Hamiltonian Dynamics

CDS 140b

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CDS

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Outline for this week

1. Introductory concepts;
2. Poisson brackets;
3. Integrability;
4. Perturbations of integrable systems.
   4.1 The KAM theorem;
   4.2 Melnikov's method.

References

Books

- J. Marsden and T. Ratiu: *Introduction to Mechanics and Symmetry*.
- V. Arnold: *Mathematical Methods of Classical Mechanics*.
- M. Tabor: *Chaos and Integrability in Nonlinear Dynamics*.
- P. Newton: *The N-vortex problem*.
- F. Verhulst.

Papers

- J. D. Meiss: *Visual Exploration of Dynamics: the Standard Map*. See `http://arxiv.org/abs/0801.0883` (has links to software used to make most of the plots in this lecture)
- J. D. Meiss: *Symplectic maps, variational principles, and transport*. Rev. Mod. Phys. **64** (1992), no. 3, pp. 795–848.

References

Software

- `GniCodes`: symplectic integration of 2nd order ODEs. Similar in use as Matlab's ode suite. See `http://www.unige.ch/~hairer/preprints/gnicodes.html`
- `StdMap`: Mac program to explore the dynamics of area preserving maps. See `http://amath.colorado.edu/faculty/jdm/stdmap.html`

Introduction

Transition to the Hamiltonian framework

- Consider a mechanical system with n degrees of freedom and described by generalised coordinates $(q^1, \ldots, q^n)$.
- Denote the kinetic energy $\frac{1}{2}mv^2$ by T, and the potential energy by $V(q)$. Define the Lagrangian L to be $T - V$.
- Define the *canonical momenta* p_i as

$$p_i = \frac{\partial L}{\partial v^i}.$$

This defines a map from velocity space with coords (q^i, v^i) to phase space with coords (q^i, p_i), called the *Legendre transformation*.

Transition to the Hamiltonian framework

The associated Hamiltonian is given by

$$H(q,p) = p_i v^i - L(q,v).$$

To express the RHS as a function of q and p only, we need to be able to invert the Legendre transformation. By the implicit function theorem, this is the case if the matrix

$$\frac{\partial^2 L}{\partial v^i \partial v^j} \tag{1}$$

is invertible.

Notes

- Not every Hamiltonian is associated to a Lagrangian in this way. See next week's class on vortex dynamics!
- Much of the above can be extended to the case where (1) is singular.

Hamilton's equations

- Variational interpretation: arise as extrema of the following *action functional*:

$$S(q(t), p(t), t) = \int p_i(t)\dot{q}^i(t) - H(q(t), p(t), t)\mathrm{d}t.$$

- Equations of motion:

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}.$$

Properties of Hamiltonian systems

Poisson brackets: definition

Let $f(q, p, t)$ be a time-dependent function on phase space. Its total derivate is

$$\begin{aligned} \dot{f} \equiv \frac{df}{dt} &= \frac{\partial f}{\partial q^i}\frac{dq^i}{dt} + \frac{\partial f}{\partial p_i}\frac{dp_i}{dt} + \frac{\partial f}{\partial t} \\ &= \frac{\partial f}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q^i} + \frac{\partial f}{\partial t} \\ &= \{f, H\} + \frac{\partial f}{\partial t}, \end{aligned}$$

where we have defined the (canonical) *Poisson bracket* of two functions f and g on phase space as

$$\{f, g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}.$$

Poisson brackets: properties

A Poisson bracket is an operation $\{\cdot, \cdot\}$ on functions satisfying the following properties:

1. $\{f, g\} = -\{g, f\}$;
2. $\{f + g, h\} = \{f, h\} + \{g, h\}$;
3. $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$;
4. $\{fg, h\} = f\{g, h\} + g\{f, h\}$.

Property 3 is called the Jacobi identity.
Properties 1, 2, and 3 make the ring of functions on $\mathbb{R}^{2n}$ into a Lie algebra.

Rewriting Hamilton's equations

- For any function f on phase space, we have

$$\frac{df}{dt} = \{f, H\}.$$

- For $f = q^i$ and $f = p_i$, we recover Hamilton's equations:

$$\dot{q}^i = \{q^i, H\} = \frac{\partial H}{\partial p_i},$$

and

$$\dot{p}_i = \{p_i, H\} = -\frac{\partial H}{\partial q^i}.$$

Conserved quantities

Definition

A function f is a conserved quantity if it Poisson commutes with the Hamiltonian:

$$\{f, H\} = 0.$$

Immediate consequences:

- If the Hamiltonian is autonomous, then it is conserved, as $\{H, H\} = 0$;
- If f and g are conserved, then so is $\{f, g\}$:

$$\{\{f, g\}, H\} = \{\{g, H\}, f\} - \{\{f, H\}, g\} = 0,$$

using the Jacobi identity. Usually this doesn't give too much information.

Not all Poisson brackets are canonical: Euler equations

- Consider a rigid body with moments of inertia (I_1, I_2, I_3) and angular velocity $\mathbf{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$. Define the angular momentum vector

$$\mathbf{\Pi} = (\Pi_1, \Pi_2, \Pi_3) = (I_1\Omega_1, I_2\Omega_2, I_3\Omega_3).$$

- The equations of motion for the rigid body (Euler equations) are

$$\dot{\mathbf{\Pi}} = \mathbf{\Pi} \times \mathbf{\Omega}.$$

or, written out in components,

$$\begin{aligned} I_1\dot{\Omega}_1 &= (I_2 - I_3)\Omega_2\Omega_3, \\ I_2\dot{\Omega}_2 &= (I_3 - I_1)\Omega_3\Omega_1, \\ I_3\dot{\Omega}_3 &= (I_1 - I_2)\Omega_1\Omega_2. \end{aligned}$$

- Clearly not canonical, since *odd* number of equations.

Euler equations: Poisson form

- Define the rigid body Poisson bracket on functions $F(\mathbf{\Pi})$, $G(\mathbf{\Pi})$ as

$$\{F, G\}_{\mathrm{r.b.}}(\mathbf{\Pi}) = -\mathbf{\Pi} \cdot (\nabla F \times \nabla G).$$

- The Euler equations are equivalent to

$$\dot{F} = \{F, H\}_{\mathrm{r.b.}},$$

where the Hamiltonian is given by

$$H = \frac{1}{2}\left(\frac{\Pi_1^2}{I_1} + \frac{\Pi_2^2}{I_2} + \frac{\Pi_3^2}{I_3}\right).$$

Other examples of Poisson brackets

- Fruitful approach: start from canonical Poisson bracket on $\mathbb{R}^{2n}$ or similar, consider a group action which leaves $\{\cdot, \cdot\}$ invariant, and define the bracket on the quotient.
- Other examples: ideal fluids, MHD, the Toda lattice, ...

(Idea for project...)

Characteristic property of Hamiltonian flows

Theorem

The flow of a Hamiltonian vector field preserves the Poisson structure:

$$\{F, G\} \circ \Phi_t = \{F \circ \Phi_t, G \circ \Phi_t\}.$$

(thm. 10.5.1 in [MandS])

Take the derivative of $u := \{F \circ \Phi_t, G \circ \Phi_t\} - \{F, G\} \circ \Phi_t$:

$$\frac{du}{dt} = \{\{F \circ \Phi_t, H\}, G \circ \Phi_t\} + \{F \circ \Phi_t, \{G \circ \Phi_t, H\}\} - \{\{F, G\} \circ \Phi_t, H\}$$

- Jacobi identity: $\frac{du}{dt} = \{u, H\}$.
- Solution: $u_t = u_0 \circ \Phi_t$, but $u_0 = 0$.

Liouville's theorem

To each Hamiltonian H, associate the following vector field on phase space:

$$X_H := \begin{pmatrix} \frac{\partial H}{\partial p_i} \\ -\frac{\partial H}{\partial q^i} \end{pmatrix}$$

Liouville's theorem

The flow of X_H preserves volume in phase space.

- Consequence of the following fact:

$$\text{flow of } \dot{x} = f(x) \text{ is volume preserving} \Longleftrightarrow \mathrm{div} f(x) = 0.$$

-

$$\mathrm{div} X_H = \frac{\partial}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial}{\partial p_i}\frac{\partial H}{\partial q^i} = 0.$$

More is true: the flow of X_H is symplectic.

Consequences of Liouville's theorem

1. In a Hamiltonian system, there are no asymptotically stable equilibria or limit cycles in phase space.
2. Poincaré's recurrence theorem:

Theorem

Let $\Phi : D \to D$ be a volume preserving diffeomorphism from a bounded region $D \subset \mathbb{R}^n$ to itself. Then for any neighborhood U in D, there is a point of U returning to U after sufficiently many iterations of Φ.

The sets $U, \Phi(U), \Phi^2(U), \ldots$ cover D and have the same volume. So, for some k and l, with $k > l$,

$$\Phi^k(U) \cap \Phi^l(U) \neq \varnothing \quad \Leftrightarrow \quad \Phi^{k-l}(U) \cap U \neq \varnothing.$$

Take $y \in U \cap \Phi^{k-l}(U)$ and put $x = \Phi^{-n}(y)$ $(n = k - l)$, then $x \in U$ and $\Phi^n(x) \in U$.

Aside: symplecticity

- Define a map $\omega : \mathbb{R}^{2N} \times \mathbb{R}^{2N} \to \mathbb{R}$ as follows

$$\omega(X, Y) = \text{signed area of parallellogram spanned by } X, Y.$$

- A map $\Phi : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ is symplectic if it preserves these areas.

Remarks

- symplecticity $\Rightarrow$ volume preservation, but not vice versa unless $2N = 2$;
- ω is a *differential form*.

Integrability and near-integrability

Integrability: definitions

Definition

A Hamiltonian system with n degrees of freedom and Hamiltonian H is called integrable if there exist n functionally independent integrals F_i, $i = 1, \ldots, n$, which are in involution: $\{F_i, F_j\} = 0$.

- The odds of randomly picking an integrable Hamiltonian among the class of all analytic functions are zero.
- Motivation: if system is integrable, one can find a canonical trafo to action-angle coordinates

$$(p, q) \mapsto (I, \phi) \quad \text{and} \quad H(p, q) \mapsto H'(I),$$

i.e. H' doesn't depend on ϕ. Trivially integrable eqns of motion:

$$\dot{I} = 0 \quad \text{and} \quad \dot{\phi} = \frac{\partial H'}{\partial I} := \omega(I).$$

Integrability: examples

- Simple example

$$H = \frac{1}{2}(p_1^2 + q_1^2) + \frac{1}{2}(p_2^2 + \omega q_2^2)$$

  Integrable; action-angle form: $H' = I_1 + I_2\omega$. Trajectories fill out 2-torus if ω is irrational.
- Rigid body (energy and angular momentum conserved);
- Kepler problem (energy, Laplace-Runge-Lenz vector, angular momentum), etc.

The Arnold-Liouville theorem

Let M_f be a level set of the F_i:

$$M_f := \{x : F_i(x) = f_i, i = 1, \dots, n\}.$$

Theorem

- *If M_f is compact and connected, then it is diffeomorphic to a smooth n-torus;*
- *M_f is invariant under the flow of H, and the motion on M_f is quasi-periodic. In angular coordinates: $\dot{\phi}_i = \omega_i(f)$.*

Important to remember: integrability $\rightarrow$ motion on invariant tori.

Proof of Arnold's theorem

Idea:

1. the n conserved quantities F_i generate n commuting vector fields X_{F_i};
2. composition of the flows of the X_{F_i} defines an action of $\mathbb{R}^n$ on M;
3. the isotropy subgroup of each point is a lattice in $\mathbb{R}^n$.

Perturbed Hamiltonian systems

Question: what can we say about systems with Hamiltonians of the form

$$H = H_0 + \epsilon H_1,$$

where H_0 is integrable, and ϵ is small?

Possible answers

- Since perturbation is small, the resulting dynamics will still be close to the original dynamics (Birkhoff averaging);
- Even a tiny perturbation destroys integrability completely, rendering the system ergodic (Fermi's point of view).

KAM theory: *sometimes* one is true, in some cases the other $\longrightarrow$ very rich picture!

Importance of near integrability

Many examples in practice

- special cases of 3-body problem;
- Motion of a charged particle in a tokamak;
- The Hénon-Heiles potential (astronomy again).

The solution

KAM theorem (after Kolmogorov-Arnold-Moser)

- One of the most important theorems in 20th century mathematical physics;
- Many deep connections with other branches of math, like number theory, analysis, etc.

Disclaimer: KAM is extremely technical! Fortunately, computer simulations give a quick and far-reaching insight (project idea).

Some definitions

Small divisors are killing us. How can we avoid them?

Definition

- Resonant torus: one for which the rotation numbers $(\omega_1, \ldots, \omega_n)$ satisfy

$$k_1\omega_1 + \cdots + k_n\omega_n = 0,$$

for some $(k_1, \ldots, k_n) \in \mathbb{Z}^n$. Otherwise, torus is non-resonant.

- Strongly non-resonant: there exist $\alpha > 0$ and $\tau > 0$ such that

$$|k_1\omega_1 + \cdots + k_n\omega_n| > \frac{\alpha}{|k|^\tau}$$

for all $k \in \mathbb{Z}^n$, $k \neq 0$.

Denote the set of all strongly non-resonant frequencies by Δ_α^τ.

The fate of resonant tori

Let T be an area preserving mapping. So T can be

1. a Poincaré mapping;
2. the time-τ advance mapping of some autonomous Hamiltonian flow (=Poincaré);
3. just any area preserving map.

Consider a $1D$ torus $\mathcal{C}$ with rational rotatation number: $\omega_1/\omega_2 = r/s \in \mathbb{Q}$. Note that every point of $\mathcal{C}$ is a fixed point of T^s.

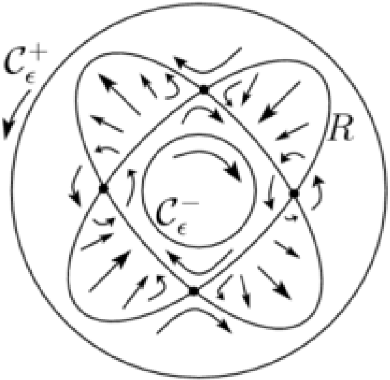
Poincaré-Birkhoff

Under small perturbations: the resonant torus breaks up and leaves $2ks$ fixed points of T^s in its wake, which are alternatingly elliptic and hyperbolic.

Technical condition: T should be a twist mapping.

The fate of resonant tori (2)

The following picture “illustrates” the proof of Poincaré-Birkhoff:



The full proof (very intuitive!) can be found in Tabor (p. 141) or Verhulst (p. 236).

Small perturbations

- So. . . all resonant tori are destroyed under arbitrarily small perturbations and give rise to chaos. Does that mean that the system is ergodic?
- Answer: an emphatic *no*. Let's find out what happens to the non-resonant tori!

The fate of the strongly non-resonant tori

Structure of the sets Ω_α^τ

Here $\Omega_\alpha^\tau := \Delta_\alpha^\tau \cap \Omega$, with $\Omega \subset \mathbb{R}^n$ compact.

Theorem

For all α and $\tau > n-1$, Ω_α^τ is a Cantor set. The complement of Ω_α^τ has Lebesgue measure of the order α.

Define

$$R_{\alpha,k}^\tau = \left\{ \omega \in \Omega : |k_1\omega_1 + \cdots + k_n\omega_n| < \frac{\alpha}{|k|^\tau} \right\}.$$

Then $\cup_{0\neq k\in\mathbb{Z}^n} R_{\alpha,k}^\tau$ is the complement of Ω_α^τ.

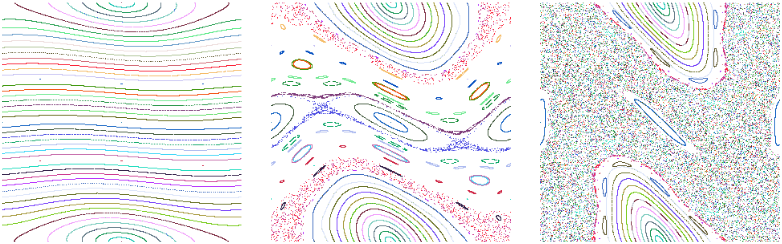
- R_α^τ is open;
- R_α^τ is dense in Ω (since $\mathbb{Q}^n \cap \Omega \subset R_\alpha^\tau$).

Sufficient to have a Cantor set.

The KAM theorem

Roughly speaking: under suitable conditions of nondegeneracy, the following holds:

- For perturbations of the order α^2, all tori in Ω_α^τ persist;
- The destroyed tori fill an part of phase space with measure of order α.



Melnikov's method

- Let $z(t)$ be a homoclinic orbit such that $\lim_{t\to\infty} = z_0$, a hyperbolic fixed point.
- Define the Melnikov function

$$M(t_0) = \int_{-\infty}^{\infty} \{H_0, H_1\}(z(t - t_0))dt.$$

Important idea: $M(t_0)$ is a measure for the distance between W^u and W^s.

Theorem

If $M(t_0)$ has simple zeros and is independent of ϵ, then, for $\epsilon > 0$ sufficiently small, W^u and W^s intersect transversely. If $M(t_0)$ remains bounded away from zero, then $W^u \cap W^s = \varnothing$.

Example: the forced pendulum

(See Marsden and Ratiu (p. 41) or Perko (p. 415) for more details)

- Equations of motion:

$$\frac{d}{dt}\begin{pmatrix}\phi\\ \dot\phi\end{pmatrix} = \begin{pmatrix}\dot\phi\\ -\sin\phi\end{pmatrix} + \epsilon\begin{pmatrix}0\\ \cos\omega t\end{pmatrix}.$$

  Hamiltonian with

$$H_0 = \frac{1}{2}\dot\phi^2 - \cos\phi \quad \text{and} \quad H_1 = \phi\cos\omega t.$$

- Homoclinic orbits for $\epsilon = 0$:

$$\begin{pmatrix}\phi(t)\\ \dot\phi(t)\end{pmatrix} = \begin{pmatrix}\pm 2\tan^{-1}(\sinh t)\\ \pm 2\,\mathrm{sech}\, t\end{pmatrix}$$

Computation of Melnikov function

$$
\begin{aligned}
M(t_0) &= \int_{-\infty}^{\infty} \{H_0, H_1\}(x(t-t_0))dt \\
&= -\int_{-\infty}^{\infty} \dot{\phi}(t-t_0)\cos\omega t dt \\
&= \mp \int_{-\infty}^{\infty} 2\,\mathrm{sech}(t-t_0)\cos\omega t dt.
\end{aligned}
$$

Change variables, and use method of residues to conclude that

$$
\begin{aligned}
M(t_0) &= \mp\left(\int_{-\infty}^{\infty} \mathrm{sech}\, t\cos\omega t dt\right)\cos\omega t_0 \\
&= \mp 2\pi\,\mathrm{sech}\left(\frac{\pi\omega}{2}\right)\cos\omega t_0.
\end{aligned}
$$

1. Conclusion: simple zeros, hence homoclinic chaos!
2. Compare with lobe dynamics for damped pendulum in LCS talk. Caution: not entirely the same system!