpha\wedge\beta)\big) = \int_H d\, i_X(\alpha\wedge\beta)$, since $d\alpha\beta$ is a 2 form on H, $d(\alpha\wedge\beta)=0$.

$= \int_S i_X(\alpha\wedge\beta)$, by Stokes' theorem

$= \int_S (i_X\alpha)\wedge\beta - \int_S \alpha\wedge(i_X\beta) = \int_S (i_X\alpha)\wedge\beta \blacksquare \int_S (i_X\beta)\wedge\alpha$, since $i_X\beta$ is a function.

$\Rightarrow \int_H (\mathcal{L}_X\alpha)\wedge\beta - \int_S (i_X\alpha)\wedge\beta = \int_H (\mathcal{L}_X\beta)\wedge\alpha - \int_S (i_X\beta)\wedge\alpha$ ✓

ⓒ let $i: H \to \mathbb{R}^3$ be embedding map, we have $\beta = i^*\alpha$

$\Rightarrow \int_H d\beta = \int_H d(i^*\alpha) = \int_H i^* d\alpha = 0$, since $d\alpha = 0$ when α is defined on $\mathbb{R}^3/\{(0,0,0)\}$.

(where we use the property $d\,i^* = i^* d$).

10/10

5. ⓐ X is parallel to N, thus X is tangent to N.

Also, $i^*(i_X r) \in \Omega^{n-1}(N)$ is actually $i_X r \in \Omega^{n-1}(M)$ restricted to N.

Thus, let $\bar{X}_1, \bar{X}_2, \cdots, \bar{X}_{n-1} \in \mathfrak{X}(N) \subset TN$, restricting to N, we have

$$i_X r(\bar{X}_1, \bar{X}_2, \cdots, \bar{X}_{n-1}) = r(X, \bar{X}_1, \bar{X}_2, \cdots, \bar{X}_{n-1})$$

Since TN is $n-1$ dimensional, $X, \bar{X}_1, \bar{X}_2, \cdots, \bar{X}_{n-1}$ are linearly dependent, thus $r(X, \bar{X}_1, \bar{X}_2, \cdots, \bar{X}_{n-1}) = 0$

$\Rightarrow i_X r(\bar{X}_1, \bar{X}_2, \cdots, \bar{X}_{n-1}) = 0$ restricted to N.

$\Rightarrow i^*(i_X r) = 0$

ⓑ let $i: \partial M \to M$ be inclusion map, X is parallel to ∂M then by the same argument as in ⓐ, we have for any n-form r on M, $i^*(i_X r) = 0$.

Also, $\int_M \mathcal{L}_X(fgM) = \int_M (\mathcal{L}_X f)gM + \int_M (\mathcal{L}_X g)fM + \int_M fg\mathcal{L}_X M$

$= \int_M (\mathcal{L}_X f)gM + \int_M (\mathcal{L}_X g)fM$, since $(\mathrm{div}_M X)_M = \mathcal{L}_X M = 0$

Also, $\int_M \mathcal{L}_X(fgM) = \int_M i_X d(fgM) + \int_M d\, i_X(fgM)$

$= \int_M d\, i_X(fgM)$, since fgM is n form, $d(fgM) = 0$

$= \int_{\partial M} i^*(i_X(fgM))$, by Stokes's Theorem.

$= 0$

$\Rightarrow \int_M (\mathcal{L}_X f)gM + \int_M (\mathcal{L}_X g)fM = 0$

$\Rightarrow \int_M (\mathcal{L}_X f)gM = -\int_M (\mathcal{L}_X g)fM$

6. ⓐ $d\beta = 2y\,dy\wedge dx + 2y\,dz\wedge dy = 0$

ⓑ i). $(1+y^2)u + 2yxv + zw = 0$, thus $i_X\beta = 0$, where $X = (u,v,w) \in D$.

let $X, Y \in D$, we have $\begin{cases} i_X\beta = 0 \\ i_Y\beta = 0 \end{cases}$

$\Rightarrow i_{[X,Y]}\beta = \mathcal{L}_X i_Y\beta - i_Y\mathcal{L}_X\beta = -i_Y\mathcal{L}_X\beta$

$= -i_Y(i_X d\beta + d\, i_X\beta) = 0$

$\Rightarrow [X,Y] \in D.$

$\Rightarrow$ D is involutive, thus integrable by Frobenius Theorem.

ii). R^3 is contractible, $d\beta = 0$, thus β is exact.

we find $\beta = d(x(1+y^2) + z^2)$.

$\frac{1}{2}$ right idea

Let $f: R^3 \to R$ with $(x,y,z) \longmapsto (x(1+y^2) + \frac{1}{2}z^2)$,

we see $Tf_{(x,y,z)} = (1+y^2, 2yx, z)$, which is full rank,

thus Tf is surjective with split kernel (since the space is finite dimensional). Take $f(x,y,z) = r$, $r \in R$, which is a regular value, by Submersion Theorem, $M = f^{-1}(r)$ is a submanifold of R^3 and $T_{(x,y,z)}M = \ker Tf_{(x,y,z)}$. ← pointwise

However, $\ker Tf_{(x,y,z)} = \{(u,v,w) \mid Tf(u,v,w) = (1+y^2)u + 2yxv + zw = 0\}$

$= D|M$, where $D|M$ means D restricted to M.

$\Rightarrow D|M$ is the tangent bundle TM of a submanifold M of R^3

$\Rightarrow$ D is integrable.

ⓒ let $X = (\frac{1}{1+y^2}, 0, 0)$, we see X is nonzero and C^∞, thus is well-defined vector field. Let F_t be the flow of X, obviously F_t is a diffeomorphism. Thus, $F_t^* i_Y\beta = i_{F_t^*Y}F_t^*\beta$, for $Y \in \mathfrak{X}(R^3)$.

Also, $\mathcal{L}_X\beta = 0$, $\Rightarrow \frac{d}{dt}F_t^*\beta = F_t^*\mathcal{L}_X\beta = 0$, $\Rightarrow F_t^*\beta = \beta$.

Thus, If $Y \in D$, $i_Y\beta = 0$ $\Rightarrow i_{F_t^*Y}F_t^*\beta = i_{F_t^*Y}\beta = 0$, $\Rightarrow F_t^*Y \in D.$

$\Rightarrow$ D is invariant under the flow F_t of X

pf: $\mathcal{L}_X\beta = i_X d\beta + d\, i_X\beta = d\, i_X\beta = d1 = 0$