The Moore-Greitzer equations model rotating stall and surge in gas turbine engines describe the
dynamics of a compression system, such as those in gas turbine engines. The three-state “MG3”
equations have the form:
\[
\begin{align*}
\frac{d\psi}{dt} &= \frac{1}{4B^2l_c} (\phi - \Phi_T(\psi)), \\
\frac{d\phi}{dt} &= \frac{1}{l_c} \left( \Phi_c(\phi) - \psi + \frac{J \partial^2 \Psi_c}{8 \partial \phi^2} \right), \\
\frac{dJ}{dt} &= \frac{2}{\mu + m} \left( \frac{\partial \Phi_c}{\partial \phi} + \frac{J \partial^3 \Phi_c}{8 \partial \phi^3} \right) J,
\end{align*}
\]
where \( \psi \) represents the pressure rise across the compressor, \( \phi \) represents the mass flow through
the compressor and \( J \) represents the amplitude squared of the first modal flow perturbation (cor-
responding to a rotating stall disturbance). For the Caltech compressor rig, the parameters and
characteristic curves are given by:
\[
B = 0.2, \quad \Phi_T(\psi) = \gamma \sqrt{\psi}, \\
l_c = 6, \quad \Psi_c(\phi) = 1 + 1.5\phi - 0.5\phi^3, \\
\mu = 1.25, \quad m = 2.
\]
The parameter \( \gamma \) represents the throttle setting and typically varies between 1 and 2.

1. Compute the bifurcation diagram for the system showing the equilibrium value(s) for \( J \) as
   a function of \( \gamma \). Your answer should match what was shown in class (i.e., make sure to get
   capture the hysteresis loop).

2. Suppose that we can modulate the throttle, so that \( \gamma = \gamma_0 + u \). Analyze the performance of
   the system using the Liaw-Abed control law \( u = kJ \). Show that if we choose \( k \) sufficiently
   large to cause the bifurcation to stall to be supercritical.

3. Suppose that we impose magnitude and rate limits on \( u \):
   \[
   |u| \leq 1, \quad |\dot{u}| \leq 1.
   \]
   Assume that we implement the control law
   \[
   \dot{u} = \alpha(J) = \begin{cases}
   \text{sat} \left( \frac{1}{\epsilon} (kJ - \text{sat}(u)) \right) & |u| < 1, \\
   0 & |u| = 1,
   \end{cases}
   \]
   where \( \text{sat}(\cdot) \) is a saturation function of magnitude 1 and \( \epsilon \) is a small constant. This control
   law limits both the magnitude and rate of the input. Using the center manifold theorem,
   compute an approximate model of the system at the bifurcation point in terms of \( u \) and \( J \)
   and use a phase portrait (computed with MATLAB or a similar tool) to describe the set of
   initial conditions for \( J \) (assuming \( u(0) = 0 \)) for which the system avoids hysteresis.