Hamiltonian aspects of fluid dynamics

CDS 140b

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Outline for this week

1. Dynamics of point vortices;
    1.1 Vorticity;
    1.2 Fluid dynamics in 2D;
    1.3 Dynamics of N vortices;
    1.4 The Kirchhoff-Routh function;
    1.5 Dynamics of $N = 1, 2, 3$ vortices;
2. Chaotic advection;
    2.1 Aref's stirring mechanism;
    2.2 The ABC flow.

Vortex dynamics

References

1. P. Newton: *The N-vortex problem. Analytical techniques*. Applied Mathematical Sciences, vol. 145. Springer-Verlag, 2001.
2. H. Aref: *Point vortex dynamics: A classical mathematics playground*. J. Math. Phys. **48**, 065401 (2007).
3. P. G. Saffmann: *Vortex Dynamics*. Cambridge Monographs on Mechanics and Applied Mathematics. Cambridge University Press, 1992.

Dynamics of an inviscid flow

1. Euler equations:

$$\frac{d\mathbf{u}}{dt} := \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p,$$

   together with the incompressibility condition $\nabla \cdot \mathbf{u} = 0$. Pressure p acts as a Lagrange multiplier for this constraint, and satisfies $\nabla^2 p = 0$.

2. Take the curl of Euler, and put $\omega = \nabla \times \mathbf{u}$:

$$\frac{d\omega}{dt} = \omega \cdot \nabla \mathbf{u}.$$

   (vorticity form of Euler eqns).

Due to the presence of p, system (1) is much more complicated than (2).

What is vorticity?

Intuitively: vorticity is a measure for the amount of rotation of the fluid.

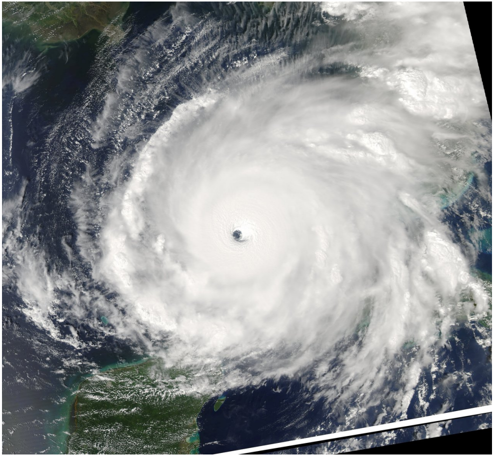
- Suppose given a flow with velocity field $\mathbf{u}(x, y, z, t)$.
- Mathematically, vorticity is a vector field ω given by

$$\omega = \nabla \times \mathbf{u}.$$

Why study vorticity?

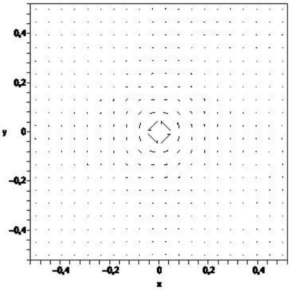
- Localised patches of vorticity appear quite often in nature;
- numerically, vortex methods are very attractive;
- vorticity equation contains just as much information as the Euler equation;
- vortices are "a classical mathematics playground" (Aref).

Hurricane Rita



Example (point vortex)

$$\mathbf{u} = \frac{1}{2\sqrt{x^2 + y^2}}(-y, x, 0) \quad \Rightarrow \quad \omega = (0, 0, \delta(x, y)).$$



This will be the building block of our subsequent treatment. Think of a point vortex as being similar to a point mass.

Fluid dynamics in 2D

We will only be concerned with 2D flows in these lectures!

- Consider a fluid with velocity $\mathbf{u}(\mathbf{x}, t) = (u_x(\mathbf{x}, t), u_y(\mathbf{x}, t))$ in $2D$.
- Fluid is incompressible if

$$\nabla \cdot \mathbf{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0.$$

- Incompressibility: there exists a stream function ψ such that

$$u_x = \frac{\partial \psi}{\partial y} \quad \text{and} \quad u_y = -\frac{\partial \psi}{\partial x}.$$

- If $\mathbf{u}$ is independent of t: steady flow.

Trajectories of fluid particles

- Motion of individual fluid particles:

$$\dot{x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \dot{y} = -\frac{\partial \psi}{\partial x}.$$

  Obvious Hamiltonian structure, with Hamiltonian ψ and conjugate variables x and y.

- Poisson form: $\dot{f} = \{f, \psi\}$ for all functions f on $\mathbb{R}^2$, and where the Poisson bracket is given by

$$\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}.$$

Result: use the heavy machinery from Hamiltonian dynamical systems to get results about fluid dynamics.

Preservation of vorticity

- Recall the equation governing the dynamics of the vorticity field. In general:

$$\frac{d\omega}{dt} = \omega \cdot \nabla \mathbf{u}.$$

- In 2D: $\mathbf{u} = (u_x, u_y, 0)$ and ω is proportional to $\mathbf{e}_z$. Therefore

$$\omega \cdot \nabla \mathbf{u} = 0.$$

  Hence

$$\frac{d\omega}{dt} = 0.$$

Result: vorticity is simply advected with the flow!

Getting $\mathbf{u}$ if ω is known

Note: in 2D, ω is a scalar.

- Fact: any vector field $\mathbf{u}$ on the whole of $\mathbb{R}^2$ can be written as

$$\mathbf{u} = \nabla\phi + \nabla \times (\psi \mathbf{e}_z),$$

  (Helmholtz-Hodge decomposition).
- Take the curl: $\nabla^2\psi = -\omega$.
- Solution of this Poisson equation gives you ψ:

$$\psi(\mathbf{x}) = -\int \frac{1}{2\pi} \log \|\mathbf{x} - \mathbf{y}\| \, \omega(\mathbf{y}) d\mathbf{y}.$$

  (Similar formulas work in 3D).
- Finally, put $\mathbf{u} = \nabla \times (\psi \mathbf{e}_z)$. This determines $\mathbf{u}$ up to a gradient of a scalar function.

Example: sum of point vortices

- Take a vorticity field of the following form:

$$\omega(\mathbf{x}) = \sum_{i=1}^{N} \Gamma_i \delta(\mathbf{x} - \mathbf{x}_i).$$

- Associated stream function:

$$\begin{aligned}\psi(\mathbf{x}) &= -\sum \Gamma_i \int \frac{1}{2\pi} \log \|\mathbf{x} - \mathbf{y}\| \, \delta(\mathbf{y} - \mathbf{x}_i(t)) d\mathbf{y} \\ &= -\sum \frac{\Gamma_i}{2\pi} \log \|\mathbf{x} - \mathbf{x}_i\| \,.\end{aligned}$$

- Velocity field:

$$\mathbf{u}(x) = \nabla \times (\psi \mathbf{e}_z) = -\sum \frac{\Gamma_i}{2\pi} \frac{(-(y - y_i), x - x_i)}{\|\mathbf{x} - \mathbf{x}_i\|^2}.$$

Dynamics of N vortices: fluid dynamics

Take again N point vortices, located at $\mathbf{x}_i(t)$, $i = 1, \ldots, N$.

- The velocity field of the fluid due these vortices is

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^{N} \nabla \times (\psi_i \mathbf{e}_z) \quad \text{where} \quad \psi_i = -\frac{\Gamma_i}{2\pi} \log \|\mathbf{x} - \mathbf{x}_i\| .$$

- Since the vortices are advected, their velocity is simply the velocity of the surrounding flow:

$$\dot{\mathbf{x}}_i(t) = \sum_{i \neq j} \nabla \times (\psi_j \mathbf{e}_z).$$

Note: we removed singular terms.

The Kirchhoff-Routh function

- The stream function for the fluid due to N vortices is

$$\psi = \sum_{i=1}^{N} \psi_i(\mathbf{x}), \quad \text{where} \quad \psi_i = -\frac{\Gamma_i}{2\pi} \log \|\mathbf{x} - \mathbf{x}_i\| .$$

- Define the Kirchhoff-Routh function H as the following function:

$$H = -\sum_{i \neq j} \frac{\Gamma_i \Gamma_j}{4\pi} \log \|\mathbf{x}_i - \mathbf{x}_j\| .$$

H is related to ψ, but without the singular contributions. Physically, H represents the kinetic energy of the N-vortex system.

Dynamics of N vortices: Hamiltonian form

Main idea: *vortex motion* = *finite-dimensional Hamiltonian system*.

1. Configuration space is $\mathbb{R}^{2N}$;
2. Hamiltonian: Kirchhoff-Routh function H.

$$\begin{cases} \Gamma_i \dot{x}_i(t) &= \frac{\partial H}{\partial y_i} \\ \Gamma_i \dot{y}_i(t) &= -\frac{\partial H}{\partial x_i} \end{cases}$$

Rescale variables to obtain "true" Hamiltonian system.
Explicitly:

$$\dot{x}_i(t) = -\frac{1}{2\pi}\sum_{i\neq j}\Gamma_j \frac{y_i - y_j}{\|\mathbf{x}_i - \mathbf{x}_j\|^2} \quad \text{and} \quad \dot{y}_i(t) = \frac{1}{2\pi}\sum_{i\neq j}\Gamma_j \frac{x_i - x_j}{\|\mathbf{x}_i - \mathbf{x}_j\|^2}$$

Poisson bracket:

$$\{f, g\} = \sum_{i=1}^{N} \frac{1}{\Gamma_i}\left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial y_i}\right).$$

Dynamics of vortices for $N = 1, 2$

$N = 1$: vortex just sits there.

$N = 2$ (see also example 1.8 in Newton)

Assume that $\Gamma_1 = \Gamma_2 = \Gamma \neq 0$.

- Two conserved quantities:

$$C = \frac{1}{2}(\mathbf{x}_1(t) + \mathbf{x}_2(t)) \quad \text{and} \quad D^2 = \|\mathbf{x}_1(t) - \mathbf{x}_2(t)\|^2 .$$

  (C: center of "mass"/barycenter/...)

- System decouples and can be rewritten in action-angle variables (R_i, θ_i) by the following canonical trafo:

$$\mathbf{x}_i - C = \sqrt{2R_i(t)}\exp(i\theta_i(t)),$$

  giving $\dot{R}_i = 0$ and $\dot{\theta}_i = 0$. Result: vortices rotate on a circle around C (integrability).

Dynamics of vortices for $N = 3, 4, \ldots$

- $N = 3$: Still integrable. Four integrals of motion: H, linear impulse $\mathcal{I}$ and angular impulse $\mathcal{L}$, where

$$\mathcal{I} = \sum_{i=1}^{N} \Gamma_i \mathbf{x}_i \quad \text{and} \quad \mathcal{L} = \sum_{i=1}^{N} \Gamma_i \left\| x_i \right\|^2 .$$

  (think Noether: invariance under time translation, spatial translation, and rotation). Three involutive quantities: H, $\mathcal{L}$, and $\mathcal{I}_x^2 + \mathcal{I}_y^2$.
- $N = 4$: Arnold-Liouville integrable if $\sum_{i=1}^{4} \Gamma_i = 0$. Nonintegrable in general.
- $N \to +\infty$: Statistical mechanics. Euler equations? Chaos?

Hamiltonian reduction

Hamiltonian

- Kinetic energy of a fluid:

$$H = \frac{1}{2}\int \|\mathbf{u}\|^2 \, d\mathbf{x} = \frac{1}{2}\int \omega\psi d\mathbf{x}.$$

- Plug in $\omega = \sum_{i=1}^N \Gamma_i \delta(\mathbf{x} - \mathbf{x}_i)$, and remove singular terms:

$$H = -\frac{1}{4\pi}\sum_{i\neq j} \Gamma_i \Gamma_j \log \|\mathbf{x}_i - \mathbf{x}_j\| .$$

Marsden and Weinstein: the passage from Euler to vortex dynamics is a special case of symplectic reduction.

Further outlook

- Generalisations and applications:
  1. Consider vortices in the presence of solid bodies: Karman vortex street, stability, etc.
  2. look at vorticity concentrated in *patches*, along *lines*, etc.
  3. quantum theory of vortices in superfluid helium.
- For (1), see Dr Kanso's lectures next week.
- *Crowds of exceedingly interesting cases present themselves.* (Kelvin 1880)

Chaotic advection

References

- V. V. Meleshko, H. Aref: *A blinking rotlet model for chaotic advection*. Phys. Fluids **8** (12), Dec. 1996, 3215-3217. Erratum: Phys. Fluids **10** (6), June 1998.
- V. I. Arnold, B. A. Khesin: *Topological Methods in Hydrodynamics*. Applied Mathematical Sciences 125. Springer (1998).

Topology of stream lines

- Let $\psi(x, y)$ be an autonomous stream function in $2D$.
- Particle trajectories are lines of constant ψ (streamlines): severely limits possible regions for chaos, ergodicity, mixing, etc.

What happens if we allow for

1. non-autonomous stream functions?
2. higher-dimensional flows?

Chaotic advection (Lagrangian chaos)

- Fluid is quite simple, but particle trajectories show remarkably complicated behaviour.
- Note: don't confuse with *Eulerian chaos* (turbulence, etc.)

Aref's stirring mechanism

Background: Stokes flow

- Consider the non-dimensional Navier-Stokes equations:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + (\mathrm{Re})^{-1} \nabla^2 \mathbf{u},$$

  where the Reynolds number $\mathrm{Re} = \frac{\rho U L}{\mu}$ gives the ratio of inertial to viscous forces.
- For very viscous flows or small length scales, inertial terms are negligible.
- Stokes equation:

$$\nabla p = (\mathrm{Re})^{-1} \nabla^2 \mathbf{u},$$

  No explicit time dependence, other than through the (possibly time dependent) boundaries.
  No-slip boundary condition: fluid sticks to boundaries.

Non-autonomous stream functions

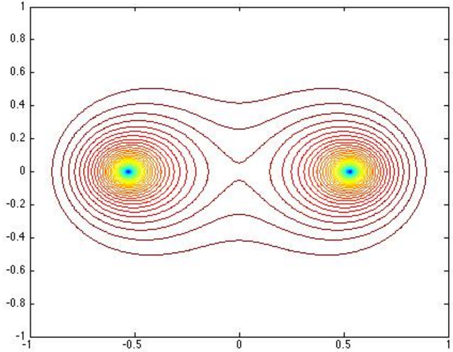
- Aref's example: viscous fluid in a cylinder, with two rotating rods (parallel to the cylinder) in the container.
- Aim: stir fluid by alternating between rotating rod #1 and rod # 2.

Stream function for one rotating rod at location $(r, theta) = (b, 0)$ (rotlet flow):

$$\psi(r,\theta) = \frac{\sigma}{2}\left(\ln \frac{r^2 - 2br\cos\theta + b^2}{a^2 - 2br\cos\theta + b^2 r^2/a^2} + \frac{(1 - r^2/a^2)(a^2 - b^2 r^2/a^2)}{a^2 - 2br\cos\theta + b^2 r^2/a^2} \right).$$

Not *terribly* complicated...

Rotlet stream lines



Each stirring rod is placed at a stagnation point in the flow of the other cylinder. Hence, they don't exert a force on each other.

Poincaré section of Aref's flow

Figures courtesy of H. Aref and V. V. Meleshko (Phys. Fluids 8).

Computation of LCS structures?

(See movies)

The ABC flow

Euler equations in 3D

- Consider again the Euler equations for an inviscid, incompressible flow, but now in three dimensions.
- Stationary flow: $\frac{\partial \mathbf{u}}{\partial t} = 0$. So

$$\mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = 0,$$

or

$$\mathbf{u} \times (\nabla \times \mathbf{u}) = \nabla \alpha,$$

where $\alpha = p + \mathbf{u}^2/2$, the Bernoulli function (first integral).
- $\alpha = 0$: so-called force-free velocity fields.

Regularity in fluid motion

- Force free velocity field: $\mathbf{u} \times (\nabla \times \mathbf{u}) = 0$. Assume that $\mathbf{u}$ vanishes nowhere.
- So, there exists a function f such that

$$\nabla \times \mathbf{u} = f\mathbf{u}.$$

- v is tangent to the level sets of $f \quad \Rightarrow \quad$ compact level surfaces of f are tori.
- The same goes for non-free force flows by looking at the level sets of α.

All this hints, *under well-defined assumptions*, at remarkably regular behaviour!

The ABC flow

- To open the door for chaos, we should tinker with these assumptions. One way out: look for $\mathbf{u}$ such that

$$\nabla \times \mathbf{u} = \lambda \mathbf{u},$$

with λ a *constant* (Beltrami fields).

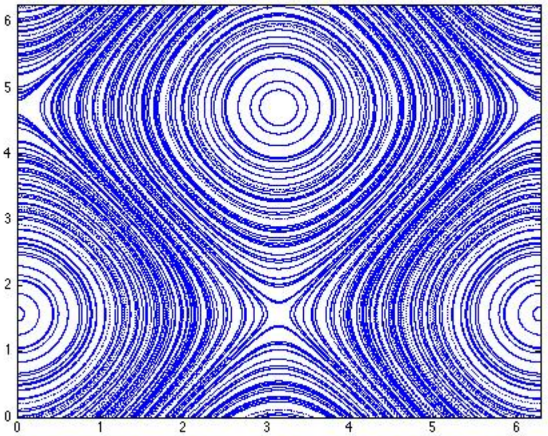
- A famous example: ABC flows on the 3-torus $\{(x, y, z) \mod 2\pi\}$ (i.e. $\mathbb{R}^3$ with periodic boundary conditions).

$$\begin{cases} v_x &= A \sin z + C \cos y, \\ v_y &= B \sin x + A \cos z, \\ v_z &= C \sin y + B \cos x. \end{cases}$$

Integrable when A, B, or C is zero, chaotic otherwise.

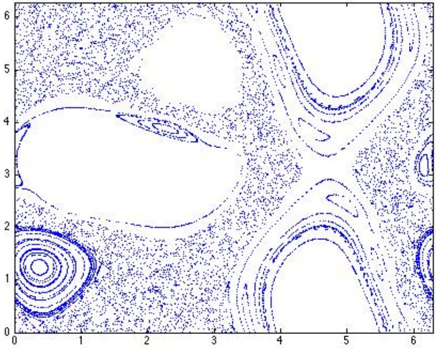
Trajectories for regular motion

$$A = 0, B = \sqrt{2/3}, C = \sqrt{1/3}.$$



Poincaré section for chaotic case

$$A = 1, B = \sqrt{2/3}, C = \sqrt{1/3}.$$



See also computation of LCS structures in Philip's lectures.