# CDS 202 Practice Final Examination

J. Marsden, March, 2008

*Attempt **four** of the following six questions.*

The exam time limit is three hours; ***no aids are permitted***.

***The exam must be turned in to the TAs by Wednesday, March 19, 2008***

***The exam has two sheets printed on both sides***

**Print Your Name**: ______________________ ←**Note!**

**The 4 questions to be graded**: ______________________ ←**Note!**

You may freely use the following properties as needed. Here $\alpha$ and $\beta$ are differential forms and $X, Y, Z$ are vector fields on a manifold $M$. (All manifolds, vector fields, and differential forms are assumed to be smooth and the manifolds are finite dimensional.)

**(a)** $\pounds_X(\alpha \wedge \beta) = (\pounds_X \alpha) \wedge \beta + \alpha \wedge (\pounds_X \beta)$

**(b)** $\pounds_{[X,Y]}\alpha = \pounds_X \pounds_Y \alpha - \pounds_Y \pounds_X \alpha$

**(c)** $\mathbf{i}_X(\alpha \wedge \beta) = (\mathbf{i}_X \alpha) \wedge \beta + (-1)^k \alpha \wedge (\mathbf{i}_X \beta)$, where $\alpha$ is a $k$-form.

**(d)** $\pounds_X \alpha = \mathbf{d}\mathbf{i}_X \alpha + \mathbf{i}_X \mathbf{d}\alpha$

**(e)** $\mathbf{i}_{[X,Y]}\beta = \pounds_X \mathbf{i}_Y \beta - \mathbf{i}_Y \pounds_X \beta$

**(f)** For $\gamma$ a one-form,

$$\mathbf{d}\gamma(X, Y) = X[\gamma(Y)] - Y[\gamma(X)] - \gamma([X, Y])$$

**(g)** For $\omega$ a two-form,

$$\begin{aligned}\mathbf{d}\omega(X, Y, Z) = {} & X[\omega(Y, Z)] - Y[\omega(X, Z)] + Z[\omega(X, Y)] \\ & - \omega([X, Y], Z) - \omega([Z, X], Y) - \omega([Y, Z], X)\end{aligned}$$

**(h)** For a one form $\alpha$ and a vector field $X$,

$$(\pounds_X \alpha)_i = X^j \frac{\partial \alpha_i}{\partial x^j} + \alpha_j \frac{\partial X^j}{\partial x^i}$$

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**1.** Consider the following vector fields $X, Y$, the one form $\alpha$ and the three form $\mu$ on $\mathbb{R}^3$:

$$
\begin{aligned}
X &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \\
Y &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \\
\alpha &= y\,dx - x\,dy + z\,dz \\
\mu &= dx \wedge dy \wedge dz
\end{aligned}
$$

(a) Compute the exterior derivative $d\alpha$ and the interior product $\mathbf{i}_X\alpha$.

(b) Compute the Lie derivative $\pounds_X\alpha$

(c) Describe the flows $F_t$ of $X$ and $G_t$ of $Y$ geometrically.

(d) Compute

$$\left.\frac{d}{dt}\right|_{t=0} F_t^*\mu \text{ and } \left.\frac{d}{dt}\right|_{t=0} G_t^*\alpha$$

(e) Compute $\left.\frac{d}{dt}\right|_{t=0} F_t^*Y$.

**2.** Let $M$ be the ellipsoidal shell in $\mathbb{R}^3$ given by $x^2 + 4y^2 + z^2 = 1$ and let $S$ be the partial ellipsoidal shell in $\mathbb{R}^3$ defined by the conditions $(x, y, z) \in M$ and $0 \le x \le 1/2$ .

(a) Show that $M$ is a smooth manifold.

(b) Argue informally that $S$ is a smooth oriented manifold with boundary; describe a specific choice of orientation.

(c) Let the one form $\alpha$ be defined on the open set $U = \mathbb{R}^3 \backslash x$-axis by

$$\alpha = \frac{zdy - ydz}{y^2 + z^2}$$

Compute $\mathbf{d}\alpha$.

(d) Let $\beta$ be the pull-back of $\alpha$ to $S$. Is $\beta$ closed? Is $\beta$ exact?

(e) Compute the integral of $\beta$ over $\partial S$.

**3.** Let $S$ be the $3 \times 3$ diagonal matrix with diagonal entries $1, 1, -2$. Let $G$ denote the set of $3 \times 3$ real matrices $A$ that satisfy $A^T S A = S$, where $A^T$ denotes the transpose of $A$.

(a) Show that, with the operation of matrix multiplication, $G$ is a Lie group.

(b) What is its dimension? Is $G$ compact?

(c) Show that the Lie algebra $\mathfrak{g}$ of $G$ may be identified with the set of $3 \times 3$ matrices $\xi$ that satisfy $\xi^T S + S\xi = 0$. What is the Lie algebra bracket?

(d) If $\alpha$ is a nonzero real number, show that the matrix

$$\xi = \begin{bmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

lies in the Lie algebra $\mathfrak{g}$. What is the one parameter subgroup of $G$ that is tangent to $\xi$ at $t = 0$?

(e) Let $\eta, \xi \in \mathfrak{g}$ be two matrices in $\mathfrak{g}$ from part (c) that commute. Let $D$ be the distribution on $G$ obtained by left translating the two dimensional vector space $V = \operatorname{span}(\eta, \xi)$ around the group. Is $D$ integrable?

**4.** (a) Let $X$ and $Y$ be the vector fields on $\mathbb{R}^3$ defined by

$$X = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \quad \text{and} \quad Y = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}.$$

Show that $X$ and $Y$ define vector fields $X_0$ and $Y_0$ on the standard two sphere $S^2$ of radius one.

i. Show that, with respect to the standard volume element on $S^2$, $\operatorname{div} X_0 = 0$ and $\operatorname{div} Y_0 = 0$.

ii. Calculate $[X_0, Y_0]$.

(b) Let $(M_1, \mu_1)$ and $(M_2, \mu_2)$ be two compact volume manifolds without boundary and let $X_1$ be a smooth vector field on $M_1$.

i. Explain how $(M_1 \times M_2, \mu_1 \times \mu_2)$ is a volume manifold with volume element $\mu_1 \times \mu_2$ determined in a natural way from $\mu_1$ and $\mu_2$.

ii. Is it true that

$$\int_{M_1 \times M_2} (\operatorname{div}_{\mu_1} X_1)\ \mu_1 \times \mu_2$$

must be zero?

**5.** (a) Let $S^1$ be the standard two sphere of radius one in $\mathbb{R}^3$ and $S^R$ the sphere of radius $R$. Let $\phi : S^1 \to S^R$ be the map that takes $\mathbf{x} \in S^1$ to $R\mathbf{x} \in S^R$. Show that $\phi$ is an orientation preserving diffeomorphism and state the change of variables formula for this map.

(b) Let the vector field $X$ on $\mathbb{R}^3$ be defined by

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$$

and let $F_t$ be its flow. Show that the flow defines, for each $t$, an orientation preserving diffeomorphism of $S^1$ to a sphere of another radius $R(t)$.

(c) Let $f(x, y, z, t)$ be a time dependent function on $\mathbb{R}^3$ and also use the notation $f$ to denote its restriction to a sphere. Let $\mu_R$ denote the standard area form on $S^R$. Find an expression for

$$\frac{d}{dt}\int_{S^{R(t)}} f\mu_{R(t)}$$

where $R(t)$ is as in part (b) and check your calculation explicitly for the function $f$ that is identically one.

**6.** (a) Consider the distribution on $\mathbb{R}^3 \setminus \{0\}$ that is given at the point $(x, y, z)$ by the set of vectors $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ satisfying $6ax + 2by + 10cz = 0$. Is this distribution integrable? If so, find the corresponding integrable manifolds.

(b) Let $\omega$ be a closed two form on a manifold $M$ and let $X$ be a vector field with a flow $F_t$ satisfying $F_t^*\omega = \omega$. Show that the distribution defined (at each point) to be the kernel of the one-form $\mathbf{i}_X\omega$ is integrable.

(c) Denote coordinates on $\mathbb{R}^{2n}$ by $(q^i, p_i)$, where $i$ ranges between 1 and $n$ and define the two-form $\omega$ by $\omega = dq^i \wedge dp_i$ (where a sum on $i$ is understood). Let $H(q, p)$ be a given function and let $X$ be the vector field such that $\mathbf{i}_X\omega = \mathbf{d}H$. Show that the conditions of part (b) hold and determine the foliation in this case.