CALTECH
Control & Dynamical Systems

Differential Forms and Stokes' Theorem

Jerrold E. Marsden

Control and Dynamical Systems, Caltech

http://www.cds.caltech.edu/˜marsden/

Differential Forms

- **Main idea:** Generalize the basic operations of vector calculus, div, grad, curl, and the integral theorems of ***Green, Gauss, and Stokes*** to manifolds of arbitrary dimension.

Differential Forms

- **Main idea:** Generalize the basic operations of vector calculus, div, grad, curl, and the integral theorems of ***Green, Gauss, and Stokes*** to manifolds of arbitrary dimension.
- **1-forms.** The term "1-form" is used in two ways—they are either members of a particular cotangent space $T_m^* M$ or else, analogous to a vector field, an assignment of a covector in $T_m^* M$ to each $m \in M$.

Differential Forms

- **Main idea:** Generalize the basic operations of vector calculus, div, grad, curl, and the integral theorems of ***Green, Gauss, and Stokes*** to manifolds of arbitrary dimension.
- 1**-forms.** The term "1-form" is used in two ways—they are either members of a particular cotangent space T_m^*M or else, analogous to a vector field, an assignment of a covector in T_m^*M to each $m \in M$.
- Basic example: differential of a real-valued function.

Differential Forms

- **Main idea:** Generalize the basic operations of vector calculus, div, grad, curl, and the integral theorems of ***Green, Gauss, and Stokes*** to manifolds of arbitrary dimension.
- **1-forms.** The term "1-form" is used in two ways—they are either members of a particular cotangent space $T_m^* M$ or else, analogous to a vector field, an assignment of a covector in $T_m^* M$ to each $m \in M$.
- Basic example: differential of a real-valued function.
- 2-*form* Ω: a map $\Omega(m) : T_m M \times T_m M \longrightarrow \mathbb{R}$ that assigns to each point $m \in M$ a skew-symmetric bilinear form on the tangent space $T_m M$ to M at m.

Differential Forms

- A k-***form*** α (or ***differential form of degree*** k) is a map

$$\alpha(m) : T_m M \times \cdots \times T_m M (k \text{ factors}) \longrightarrow \mathbb{R},$$

which, for each $m \in M$, is a skew-symmetric k-multilinear map on the tangent space $T_m M$ to M at m.

Differential Forms

- A k-***form*** α (or ***differential form of degree*** k) is a map

$$\alpha(m) : T_mM \times \cdots \times T_mM (k \text{ factors}) \longrightarrow \mathbb{R},$$

which, for each $m \in M$, is a skew-symmetric k-multilinear map on the tangent space T_mM to M at m.

- Without the skew-symmetry assumption, α would be a $(0, k)$-***tensor***.

Differential Forms

- A k-***form*** α (or ***differential form of degree*** k) is a map

$$\alpha(m) : T_mM \times \cdots \times T_mM(k \text{ factors}) \longrightarrow \mathbb{R},$$

which, for each $m \in M$, is a skew-symmetric k-multilinear map on the tangent space T_mM to M at m.

- Without the skew-symmetry assumption, α would be a $(0, k)$-***tensor***.
- A map $\alpha : V \times \cdots \times V$ (V is a vector space and there are k factors) $\longrightarrow \mathbb{R}$ is ***multilinear*** when it is linear in each of its factors.

Differential Forms

- A k-***form*** α (or ***differential form of degree*** k) is a map

$$\alpha(m) : T_mM \times \cdots \times T_mM (k \text{ factors}) \longrightarrow \mathbb{R},$$

which, for each $m \in M$, is a skew-symmetric k-multilinear map on the tangent space T_mM to M at m.

- Without the skew-symmetry assumption, α would be a $(0, k)$-***tensor***.

- A map $\alpha : V \times \cdots \times V$ (V is a vector space and there are k factors) $\longrightarrow \mathbb{R}$ is ***multilinear*** when it is linear in each of its factors.

- It is is ***skew*** (or ***alternating***) when it changes sign whenever two of its arguments are interchanged

Differential Forms

- Why is skew-symmetry important? Some examples where it is implicitly used

Differential Forms

- Why is skew-symmetry important? Some examples where it is implicitly used
  - Determinants and integration: Jacobian determinants in the change of variables theorem.

Differential Forms

- Why is skew-symmetry important? Some examples where it is implicitly used
  - Determinants and integration: Jacobian determinants in the change of variables theorem.
  - Cross products and the curl

Differential Forms

- Why is skew-symmetry important? Some examples where it is implicitly used
  - Determinants and integration: Jacobian determinants in the change of variables theorem.
  - Cross products and the curl
  - Orientation or "handedness"

Differential Forms

- Let $x^1, \ldots, x^n$ denote coordinates on M, let

$$\{e_1, \ldots, e_n\} = \{\partial/\partial x^1, \ldots, \partial/\partial x^n\}$$

be the corresponding basis for $T_m M$.

Differential Forms

- Let $x^1, \ldots, x^n$ denote coordinates on M, let
$$\{e_1, \ldots, e_n\} = \{\partial/\partial x^1, \ldots, \partial/\partial x^n\}$$
be the corresponding basis for $T_m M$.
- Let $\{e^1, \ldots, e^n\} = \{dx^1, \ldots, dx^n\}$ be the dual basis for $T_m^* M$.

Differential Forms

- Let $x^1, \ldots, x^n$ denote coordinates on M, let
$$\{e_1, \ldots, e_n\} = \{\partial/\partial x^1, \ldots, \partial/\partial x^n\}$$
be the corresponding basis for $T_m M$.
- Let $\{e^1, \ldots, e^n\} = \{dx^1, \ldots, dx^n\}$ be the dual basis for $T_m^* M$.
- At each $m \in M$, we can write a 2-form as
$$\Omega_m(v, w) = \Omega_{ij}(m) v^i w^j,$$
where
$$\Omega_{ij}(m) = \Omega_m \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right),$$

Differential Forms

- Let $x^1, \ldots, x^n$ denote coordinates on M, let
$$\{e_1, \ldots, e_n\} = \{\partial/\partial x^1, \ldots, \partial/\partial x^n\}$$
be the corresponding basis for $T_m M$.
- Let $\{e^1, \ldots, e^n\} = \{dx^1, \ldots, dx^n\}$ be the dual basis for $T_m^* M$.
- At each $m \in M$, we can write a 2-form as
$$\Omega_m(v, w) = \Omega_{ij}(m) v^i w^j,$$
where
$$\Omega_{ij}(m) = \Omega_m\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right),$$
- Similarly for k-forms.

Tensor and Wedge Products

- If α is a $(0,k)$-tensor on a manifold M and β is a $(0,l)$-tensor, their ***tensor product*** (sometimes called the ***outer product***), $\alpha \otimes \beta$ is the $(0,k+l)$-tensor on M defined by

$$(\alpha \otimes \beta)_m(v_1, \ldots, v_{k+l}) \\ = \alpha_m(v_1, \ldots, v_k)\beta_m(v_{k+1}, \ldots, v_{k+l})$$

at each point $m \in M$.

Tensor and Wedge Products

- If α is a $(0, k)$-tensor on a manifold M and β is a $(0, l)$-tensor, their ***tensor product*** (sometimes called the ***outer product***), $\alpha \otimes \beta$ is the $(0, k+l)$-tensor on M defined by

$$(\alpha \otimes \beta)_m(v_1, \ldots, v_{k+l}) = \alpha_m(v_1, \ldots, v_k)\beta_m(v_{k+1}, \ldots, v_{k+l})$$

  at each point $m \in M$.
- Outer product of two vectors is a *matrix*

Tensor and Wedge Products

- If t is a $(0,p)$-tensor, define the ***alternation operator*** $\mathbf{A}$ acting on t by

$$\mathbf{A}(t)(v_1,\ldots,v_p) = \frac{1}{p!}\sum_{\pi\in S_p} \operatorname{sgn}(\pi)t(v_{\pi(1)},\ldots,v_{\pi(p)}),$$

where $\operatorname{sgn}(\pi)$ is the ***sign*** of the permutation π,

$$\operatorname{sgn}(\pi) = \begin{cases} +1 \text{ if } \pi \text{ is even }, \\ -1 \text{ if } \pi \text{ is odd }, \end{cases}$$

and S_p is the group of all permutations of the set $\{1,2,\ldots,p\}$.

Tensor and Wedge Products

- If t is a $(0,p)$-tensor, define the ***alternation operator*** $\mathbf{A}$ acting on t by

$$\mathbf{A}(t)(v_1, \ldots, v_p) = \frac{1}{p!} \sum_{\pi \in S_p} \mathrm{sgn}(\pi) t(v_{\pi(1)}, \ldots, v_{\pi(p)}),$$

where $\mathrm{sgn}(\pi)$ is the ***sign*** of the permutation π,

$$\mathrm{sgn}(\pi) = \begin{cases} +1 \text{ if } \pi \text{ is even }, \\ -1 \text{ if } \pi \text{ is odd }, \end{cases}$$

and S_p is the group of all permutations of the set $\{1, 2, \ldots, p\}$.

- The operator $\mathbf{A}$ therefore ***skew-symmetrizes*** p-multilinear maps.

Tensor and Wedge Products

- If α is a k-form and β is an l-form on M, their ***wedge product*** $\alpha \wedge \beta$ is the $(k+l)$-form on M defined by

$$\alpha \wedge \beta = \frac{(k+l)!}{k!\, l!} \mathbf{A}(\alpha \otimes \beta).$$

Tensor and Wedge Products

- If α is a k-form and β is an l-form on M, their ***wedge product*** $\alpha \wedge \beta$ is the $(k+l)$-form on M defined by

$$\alpha \wedge \beta = \frac{(k+l)!}{k!\, l!} \mathbf{A}(\alpha \otimes \beta).$$

- One has to be careful here as some authors use different conventions.

Tensor and Wedge Products

- If α is a k-form and β is an l-form on M, their ***wedge product*** $\alpha \wedge \beta$ is the $(k+l)$-form on M defined by
$$\alpha \wedge \beta = \frac{(k+l)!}{k!\,l!}\mathbf{A}(\alpha \otimes \beta).$$
- One has to be careful here as some authors use different conventions.
- **Examples:** if α and β are one-forms, then
$$(\alpha \wedge \beta)(v_1, v_2) = \alpha(v_1)\beta(v_2) - \alpha(v_2)\beta(v_1),$$

Tensor and Wedge Products

- If α is a k-form and β is an l-form on M, their ***wedge product*** $\alpha \wedge \beta$ is the $(k+l)$-form on M defined by
$$\alpha \wedge \beta = \frac{(k+l)!}{k!\,l!}\mathbf{A}(\alpha \otimes \beta).$$
- One has to be careful here as some authors use different conventions.
- **Examples:** if α and β are one-forms, then
$$(\alpha \wedge \beta)(v_1, v_2) = \alpha(v_1)\beta(v_2) - \alpha(v_2)\beta(v_1),$$
- If α is a 2-form and β is a 1-form,
$$\begin{aligned}(\alpha \wedge \beta)&(v_1, v_2, v_3)\\ &= \alpha(v_1, v_2)\beta(v_3) - \alpha(v_1, v_3)\beta(v_2) + \alpha(v_2, v_3)\beta(v_1).\end{aligned}$$

Tensor and Wedge Products

□ Wedge product properties:

(i) ***Associative:*** $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$.

(ii) ***Bilinear:***

$$(a\alpha_1 + b\alpha_2) \wedge \beta = a(\alpha_1 \wedge \beta) + b(\alpha_2 \wedge \beta),$$
$$\alpha \wedge (c\beta_1 + d\beta_2) = c(\alpha \wedge \beta_1) + d(\alpha \wedge \beta_2).$$

(iii) ***Anticommutative:*** $\alpha \wedge \beta = (-1)^{kl}\beta \wedge \alpha$, where α is a k-form and β is an l-form.

Tensor and Wedge Products

- **Wedge product properties:**

  (i) ***Associative:*** $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$.

  (ii) ***Bilinear:***

$$(a\alpha_1 + b\alpha_2) \wedge \beta = a(\alpha_1 \wedge \beta) + b(\alpha_2 \wedge \beta),$$
$$\alpha \wedge (c\beta_1 + d\beta_2) = c(\alpha \wedge \beta_1) + d(\alpha \wedge \beta_2).$$

  (iii) ***Anticommutative:*** $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$, where α is a k-form and β is an l-form.

- **Coordinate Representation:** Use dual basis dx^i; a k-form can be written

$$\alpha = \alpha_{i_1 \dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

  where the sum is over all i_j satisfying $i_1 < \cdots < i_k$.

Pull-Back and Push-Forward

- $\varphi : M \to N$, a smooth map and α a k-form on N.

Pull-Back and Push-Forward

- $\varphi : M \to N$, a smooth map and α a k-form on N.
- ***Pull-back:*** $\varphi^*\alpha$ of α by φ: the k-form on M

$$(\varphi^*\alpha)_m(v_1, \ldots, v_k) = \alpha_{\varphi(m)}(T_m\varphi \cdot v_1, \ldots, T_m\varphi \cdot v_k).$$

Pull-Back and Push-Forward

- $\varphi : M \to N$, a smooth map and α a k-form on N.
- ***Pull-back:*** $\varphi^*\alpha$ of α by φ: the k-form on M

$$(\varphi^*\alpha)_m(v_1, \ldots, v_k) = \alpha_{\varphi(m)}(T_m\varphi \cdot v_1, \ldots, T_m\varphi \cdot v_k).$$

- ***Push-forward*** (if φ is a diffeomorphism): $\varphi_* = (\varphi^{-1})^*$.

Pull-Back and Push-Forward

- $\varphi : M \to N$, a smooth map and α a k-form on N.
- ***Pull-back:*** $\varphi^*\alpha$ of α by φ: the k-form on M

$$(\varphi^*\alpha)_m(v_1, \ldots, v_k) = \alpha_{\varphi(m)}(T_m\varphi \cdot v_1, \ldots, T_m\varphi \cdot v_k).$$

- ***Push-forward*** (if φ is a diffeomorphism): $\varphi_* = (\varphi^{-1})^*$.
- The pull-back of a wedge product is the wedge product of the pull-backs:

$$\varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta.$$

Interior Products

- Let α be a k-form on a manifold M and X a vector field.

Interior Products

- Let α be a k-form on a manifold M and X a vector field.
- The ***interior product*** $\mathbf{i}_X\alpha$ (sometimes called the ***contraction*** of X and α and written, using the "hook" notation, as $X \lrcorner\, \alpha$) is defined by

$$(\mathbf{i}_X\alpha)_m(v_2, \ldots, v_k) = \alpha_m(X(m), v_2, \ldots, v_k).$$

Interior Products

- Let α be a k-form on a manifold M and X a vector field.
- The ***interior product*** $\mathbf{i}_X\alpha$ (sometimes called the ***contraction*** of X and α and written, using the "hook" notation, as $X \lrcorner \alpha$) is defined by

$$(\mathbf{i}_X\alpha)_m(v_2, \ldots, v_k) = \alpha_m(X(m), v_2, \ldots, v_k).$$

- **Product Rule-Like Property.** Let α be a k-form and β a 1-form on a manifold M. Then

$$\mathbf{i}_X(\alpha \wedge \beta) = (\mathbf{i}_X\alpha) \wedge \beta + (-1)^k \alpha \wedge (\mathbf{i}_X\beta).$$

or, in the hook notation,

$$X \lrcorner (\alpha \wedge \beta) = (X \lrcorner \alpha) \wedge \beta + (-1)^k \alpha \wedge (X \lrcorner \beta).$$

Exterior Derivative

- The ***exterior derivative*** $\mathbf{d}\alpha$ of a k-form α is the $(k+1)$-form determined by the following properties:

Exterior Derivative

- The ***exterior derivative*** $\mathbf{d}\alpha$ of a k-form α is the $(k+1)$-form determined by the following properties:
  - If $\alpha = f$ is a 0-form, then $\mathbf{d}f$ is the differential of f.

Exterior Derivative

- The ***exterior derivative*** $\mathbf{d}\alpha$ of a k-form α is the $(k+1)$-form determined by the following properties:
  - If $\alpha = f$ is a 0-form, then $\mathbf{d}f$ is the differential of f.
  - $\mathbf{d}\alpha$ is ***linear*** in α—for all real numbers c_1 and c_2,

$$\mathbf{d}(c_1\alpha_1 + c_2\alpha_2) = c_1\mathbf{d}\alpha_1 + c_2\mathbf{d}\alpha_2.$$

Exterior Derivative

- The ***exterior derivative*** $\mathbf{d}\alpha$ of a k-form α is the $(k+1)$-form determined by the following properties:
  - If $\alpha = f$ is a 0-form, then $\mathbf{d}f$ is the differential of f.
  - $\mathbf{d}\alpha$ is ***linear*** in α—for all real numbers c_1 and c_2,

$$\mathbf{d}(c_1\alpha_1 + c_2\alpha_2) = c_1\mathbf{d}\alpha_1 + c_2\mathbf{d}\alpha_2.$$

  - $\mathbf{d}\alpha$ satisfies the ***product rule***—

$$\mathbf{d}(\alpha \wedge \beta) = \mathbf{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathbf{d}\beta,$$

    where α is a k-form and β is an l-form.

Exterior Derivative

- The ***exterior derivative*** $\mathbf{d}\alpha$ of a k-form α is the $(k+1)$-form determined by the following properties:
  - If $\alpha = f$ is a 0-form, then $\mathbf{d}f$ is the differential of f.
  - $\mathbf{d}\alpha$ is ***linear*** in α—for all real numbers c_1 and c_2,

$$\mathbf{d}(c_1\alpha_1 + c_2\alpha_2) = c_1\mathbf{d}\alpha_1 + c_2\mathbf{d}\alpha_2.$$

  - $\mathbf{d}\alpha$ satisfies the ***product rule***—

$$\mathbf{d}(\alpha \wedge \beta) = \mathbf{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathbf{d}\beta,$$

    where α is a k-form and β is an l-form.
  - $\mathbf{d}^2 = 0$, that is, $\mathbf{d}(\mathbf{d}\alpha) = 0$ for any k-form α.

Exterior Derivative

- The ***exterior derivative*** $\mathbf{d}\alpha$ of a k-form α is the $(k+1)$-form determined by the following properties:
  - If $\alpha = f$ is a 0-form, then $\mathbf{d}f$ is the differential of f.
  - $\mathbf{d}\alpha$ is ***linear*** in α—for all real numbers c_1 and c_2,
  $$\mathbf{d}(c_1\alpha_1 + c_2\alpha_2) = c_1\mathbf{d}\alpha_1 + c_2\mathbf{d}\alpha_2.$$
  - $\mathbf{d}\alpha$ satisfies the ***product rule***—
  $$\mathbf{d}(\alpha \wedge \beta) = \mathbf{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathbf{d}\beta,$$
  where α is a k-form and β is an l-form.
  - $\mathbf{d}^2 = 0$, that is, $\mathbf{d}(\mathbf{d}\alpha) = 0$ for any k-form α.
  - $\mathbf{d}$ is a ***local operator***, that is, $\mathbf{d}\alpha(m)$ depends only on α restricted to any open neighborhood of m; that is, if U is open in M, then
  $$\mathbf{d}(\alpha|U) = (\mathbf{d}\alpha)|U.$$

Exterior Derivative

- If α is a k-form given in coordinates by

$$\alpha = \alpha_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \quad (\text{sum on } i_1 < \cdots < i_k),$$

then the coordinate expression for the exterior derivative is

$$\mathbf{d}\alpha = \frac{\partial \alpha_{i_1 \ldots i_k}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

with a sum over j and $i_1 < \cdots < i_k$

Exterior Derivative

- If α is a k-form given in coordinates by

$$\alpha = \alpha_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \quad (\text{sum on } i_1 < \cdots < i_k),$$

  then the coordinate expression for the exterior derivative is

$$\mathbf{d}\alpha = \frac{\partial \alpha_{i_1 \ldots i_k}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

  with a sum over j and $i_1 < \cdots < i_k$

- This formula is easy to remember from the properties.

Exterior Derivative

Properties.

- Exterior differentiation commutes with pull-back, that is,

$$\mathbf{d}(\varphi^*\alpha) = \varphi^*(\mathbf{d}\alpha),$$

where α is a k-form on a manifold N and $\varphi : M \to N$.

Exterior Derivative

□ Properties.

- Exterior differentiation commutes with pull-back, that is,

$$\mathbf{d}(\varphi^*\alpha) = \varphi^*(\mathbf{d}\alpha),$$

  where α is a k-form on a manifold N and $\varphi : M \longrightarrow N$.
- A k-form α is called ***closed*** if $\mathbf{d}\alpha = 0$ and is ***exact*** if there is a $(k-1)$-form β such that $\alpha = \mathbf{d}\beta$.

Exterior Derivative

□ Properties.

- Exterior differentiation commutes with pull-back, that is,

$$\mathbf{d}(\varphi^*\alpha) = \varphi^*(\mathbf{d}\alpha),$$

  where α is a k-form on a manifold N and $\varphi : M \longrightarrow N$.
- A k-form α is called ***closed*** if $\mathbf{d}\alpha = 0$ and is ***exact*** if there is a $(k-1)$-form β such that $\alpha = \mathbf{d}\beta$.
- $\mathbf{d}^2 = 0 \Rightarrow$ an exact form is closed (but the converse need not hold—we recall the standard vector calculus example shortly)

Exterior Derivative

□ Properties.

- Exterior differentiation commutes with pull-back, that is,

$$\mathbf{d}(\varphi^*\alpha) = \varphi^*(\mathbf{d}\alpha),$$

  where α is a k-form on a manifold N and $\varphi : M \longrightarrow N$.
- A k-form α is called ***closed*** if $\mathbf{d}\alpha = 0$ and is ***exact*** if there is a $(k-1)$-form β such that $\alpha = \mathbf{d}\beta$.
- $\mathbf{d}^2 = 0 \Rightarrow$ an exact form is closed (but the converse need not hold—we recall the standard vector calculus example shortly)
- **Poincaré Lemma** A closed form is ***locally exact***; that is, if $\mathbf{d}\alpha = 0$, there is a neighborhood about each point on which $\alpha = \mathbf{d}\beta$.

Vector Calculus

□ **Sharp and Flat** (Using standard coordinates in $\mathbb{R}^3$)

(a) $v^\flat = v^1\,dx + v^2\,dy + v^3\,dz$, the one-form corresponding to the vector $v = v^1\mathbf{e}_1 + v^2\mathbf{e}_2 + v^3\mathbf{e}_3$.

(b) $\alpha^\sharp = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3$, the vector corresponding to the one-form $\alpha = \alpha_1\,dx + \alpha_2\,dy + \alpha_3\,dz$.

Vector Calculus

□ Sharp and Flat (Using standard coordinates in $\mathbb{R}^3$)

(a) $v^\flat = v^1\,dx + v^2\,dy + v^3\,dz$, the one-form corresponding to the vector $v = v^1\mathbf{e}_1 + v^2\mathbf{e}_2 + v^3\mathbf{e}_3$.

(b) $\alpha^\sharp = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3$, the vector corresponding to the one-form $\alpha = \alpha_1\,dx + \alpha_2\,dy + \alpha_3\,dz$.

□ Hodge Star Operator

(a) $*1 = dx \wedge dy \wedge dz$.

(b) $*dx = dy \wedge dz$, $*dy = -dx \wedge dz$, $*dz = dx \wedge dy$,
$*(dy \wedge dz) = dx$, $*(dx \wedge dz) = -dy$, $*(dx \wedge dy) = dz$.

(c) $*(dx \wedge dy \wedge dz) = 1$.

Vector Calculus

Sharp and Flat (Using standard coordinates in $\mathbb{R}^3$)

(a) $v^\flat = v^1\,dx + v^2\,dy + v^3\,dz$, the one-form corresponding to the vector $v = v^1\mathbf{e}_1 + v^2\mathbf{e}_2 + v^3\mathbf{e}_3$.

(b) $\alpha^\sharp = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3$, the vector corresponding to the one-form $\alpha = \alpha_1\,dx + \alpha_2\,dy + \alpha_3\,dz$.

Hodge Star Operator

(a) $*1 = dx \wedge dy \wedge dz$.

(b) $*dx = dy \wedge dz$, $*dy = -dx \wedge dz$, $*dz = dx \wedge dy$, $*(dy \wedge dz) = dx$, $*(dx \wedge dz) = -dy$, $*(dx \wedge dy) = dz$.

(c) $*(dx \wedge dy \wedge dz) = 1$.

Cross Product and Dot Product

(a) $v \times w = [*(v^\flat \wedge w^\flat)]^\sharp$.

(b) $(v \cdot w)dx \wedge dy \wedge dz = v^\flat \wedge *(w^\flat)$.

Vector Calculus

- **Gradient** $\nabla f = \operatorname{grad} f = (\mathbf{d} f)^\sharp.$

Vector Calculus

- **Gradient** $\nabla f = \operatorname{grad} f = (\mathbf{d}f)^\sharp.$
- **Curl** $\nabla \times F = \operatorname{curl} F = [*(\mathbf{d}F^\flat)]^\sharp.$

Vector Calculus

- **Gradient** $\nabla f = \operatorname{grad} f = (\mathbf{d}f)^\sharp.$
- **Curl** $\nabla \times F = \operatorname{curl} F = [*(\mathbf{d}F^\flat)]^\sharp.$
- **Divergence** $\nabla \cdot F = \operatorname{div} F = *\mathbf{d}(*F^\flat).$

Lie Derivative

- **Dynamic definition:** Let α be a k-form and X be a vector field with flow φ_t. The ***Lie derivative*** of α along X is

$$\pounds_X \alpha = \lim_{t\to 0} \frac{1}{t}[(\varphi_t^* \alpha) - \alpha] = \frac{d}{dt} \varphi_t^* \alpha \bigg|_{t=0}.$$

Lie Derivative

- **Dynamic definition:** Let α be a k-form and X be a vector field with flow φ_t. The ***Lie derivative*** of α along X is

$$\pounds_X \alpha = \lim_{t \to 0} \frac{1}{t}[(\varphi_t^* \alpha) - \alpha] = \frac{d}{dt} \varphi_t^* \alpha \bigg|_{t=0}.$$

- Extend to non-zero values of t:

$$\frac{d}{dt} \varphi_t^* \alpha = \varphi_t^* \pounds_X \alpha.$$

Lie Derivative

- **Dynamic definition:** Let α be a k-form and X be a vector field with flow φ_t. The ***Lie derivative*** of α along X is

$$\pounds_X \alpha = \lim_{t \to 0} \frac{1}{t}[(\varphi_t^* \alpha) - \alpha] = \frac{d}{dt} \varphi_t^* \alpha \Big|_{t=0}.$$

- Extend to non-zero values of t:

$$\frac{d}{dt} \varphi_t^* \alpha = \varphi_t^* \pounds_X \alpha.$$

- **Time-dependent** vector fields

$$\frac{d}{dt} \varphi_{t,s}^* \alpha = \varphi_{t,s}^* \pounds_X \alpha.$$

Lie Derivative

- **Real Valued Functions.** The ***Lie derivative of f along*** X is the ***directional derivative***

$$\pounds_X f = X[f] := \mathbf{d} f \cdot X. \tag{1}$$

Lie Derivative

- **Real Valued Functions.** The ***Lie derivative of*** f ***along*** X is the ***directional derivative***

$$\pounds_X f = X[f] := \mathbf{d}f \cdot X. \tag{1}$$

- In coordinates

$$\pounds_X f = X^i \frac{\partial f}{\partial x^i}.$$

Lie Derivative

- **Real Valued Functions.** The ***Lie derivative of*** f ***along*** X is the ***directional derivative***

$$\pounds_X f = X[f] := \mathbf{d}f \cdot X. \tag{1}$$

- In coordinates

$$\pounds_X f = X^i \frac{\partial f}{\partial x^i}.$$

- **Useful Notation.**

$$X = X^i \frac{\partial}{\partial x^i}.$$

Lie Derivative

- **Real Valued Functions.** The ***Lie derivative of*** f ***along*** X is the ***directional derivative***

$$\pounds_X f = X[f] := \mathbf{d}f \cdot X. \tag{1}$$

- In coordinates

$$\pounds_X f = X^i \frac{\partial f}{\partial x^i}.$$

- **Useful Notation.**

$$X = X^i \frac{\partial}{\partial x^i}.$$

- **Operator notation:** $X[f] = \mathbf{d}f \cdot X$

Lie Derivative

- **Real Valued Functions.** The ***Lie derivative of*** f ***along*** X is the ***directional derivative***

$$\pounds_X f = X[f] := \mathbf{d}f \cdot X. \tag{1}$$

- In coordinates

$$\pounds_X f = X^i \frac{\partial f}{\partial x^i}.$$

- **Useful Notation.**

$$X = X^i \frac{\partial}{\partial x^i}.$$

- **Operator notation:** $X[f] = \mathbf{d}f \cdot X$
- The operator is a ***derivation***; that is, the product rule holds.

Lie Derivative

- **Pull-back.** If Y is a vector field on a manifold N and $\varphi : M \to N$ is a diffeomorphism, the ***pull-back*** $\varphi^* Y$ is a vector field on M defined by

$$(\varphi^* Y)(m) = \left(T_m \varphi^{-1} \circ Y \circ \varphi \right)(m).$$

Lie Derivative

- **Pull-back.** If Y is a vector field on a manifold N and $\varphi : M \to N$ is a diffeomorphism, the ***pull-back*** $\varphi^* Y$ is a vector field on M defined by

$$(\varphi^* Y)(m) = \left(T_m \varphi^{-1} \circ Y \circ \varphi\right)(m).$$

- **Push-forward.** For a diffeomorphism φ, the ***push-forward*** is defined, as for forms, by $\varphi_* = (\varphi^{-1})^*$.

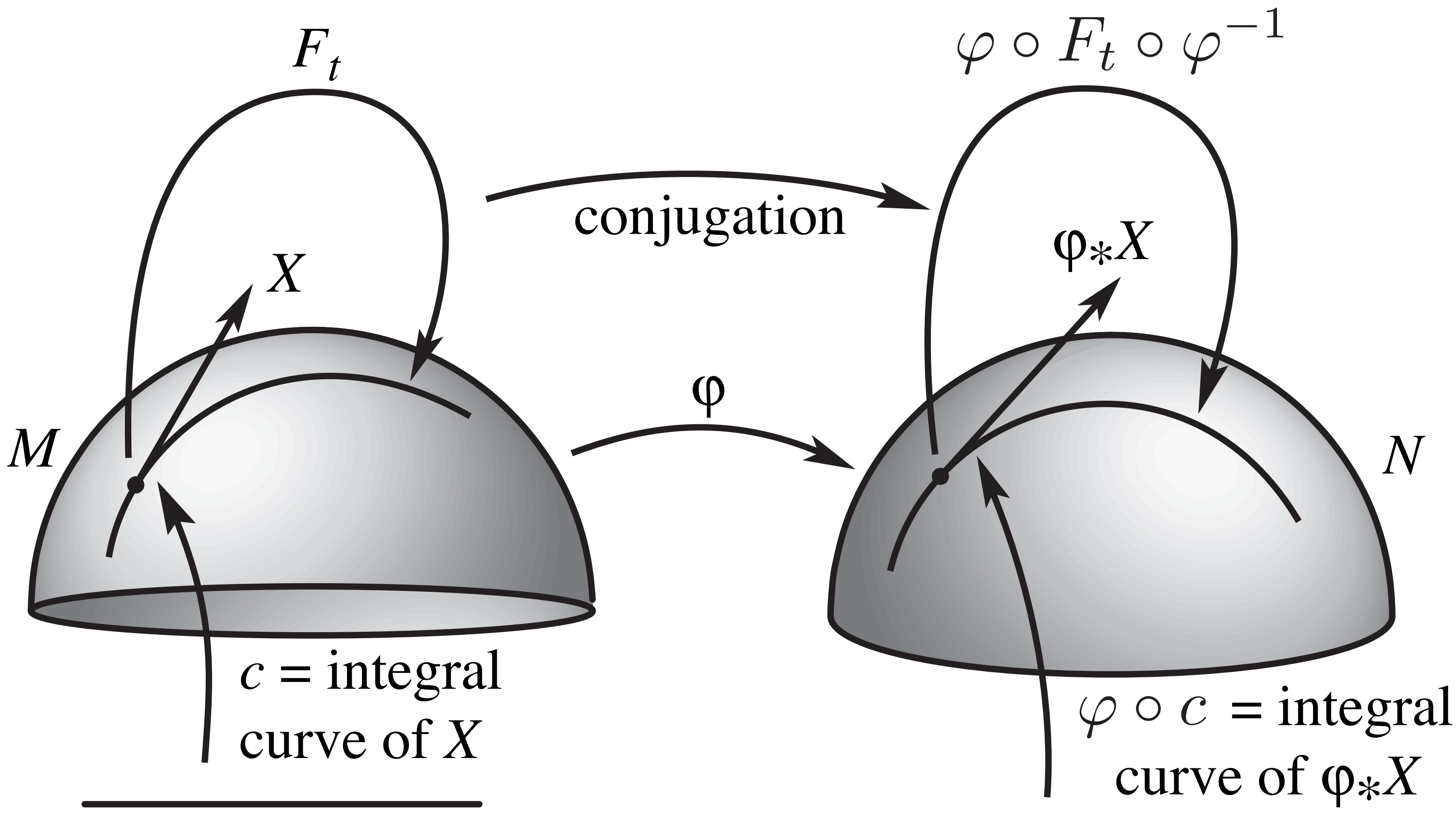
Lie Derivative

- **Pull-back.** If Y is a vector field on a manifold N and $\varphi : M \to N$ is a diffeomorphism, the ***pull-back*** $\varphi^* Y$ is a vector field on M defined by

$$(\varphi^* Y)(m) = \left(T_m \varphi^{-1} \circ Y \circ \varphi\right)(m).$$

- **Push-forward.** For a diffeomorphism φ, the ***push-forward*** is defined, as for forms, by $\varphi_* = (\varphi^{-1})^*$.
- Flows of X and $\varphi_* X$ related by conjugation.

Lie Derivative



Jacobi–Lie Bracket

- The Lie derivative on functions is a ***derivation***; conversely, derivations determine vector fields.

Jacobi–Lie Bracket

- The Lie derivative on functions is a ***derivation***; conversely, derivations determine vector fields.
- The commutator is a derivation

$$f \mapsto X[Y[f]] - Y[X[f]] = [X, Y][f],$$

which determines the unique vector field $[X, Y]$ the ***Jacobi–Lie bracket*** of X and Y.

Jacobi–Lie Bracket

- The Lie derivative on functions is a ***derivation***; conversely, derivations determine vector fields.
- The commutator is a derivation

$$f \mapsto X[Y[f]] - Y[X[f]] = [X,Y][f],$$

which determines the unique vector field $[X,Y]$ the ***Jacobi–Lie bracket*** of X and Y.

- $\pounds_X Y = [X,Y]$, ***Lie derivative*** of Y along X.

Jacobi–Lie Bracket

- The Lie derivative on functions is a ***derivation***; conversely, derivations determine vector fields.
- The commutator is a derivation

$$f \mapsto X[Y[f]] - Y[X[f]] = [X, Y][f],$$

which determines the unique vector field $[X, Y]$ the ***Jacobi–Lie bracket*** of X and Y.
- $\pounds_X Y = [X, Y]$, ***Lie derivative*** of Y along X.
- The analog of the Lie derivative formula holds.

Jacobi–Lie Bracket

- The Lie derivative on functions is a ***derivation***; conversely, derivations determine vector fields.
- The commutator is a derivation

$$f \mapsto X[Y[f]] - Y[X[f]] = [X, Y][f],$$

which determines the unique vector field $[X, Y]$ the ***Jacobi–Lie bracket*** of X and Y.
- $\pounds_X Y = [X, Y]$, ***Lie derivative*** of Y along X.
- The analog of the Lie derivative formula holds.
- Coordinates:

$$(\pounds_X Y)^j = X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} = (X \cdot \nabla) Y^j - (Y \cdot \nabla) X^j,$$

Jacobi–Lie Bracket

- The formula for $[X,Y] = \pounds_X Y$ can be remembered by writing

$$\left[X^i \frac{\partial}{\partial x^i}, Y^j \frac{\partial}{\partial x^j}\right] = X^i \frac{\partial Y^j}{\partial x^i}\frac{\partial}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j}\frac{\partial}{\partial x^i}.$$

Algebraic Approach.

- **Program:** Extend the definition of the Lie derivative from functions and vector fields to differential forms, by requiring that the Lie derivative be a derivation

Algebraic Approach.

- **Program:** Extend the definition of the Lie derivative from functions and vector fields to differential forms, by requiring that the Lie derivative be a derivation
- **Example.** For a 1-form α,

$$\pounds_X\langle\alpha, Y\rangle = \langle\pounds_X\alpha, Y\rangle + \langle\alpha, \pounds_X Y\rangle \,,$$

where X, Y are vector fields and $\langle\alpha, Y\rangle = \alpha(Y)$.

Algebraic Approach.

- **Program:** Extend the definition of the Lie derivative from functions and vector fields to differential forms, by requiring that the Lie derivative be a derivation
- **Example.** For a 1-form α,

$$\pounds_X \langle \alpha, Y \rangle = \langle \pounds_X \alpha, Y \rangle + \langle \alpha, \pounds_X Y \rangle \,,$$

where X, Y are vector fields and $\langle \alpha, Y \rangle = \alpha(Y)$.

- More generally, determine $\pounds_X \alpha$ by

$$\pounds_X(\alpha(Y_1, \ldots, Y_k))$$

$$= (\pounds_X \alpha)(Y_1, \ldots, Y_k) + \sum_{i=1}^{k} \alpha(Y_1, \ldots, \pounds_X Y_i, \ldots, Y_k).$$

Equivalence

- ***The dynamic and algebraic definitions of the Lie derivative of a differential k-form are equivalent.***

Equivalence

- ***The dynamic and algebraic definitions of the Lie derivative of a differential k-form are equivalent.***
- The Lie derivative formalism holds for all tensors, not just differential forms.

Equivalence

- ***The dynamic and algebraic definitions of the Lie derivative of a differential k-form are equivalent.***
- The Lie derivative formalism holds for all tensors, not just differential forms.
- Very useful in all areas of mechanics: eg, the rate of strain tensor in elasticity is a Lie derivative and the vorticity advection equation in fluid dynamics are both Lie derivative equations.

Properties

- **Cartan's Magic Formula.** For X a vector field and α a k-form

$$\pounds_X \alpha = \mathbf{d}\mathbf{i}_X \alpha + \mathbf{i}_X \mathbf{d}\alpha,$$

Properties

- **Cartan's Magic Formula.** For X a vector field and α a k-form

$$\pounds_X \alpha = \mathbf{d}\mathbf{i}_X \alpha + \mathbf{i}_X \mathbf{d}\alpha,$$

- In the "hook" notation,

$$\pounds_X \alpha = \mathbf{d}(X \,\lrcorner\, \alpha) + X \,\lrcorner\, \mathbf{d}\alpha.$$

Properties

- **Cartan's Magic Formula.** For X a vector field and α a k-form

$$\pounds_X \alpha = \mathbf{d}\mathbf{i}_X \alpha + \mathbf{i}_X \mathbf{d}\alpha,$$

- In the "hook" notation,

$$\pounds_X \alpha = \mathbf{d}(X \,\lrcorner\, \alpha) + X \,\lrcorner\, \mathbf{d}\alpha.$$

- If $\varphi : M \to N$ is a diffeomorphism, then

$$\varphi^* \pounds_Y \beta = \pounds_{\varphi^* Y} \varphi^* \beta$$

for $Y \in \mathfrak{X}(N)$ and $\beta \in \Omega^k(M)$.

Properties

- **Cartan's Magic Formula.** For X a vector field and α a k-form

$$\pounds_X \alpha = \mathbf{d}\mathbf{i}_X \alpha + \mathbf{i}_X \mathbf{d}\alpha,$$

- In the "hook" notation,

$$\pounds_X \alpha = \mathbf{d}(X \,\lrcorner\, \alpha) + X \,\lrcorner\, \mathbf{d}\alpha.$$

- If $\varphi : M \to N$ is a diffeomorphism, then

$$\varphi^* \pounds_Y \beta = \pounds_{\varphi^* Y} \varphi^* \beta$$

for $Y \in \mathfrak{X}(N)$ and $\beta \in \Omega^k(M)$.

- Many other useful identities, such as

$$\mathbf{d}\Theta(X, Y) = X[\Theta(Y)] - Y[\Theta(X)] - \Theta([X, Y]).$$

Volume Forms and Divergence

- An n-manifold M is ***orientable*** if there is a nowhere-vanishing n-form μ on it; μ is a ***volume form***

Volume Forms and Divergence

- An n-manifold M is ***orientable*** if there is a nowhere-vanishing n-form μ on it; μ is a ***volume form***
- Two volume forms μ_1 and μ_2 on M define the same ***orientation*** if $\mu_2 = f\mu_1$, where $f > 0$.

Volume Forms and Divergence

- An n-manifold M is ***orientable*** if there is a nowhere-vanishing n-form μ on it; μ is a ***volume form***
- Two volume forms μ_1 and μ_2 on M define the same ***orientation*** if $\mu_2 = f\mu_1$, where $f > 0$.
- **Oriented Basis.** A basis $\{v_1, \ldots, v_n\}$ of T_mM is ***positively oriented*** relative to the volume form μ on M if $\mu(m)(v_1, \ldots, v_n) > 0$.

Volume Forms and Divergence

- An n-manifold M is ***orientable*** if there is a nowhere-vanishing n-form μ on it; μ is a ***volume form***
- Two volume forms μ_1 and μ_2 on M define the same ***orientation*** if $\mu_2 = f\mu_1$, where $f > 0$.
- **Oriented Basis.** A basis $\{v_1, \ldots, v_n\}$ of T_mM is ***positively oriented*** relative to the volume form μ on M if $\mu(m)(v_1, \ldots, v_n) > 0$.
- **Divergence.** If μ is a volume form, there is a function, called the ***divergence*** of X relative to μ and denoted by $\operatorname{div}_\mu(X)$ or simply $\operatorname{div}(X)$, such that

$$\pounds_X \mu = \operatorname{div}_\mu(X)\mu.$$

Volume Forms and Divergence

- Dynamic approach to Lie derivatives $\Rightarrow \operatorname{div}_\mu(X) = 0$ if and only if $F_t^* \mu = \mu$, where F_t is the flow of X (that is, F_t is ***volume preserving***.)

Volume Forms and Divergence

- Dynamic approach to Lie derivatives $\Rightarrow \mathrm{div}_\mu(X) = 0$ if and only if $F_t^*\mu = \mu$, where F_t is the flow of X (that is, F_t is ***volume preserving***.)
- If $\varphi : M \longrightarrow M$, there is a function, called the ***Jacobian*** of φ and denoted by $J_\mu(\varphi)$ or simply $J(\varphi)$, such that

$$\varphi^*\mu = J_\mu(\varphi)\mu.$$

Volume Forms and Divergence

- Dynamic approach to Lie derivatives $\Rightarrow \operatorname{div}_\mu(X) = 0$ if and only if $F_t^* \mu = \mu$, where F_t is the flow of X (that is, F_t is ***volume preserving***.)
- If $\varphi : M \longrightarrow M$, there is a function, called the ***Jacobian*** of φ and denoted by $J_\mu(\varphi)$ or simply $J(\varphi)$, such that

$$\varphi^* \mu = J_\mu(\varphi)\mu.$$

- Consequence: *φ is volume preserving if and only if $J_\mu(\varphi) = 1$.*

Frobenius' Theorem

- A vector subbundle (a regular distribution) $E \subset TM$ is ***involutive*** if for any two vector fields X, Y on M with values in E, the Jacobi–Lie bracket $[X, Y]$ is also a vector field with values in E.

Frobenius' Theorem

- A vector subbundle (a regular distribution) $E \subset TM$ is ***involutive*** if for any two vector fields X, Y on M with values in E, the Jacobi–Lie bracket $[X, Y]$ is also a vector field with values in E.
- E is ***integrable*** if for each $m \in M$ there is a local submanifold of M containing m such that its tangent bundle equals E restricted to this submanifold.

Frobenius' Theorem

- A vector subbundle (a regular distribution) $E \subset TM$ is ***involutive*** if for any two vector fields X, Y on M with values in E, the Jacobi–Lie bracket $[X, Y]$ is also a vector field with values in E.
- E is ***integrable*** if for each $m \in M$ there is a local submanifold of M containing m such that its tangent bundle equals E restricted to this submanifold.
- If E is integrable, the local integral manifolds can be extended to a maximal integral manifold. The collection of these forms a ***foliation***.

Frobenius' Theorem

- A vector subbundle (a regular distribution) $E \subset TM$ is ***involutive*** if for any two vector fields X, Y on M with values in E, the Jacobi–Lie bracket $[X, Y]$ is also a vector field with values in E.
- E is ***integrable*** if for each $m \in M$ there is a local submanifold of M containing m such that its tangent bundle equals E restricted to this submanifold.
- If E is integrable, the local integral manifolds can be extended to a maximal integral manifold. The collection of these forms a ***foliation***.
- **Frobenius theorem:** E is involutive if and only if it is integrable.

Stokes' Theorem

- **Idea:** Integral of an n-form μ on an oriented n-manifold M: pick a covering by coordinate charts and sum up the ordinary integrals of $f(x^1, \ldots, x^n)\, dx^1 \cdots dx^n$, where

$$\mu = f(x^1, \ldots, x^n)\, dx^1 \wedge \cdots \wedge dx^n$$

(don't count overlaps twice).

Stokes' Theorem

- **Idea:** Integral of an n-form μ on an oriented n-manifold M: pick a covering by coordinate charts and sum up the ordinary integrals of $f(x^1, \ldots, x^n)\, dx^1 \cdots dx^n$, where

$$\mu = f(x^1, \ldots, x^n)\, dx^1 \wedge \cdots \wedge dx^n$$

  (don't count overlaps twice).
- The change of variables formula guarantees that the result, denoted by $\int_M \mu$, is well-defined.

Stokes' Theorem

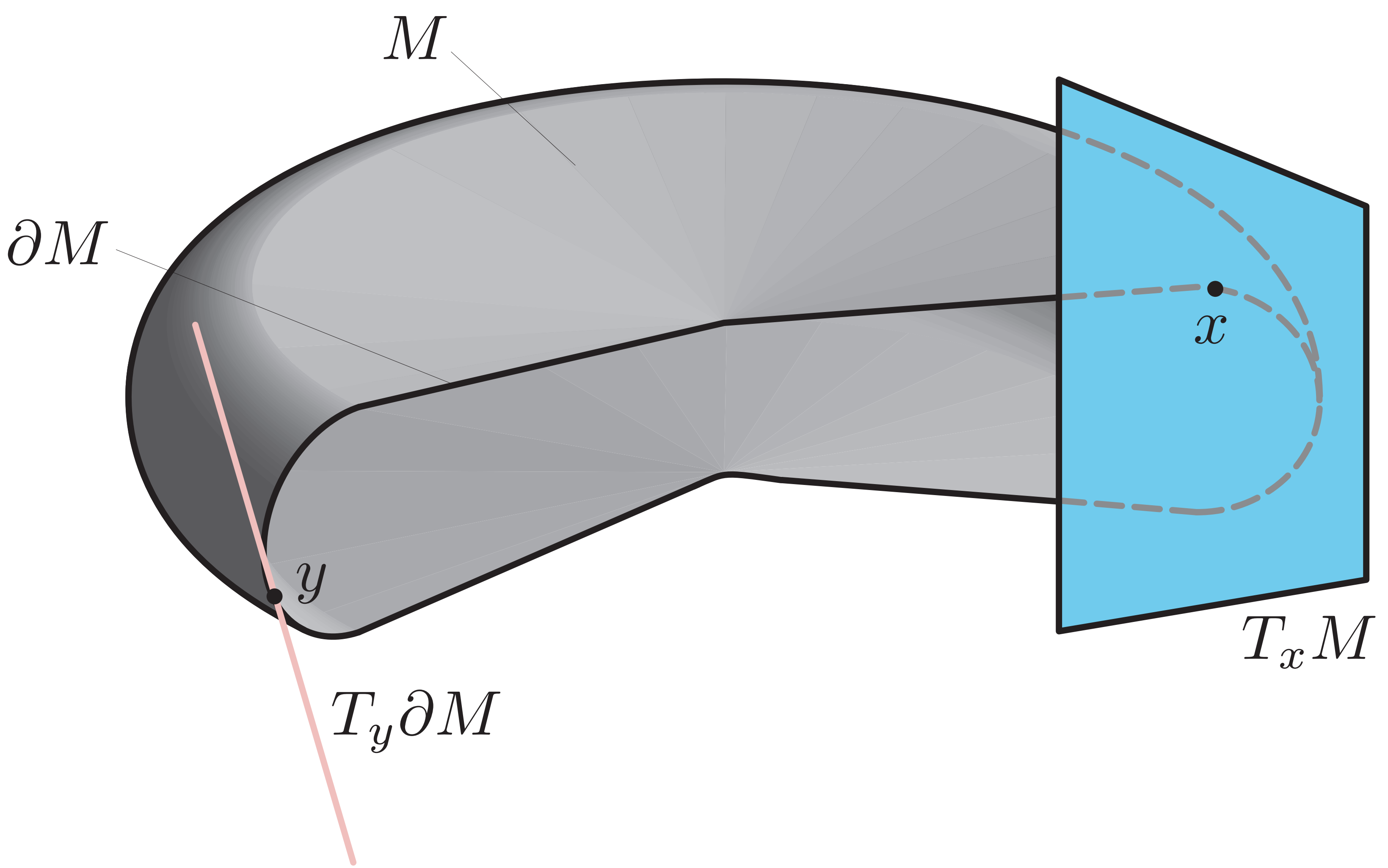
- **Idea:** Integral of an n-form μ on an oriented n-manifold M: pick a covering by coordinate charts and sum up the ordinary integrals of $f(x^1, \ldots, x^n)\, dx^1 \cdots dx^n$, where

$$\mu = f(x^1, \ldots, x^n)\, dx^1 \wedge \cdots \wedge dx^n$$

(don't count overlaps twice).

- The change of variables formula guarantees that the result, denoted by $\int_M \mu$, is well-defined.
- **Oriented manifold with boundary:** the boundary, ∂M, inherits a compatible orientation: generalizes the relation between the orientation of a surface and its boundary in the classical Stokes' theorem in $\mathbb{R}^3$.

Stokes' Theorem



Stokes' Theorem

- **Stokes' Theorem** Suppose that M is a compact, oriented k-dimensional manifold with boundary ∂M. Let α be a smooth $(k-1)$-form on M. Then
$$\int_M \mathbf{d}\alpha = \int_{\partial M} \alpha.$$

Stokes' Theorem

- **Stokes' Theorem** Suppose that M is a compact, oriented k-dimensional manifold with boundary ∂M. Let α be a smooth $(k-1)$-form on M. Then

$$\int_M \mathbf{d}\alpha = \int_{\partial M} \alpha.$$

- Special cases: The classical vector calculus theorems of Green, Gauss and Stokes.

Stokes' Theorem

(a) Fundamental Theorem of Calculus.

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

(b) Green's Theorem. For a region $\Omega \subset \mathbb{R}^2$,

$$\iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_{\partial\Omega} P\,dx + Q\,dy.$$

(c) Divergence Theorem. For a region $\Omega \subset \mathbb{R}^3$,

$$\iiint_\Omega \operatorname{div} \mathbf{F}\,dV = \iint_{\partial\Omega} \mathbf{F} \cdot n\,dA.$$

Stokes' Theorem

(d) **Classical Stokes' Theorem.** For a surface $S \subset \mathbb{R}^3$,

$$\iint_S \left\{ \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy \right\} = \iint_S \mathbf{n} \cdot \operatorname{curl} \mathbf{F} \, dA = \int_{\partial S} P\, dx + Q\, dy + R\, dz,$$

where $\mathbf{F} = (P, Q, R)$.

Stokes' Theorem

- **Poincaré lemma:** generalizes vector calculus theorems: if curl $\mathbf{F} = 0$, then $\mathbf{F} = \nabla f$, and if div $\mathbf{F} = 0$, then $\mathbf{F} = \nabla \times \mathbf{G}$.

Stokes' Theorem

- **Poincaré lemma:** generalizes vector calculus theorems: if $\operatorname{curl} \mathbf{F} = 0$, then $\mathbf{F} = \nabla f$, and if $\operatorname{div} \mathbf{F} = 0$, then $\mathbf{F} = \nabla \times \mathbf{G}$.
- Recall: *if α is closed, then locally α is exact; that is, if $\mathbf{d}\alpha = 0$, then locally $\alpha = \mathbf{d}\beta$ for some β.*

Stokes' Theorem

- **Poincaré lemma:** generalizes vector calculus theorems: if curl $\mathbf{F} = 0$, then $\mathbf{F} = \nabla f$, and if div $\mathbf{F} = 0$, then $\mathbf{F} = \nabla \times \mathbf{G}$.
- Recall: *if α is closed, then locally α is exact; that is, if $\mathbf{d}\alpha = 0$, then locally $\alpha = \mathbf{d}\beta$ for some β.*
- **Calculus Examples:** need not hold globally:

$$\alpha = \frac{xdy - ydx}{x^2 + y^2}$$

is closed (or as a vector field, has zero curl) but is not exact (not the gradient of any function on $\mathbb{R}^2$ minus the origin).

Change of Variables

- M and N oriented n-manifolds; $\varphi : M \longrightarrow N$ an orientation-preserving diffeomorphism, α an n-form on N (with, say, compact support), then

$$\int_M \varphi^* \alpha = \int_N \alpha.$$

Identities for Vector Fields and Forms

- Vector fields on M with the bracket $[X, Y]$ form a ***Lie algebra***; that is, $[X, Y]$ is real bilinear, skew-symmetric, and ***Jacobi's identity*** holds:
$$[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0.$$
Locally,
$$[X, Y] = (X \cdot \nabla)Y - (Y \cdot \nabla)X,$$
and on functions,
$$[X, Y][f] = X[Y[f]] - Y[X[f]].$$

- For diffeomorphisms φ and ψ,
$$\varphi_*[X, Y] = [\varphi_* X, \varphi_* Y] \quad \text{and} \quad (\varphi \circ \psi)_* X = \varphi_* \psi_* X.$$

- $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$ and $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$ for k- and l-forms α and β.

- For maps φ and ψ,
$$\varphi^*(\alpha \wedge \beta) = \varphi^* \alpha \wedge \varphi^* \beta \quad \text{and} \quad (\varphi \circ \psi)^* \alpha = \psi^* \varphi^* \alpha.$$

Identities for Vector Fields and Forms

- $\mathbf{d}$ is a real linear map on forms, $\mathbf{dd}\alpha = 0$, and
$$\mathbf{d}(\alpha \wedge \beta) = \mathbf{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathbf{d}\beta$$
for α a k-form.
- For α a k-form and $X_0, \ldots, X_k$ vector fields,
$$(\mathbf{d}\alpha)(X_0, \ldots, X_k) = \sum_{i=0}^{k} (-1)^i X_i[\alpha(X_0, \ldots, \hat{X}_i, \ldots, X_k)]$$
$$+ \sum_{0 \le i < j \le k} (-1)^{i+j} \alpha([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k),$$
where $\hat{X}_i$ means that X_i is omitted. Locally,
$$\mathbf{d}\alpha(x)(v_0, \ldots, v_k) = \sum_{i=0}^{k} (-1)^i \mathbf{D}\alpha(x) \cdot v_i(v_0, \ldots, \hat{v}_i, \ldots, v_k).$$
- For a map φ,
$$\varphi^* \mathbf{d}\alpha = \mathbf{d}\varphi^* \alpha.$$

Identities for Vector Fields and Forms

- **Poincaré Lemma.** If $\mathbf{d}\alpha = 0$, then the k-form α is locally exact; that is, there is a neighborhood U about each point on which $\alpha = \mathbf{d}\beta$. This statement is global on contractible manifolds or more generally if $H^k(M) = 0$.
- $\mathbf{i}_X\alpha$ is real bilinear in X, α, and for $h : M \to \mathbb{R}$,

$$\mathbf{i}_{hX}\alpha = h\mathbf{i}_X\alpha = \mathbf{i}_X h\alpha.$$

Also, $\mathbf{i}_X\mathbf{i}_X\alpha = 0$ and

$$\mathbf{i}_X(\alpha \wedge \beta) = \mathbf{i}_X\alpha \wedge \beta + (-1)^k \alpha \wedge \mathbf{i}_X\beta$$

for α a k-form.

- For a diffeomorphism φ,

$$\varphi^*(\mathbf{i}_X\alpha) = \mathbf{i}_{\varphi^*X}(\varphi^*\alpha), \quad \text{i.e.,} \quad \varphi^*(X \,\lrcorner\, \alpha) = (\varphi^*X) \,\lrcorner\, (\varphi^*\alpha).$$

- If $f : M \to N$ is a mapping and Y is f-related to X, that is,

$$Tf \circ X = Y \circ f,$$

Identities for Vector Fields and Forms

then

$$\mathbf{i}_X f^*\alpha = f^*\mathbf{i}_Y\alpha; \quad \text{i.e.,} \quad X \mathbin{\lrcorner} (f^*\alpha) = f^*(Y \mathbin{\lrcorner} \alpha).$$

- $\pounds_X\alpha$ is real bilinear in X, α and

$$\pounds_X(\alpha \wedge \beta) = \pounds_X\alpha \wedge \beta + \alpha \wedge \pounds_X\beta.$$

- **Cartan's Magic Formula:**

$$\pounds_X\alpha = \mathbf{d}\mathbf{i}_X\alpha + \mathbf{i}_X\mathbf{d}\alpha = \mathbf{d}(X \mathbin{\lrcorner} \alpha) + X \mathbin{\lrcorner} \mathbf{d}\alpha.$$

- For a diffeomorphism φ,

$$\varphi^*\pounds_X\alpha = \pounds_{\varphi^*X}\varphi^*\alpha.$$

If $f : M \to N$ is a mapping and Y is f-related to X, then

$$\pounds_Y f^*\alpha = f^*\pounds_X\alpha.$$

Identities for Vector Fields and Forms

- $$(\pounds_X \alpha)(X_1, \ldots, X_k) = X[\alpha(X_1, \ldots, X_k)] - \sum_{i=0}^{k} \alpha(X_1, \ldots, [X, X_i], \ldots, X_k).$$

  Locally,

  $$(\pounds_X \alpha)(x) \cdot (v_1, \ldots, v_k) = (\mathbf{D}\alpha_x \cdot X(x))(v_1, \ldots, v_k) + \sum_{i=0}^{k} \alpha_x(v_1, \ldots, \mathbf{D}X_x \cdot v_i, \ldots, v_k).$$

- More identities:
  - $\pounds_{fX}\alpha = f\pounds_X\alpha + \mathbf{d}f \wedge \mathbf{i}_X\alpha;$
  - $\pounds_{[X,Y]}\alpha = \pounds_X\pounds_Y\alpha - \pounds_Y\pounds_X\alpha;$
  - $\mathbf{i}_{[X,Y]}\alpha = \pounds_X\mathbf{i}_Y\alpha - \mathbf{i}_Y\pounds_X\alpha;$
  - $\pounds_X\mathbf{d}\alpha = \mathbf{d}\pounds_X\alpha;$
  - $\pounds_X\mathbf{i}_X\alpha = \mathbf{i}_X\pounds_X\alpha;$

Identities for Vector Fields and Forms

- $\pounds_X(\alpha \wedge \beta) = \pounds_X\alpha \wedge \beta + \alpha \wedge \pounds_X\beta$.

Identities for Vector Fields and Forms

∘ **Coordinate formulas:** for $X = X^l \partial/\partial x^l$, and

$$\alpha = \alpha_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where $i_1 < \cdots < i_k$:

- $$\mathbf{d}\alpha = \left(\frac{\partial \alpha_{i_1 \ldots i_k}}{\partial x^l} \right) dx^l \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

- $$\mathbf{i}_X \alpha = X^l \alpha_{l i_2 \ldots i_k} dx^{i_2} \wedge \cdots \wedge dx^{i_k},$$

- $$\pounds_X \alpha = X^l \left(\frac{\partial \alpha_{i_1 \ldots i_k}}{\partial x^l} \right) dx^{i_1} \wedge \cdots \wedge dx^{i_k} + \alpha_{l i_2 \ldots i_k} \left(\frac{\partial X^l}{\partial x^{i_1}} \right) dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k} + \ldots .$$