CDS 140b Winter 2008 Week 6
Spherical Tippe Top Inversion as a Dissipation-Induced Instability

Due: Thursday, Feb 21, 2008

This assignment references the preprint "Dissipation-Induced Heteroclinic Orbits in Tippe Tops" which accompanies this homework.

Problem A [Quadratic Lagrangians]

Let $M \in L(\mathbb{R}^n, \mathbb{R}^n)$ be a symmetric positive-definite matrix. Consider a mechanical system with quadratic Lagrangian $L : \mathbb{R}^{2n} \to \mathbb{R}$:

$$L(\mathbf{q}, \dot{\mathbf{q}}) = \dot{\mathbf{q}}^T M \dot{\mathbf{q}} + \dot{\mathbf{q}}^T G \mathbf{q} - \mathbf{q}^T K \mathbf{q}.$$

Show that *the only* conservative forces derivable from such a quadratic Lagrangian are gyroscopic and potential. (Recall a gyroscopic force is a force that can be written as $A\dot{\mathbf{q}}$ where A is skew-symmetric and a potential force is a force that can be written as $B\mathbf{q}$ where B is symmetric.)

Problem B [Linear Hamiltonian Systems]

Let $\mathbf{q}(t) \in \mathbb{R}^2$. Consider $A, B \in L(\mathbb{R}^2, \mathbb{R}^2)$ where A is skew-symmetric and B is symmetric. Verify that the following differential equations are Hamiltonian:

$$\ddot{\mathbf{q}} + A\dot{\mathbf{q}} + B\mathbf{q} = 0. \qquad (*)$$

Show that if σ is an eigenvalue of (*) then so are $1/\sigma$, $\bar{\sigma}$ and $1/\bar{\sigma}$.

Problem C [Conservative Spherical Tippe Top]

Let $a, b \in \mathbb{R}$. Recall the governing equations of the conservative spherical tippe top admit the following momentum invariant:

$$J = a\boldsymbol{\pi}^T \boldsymbol{\xi}_3 + b\boldsymbol{\pi}^T \mathbf{e}_3.$$

Show that as a consequence of this invariance, the conservative spherical tippe top cannot capture tippe top inversion. That is, there exists no time T such that $\boldsymbol{\xi}_3(T) \approx \mathbf{e}_3$ given that one initializes the system in a neighborhood of the noninverted state: $\boldsymbol{\xi}_3(0) \approx -\mathbf{e}_3$ and $\boldsymbol{\pi}(0) \approx \sigma \mathrm{Fr}\mathbf{e}_3$.

Problem D [Jellett Momentum Map]

Recall the governing equations of the spherical tippe top are given by:

$$\begin{cases} \dot{\mathbf{x}} &= \mathbf{v}, \\ \mu\dot{\mathbf{v}} &= (\lambda - \mu)\mathbf{e}_3 + \mathbf{F}_f, \\ \mathbf{x}^{\mathrm{T}}\mathbf{e}_3 &= 1 + \epsilon\mathbf{e}_3^{\mathrm{T}}\boldsymbol{\xi}_3, \\ \dot{\boldsymbol{\xi}}_3 &= \boldsymbol{\omega} \times \boldsymbol{\xi}_3, \\ \dot{\boldsymbol{\pi}} &= -\lambda\epsilon\boldsymbol{\xi}_3 \times \mathbf{e}_3 + \mathbf{q} \times (\mathbf{F}_f), \\ \boldsymbol{\pi} &= \boldsymbol{\omega} + (\sigma - 1)(\boldsymbol{\omega}^{\mathrm{T}}\boldsymbol{\xi}_3)\boldsymbol{\xi}_3. \end{cases} \tag{**}$$

Prove that the Jellett momentum map:

$$\mathcal{J} = -\boldsymbol{\pi}^T\mathbf{q}$$

is invariant under the flow of (**). Discuss its role in the analysis of the spherical tippe top's inversion.

Problem E [Relative Equilibria]

Show that the fully nonlinear rotational equations of the spherical tippe top:

$$\begin{cases} \dot{\boldsymbol{\xi}}_3 &= \boldsymbol{\pi} \times \boldsymbol{\xi}_3, \\ \dot{\boldsymbol{\pi}} &= -\mu\epsilon\boldsymbol{\xi}_3 \times \mathbf{e}_3 + \nu\mathbf{q} \times \mathbf{q} \times \boldsymbol{\pi} - \nu\frac{\sigma-1}{\sigma}(\boldsymbol{\pi}^{\mathrm{T}}\boldsymbol{\xi}_3)\mathbf{q} \times \mathbf{q} \times \boldsymbol{\xi}_3, \end{cases}$$

admit the following fixed points:

$$\boldsymbol{\pi} = \pi_0\mathbf{e}_3, \quad \boldsymbol{\xi}_3 = n_0\mathbf{e}_3, \quad n_0^2 = 1,$$

where n_0 and π_0 are real constants.

Project Ideas

1. Write a "article review" of the preprint.
2. Simulate tippe top inversion using variational rigid body integrators coupled to Sigrid Leyendecker's discrete null-space method to handle the external surface constraint.
3. Analyze stochastic resonance in tippe top inversion as discussed in the preprint (cf. §6).
4. Model dynamic stability of an inverted pendulum with an autonomous mechanical system.