CDS 202 Final Examination

J. Marsden, March, 2002

*Attempt **five** of the following six questions.*

*This exam has **four** pages including this cover page.*

The exam time limit is three hours; no aids are permitted.

The exam must be turned in by 5pm on Tuesday, March 19.

Print Your Name: _____ ←**Note!**

The 5 questions to be graded: _____ ←**Note!**

You may freely use the following properties as needed. Here α and β are differential forms and X, Y, Z are vector fields on a manifold M. (In the exam, all manifolds, vector fields, and differential forms are assumed to be smooth and the manifolds are finite dimensional.)

(a) $\pounds_X(\alpha \wedge \beta) = (\pounds_X \alpha) \wedge \beta + \alpha \wedge (\pounds_X \beta)$

(b) $\pounds_{[X,Y]}\alpha = \pounds_X \pounds_Y \alpha - \pounds_Y \pounds_X \alpha$

(c) $\mathbf{i}_X(\alpha \wedge \beta) = (\mathbf{i}_X \alpha) \wedge \beta + (-1)^k \alpha \wedge (\mathbf{i}_X \beta)$, where α is a k-form.

(d) $\pounds_X \alpha = \mathbf{d}\mathbf{i}_X \alpha + \mathbf{i}_X \mathbf{d}\alpha$

(e) $\mathbf{i}_{[X,Y]}\beta = \pounds_X \mathbf{i}_Y \beta - \mathbf{i}_Y \pounds_X \beta$

(f) For γ a one-form,

$$\mathbf{d}\gamma(X, Y) = X[\gamma(Y)] - Y[\gamma(X)] - \gamma([X, Y])$$

(g) For ω a two-form,

$$\begin{aligned}\mathbf{d}\omega(X, Y, Z) = X[\omega(Y, Z)] - Y[\omega(X, Z)] + Z[\omega(X, Y) \\ - \omega([X, Y], Z) - \omega([Z, X], Y) - \omega([Y, Z], X)\end{aligned}$$

(h) For a one form α and a vector field X,

$$(\pounds_X \alpha)_i = X^j \frac{\partial \alpha_i}{\partial x^j} + \alpha_j \frac{\partial X^j}{\partial x^i}$$

1. Consider the two-tensor on $\mathbb{R}^2$ given by

$$g = g_{11} dx \otimes dx + g_{12} dx \otimes dy + g_{21} dy \otimes dx + g_{22} dy \otimes dy,$$

in standard coordinates (x, y) on $\mathbb{R}^2$, where the matrix g_{ij} is given by

$$g_{ij} = \begin{pmatrix} 1 + 2x^2 & x \\ x & 1 + y^2 \end{pmatrix}$$

and the vector field X on $\mathbb{R}^2$ defined by

$$X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$$

(a) Compute the Lie derivative $\pounds_X g$.

(b) Find the flow F_t of X.

(c) Compute $\left. \frac{d}{dt} \right|_{t=0} F_t^* g$.

2. (a) Let $M = S^2$ be the standard two sphere of radius 1 in $\mathbb{R}^3$. Let α be the one form on $\mathbb{R}^3$ minus the origin defined by

$$\alpha = \frac{x dy - y dx}{x^2 + y^2} + dz$$

i. Compute $\mathbf{d}\alpha$ and interpret your result in standard vector calculus language.

ii. Let $i : M \to \mathbb{R}^3$ be the inclusion and let $\beta = i^* \alpha$. Is β closed? exact?

(b) Let the Lie group S^1 act on M by rotations around the z-axis. Specifically, for $\theta \in S^1$, regarded as angles mod 2π, the action of θ is by rotation about the z-axis through the angle θ. Let X be the infinitesimal generator corresponding to the Lie algebra element 1 and let μ be the standard volume element on M.

i. Compute X

ii. Compute $\mathbf{i}_X \mu$

iii. Compute $\operatorname{div}_\mu X$

3. Let O(4) denote the set of 4×4 real orthogonal matrices.

(a) Show that O(4) is a manifold and a Lie group; what is its dimension? is it connected?

(b) Show that its Lie algebra consists of 4×4 skew matrices

(c) Let ξ and η be the Lie algebra elements

$$\xi = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \qquad \eta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Compute the Lie algebra bracket $[\xi, \eta]$.

(d) Let X_ξ and X_η be the left invariant vector fields on O(4) equaling ξ and η at the identity and let D be the distribution that is spanned by X_ξ and X_η. Is D integrable?

4. Let H denote the upper hemisphere in $\mathbb{R}^3$ defined by

$$H = \left\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \quad \text{and} \quad z \geq 0\right\}$$

and let $S = \partial H$ be its boundary.

(a) Give H and S consistent orientations; illustrate with a figure.

(b) Let α and β be one forms on H and let X be a vector field on H. Is it true that

$$\int_H (\pounds_X \alpha) \wedge \beta - \int_S (\mathbf{i}_X \alpha) \wedge \beta = \int_H (\pounds_X \beta) \wedge \alpha - \int_S (\mathbf{i}_X \beta) \wedge \alpha?$$

(c) Let α be the one form on $\mathbb{R}^3$ defined by

$$\alpha = \frac{xdy - ydx}{x^2 + y^2} + dz$$

and let β be α pulled back to H. Compute

$$\int_H \mathbf{d}\beta.$$

5. (a) Let X be a vector field on a manifold M of dimension n and let N be a submanifold of dimension $n-1$. Suppose that X is parallel to N. Let γ be an n-form on M and let $i : N \to M$ be the embedding map. Is it true that $i^* (\mathbf{i}_X \gamma) = 0$?

(b) Suppose that (M, μ) is a volume manifold with boundary and X is a divergence free vector field on M that is parallel to the boundary. For functions f, g on M, is it true that

$$\int_M (\pounds_X f)\, g\mu = -\int_M (\pounds_X g)\, f\mu?$$

6. Let the one form β on $\mathbb{R}^3$ be defined by

$$\beta = (1 + y^2)dx + 2xydy + zdz$$

and let D be the distribution defined at the point (x, y, z) to be the set of vectors with components (u, v, w) such that

$$(1 + y^2)u + 2yxv + zw = 0$$

(a) Compute the exterior derivative $\mathbf{d}\beta$

(b) Is the distribution D integrable? Answer this in two ways

i. By using the Frobenius theorem directly

ii. By using a property of β

(c) Find a nonzero vector field X on $\mathbb{R}^3$ such that $\pounds_X \beta = 0$. [HINT: Try the vector field $X = (U, 0, 0)$, where $U(x, y, z) = 1/(1 + y^2)$.] Show that, in an appropriate sense, the distribution D is invariant under the flow F_t of X.