CDS 140b – Introduction to Dynamics

Eva Kanso, kanso@usc.edu

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1 Hw problems

1. Consider a rigid body $\mathcal{B}$ submerged in an infinite volume of inviscid, incompressible fluid that is assumed to remain irrotational at all time. That is, the fluid velocity is given by $\mathbf{u} = \nabla\phi$ where ϕ is solution to Laplace's equation $\Delta\phi = 0$ subject to boundary conditions:

$$\nabla\phi \cdot \mathbf{n}|_{\partial\mathcal{B}} = \text{ normal velocity of body}, \quad \nabla\phi|_{\infty} = 0$$

The kinetic energy of the solid-fluid system is:

$$T = \underbrace{\sum_i \frac{1}{2}\left(\Omega^T \mathbb{I}\Omega \; + \; mV^T V\right)}_{T_s\text{: Kinetic Energy of the Solids}} \quad + \quad \underbrace{\frac{1}{2}\int_{\mathcal{F}} \rho_f \, \mathbf{u}\cdot\mathbf{u}\,\mathrm{dv}}_{T_f\text{: Kinetic Energy of Fluid}}$$

where Ω and V are the body's angular and linear velocities expressed in body-fixed frame (say, chosen to coincide with the body's principal axis).

   (a) Show that ϕ can be written as in terms of velocity potentials φ_i and χ_i $(i = 1, 2, 3)$

$$\phi = \varphi_i V_i + \chi_i \Omega_i$$

   where φ_i and χ_i are solutions to Laplace's equations subject to properly chosen boundary conditions.

   (b) Use the above expression for ϕ to show that the kinetic energy of the fluid T_f can be rewritten as

$$T_f = \frac{1}{2}\begin{pmatrix} \Omega^T & V^T \end{pmatrix} \Theta_f \begin{pmatrix} \Omega \\ V \end{pmatrix}$$

   where Θ_f is a 6×6 symmetric matrix with elements determined by the geometry and configuration of the body and the fluid density.

2. Consider an ellipsoidal rigid body moving in potential flow. The motion is governed by Kirchhoff's equations given here in component form

$$\dot{\Pi}_1 = \frac{I_2 - I_3}{I_2 I_3}\Pi_2\Pi_3 + \frac{m_2 - m_3}{m_2 m_3}P_2 P_3 \;, \qquad \dot{P}_1 = \frac{P_2\Pi_3}{I_3} - \frac{P_3\Pi_2}{I_2}$$

$$\dot{\Pi}_2 = \frac{I_3 - I_1}{I_3 I_1}\Pi_3\Pi_1 + \frac{m_3 - m_1}{m_3 m_1}P_3 P_1 \;, \qquad \dot{P}_2 = \frac{P_3\Pi_1}{I_1} - \frac{P_1\Pi_3}{I_3}$$

$$\dot{\Pi}_3 = \frac{I_1 - I_2}{I_1 I_2}\Pi_1\Pi_2 + \frac{m_1 - m_2}{m_1 m_2}P_1 P_2 \;, \qquad \dot{P}_3 = \frac{P_1\Pi_2}{I_2} - \frac{P_2\Pi_1}{I_1}$$

(a) Verify that $\Pi_e = (0, 0, \Pi_3^o)$ and $P_e = (0, 0, P_3^o)$ is an equilibrium of the above equations.

(b) Assume that $I_1 = I_2$ and $m_1 = m_2$ and that $I_3 < I_1 = I_2$. Use the energy-Casimir method to show that the equilibrium (Π_e, P_e) is nonlinearly stable *(hint: see lecture notes and Leonard 1997)*.

3. Consider a circular cylinder interacting with N point vortices of strength Γ_k, $k = 1, \ldots, N$ such that $\sum_k \Gamma_k = 0$. The equations of motion governing the body-vortex dynamics are

$$\begin{aligned} \dot{\Pi} + V \times P &= 0 \\ \dot{P} + \Omega \times P &= 0 \\ \Gamma_k \left(\frac{d\mathrm{X}_k}{dt} + \Omega \times \mathrm{X}_k + V \right) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left(\frac{\partial W}{\partial \mathrm{X}_k} \right). \end{aligned}$$

Show that these equations are Hamiltonian equations for the Hamiltonian function $H(\Pi, P, \mathrm{X}_k)$

$$H = \frac{1}{2}(\Pi, P)^T \mathbb{I}^{-1} (\Pi, P) - (\Pi^{\mathrm{v}}, P^{\mathrm{v}})^T \mathbb{I}^{-1} (\Pi, P) + \frac{1}{2}(\Pi^{\mathrm{v}}, P^{\mathrm{v}})^T \mathbb{I}^{-1} (\Pi^{\mathrm{v}}, P^{\mathrm{v}}) + W^{\mathrm{v}}(\mathrm{X}_k),$$

where $\mathbb{I} = \mathbb{M}^b + \mathbb{M}^f$ and $W^{\mathrm{v}} = W - \sum_k \Gamma_k \psi^{\mathrm{b}}$ and the bracket

$$\{F, G\}(\Pi, P, \mathrm{X}_k) = \{F, G\}_{\text{submerged body}} + \{F, G\}_{\text{point vortices}}$$

Hint: see lecture notes, Shashikanth et al. [2002] and Shashikanth [2005].

2 Project ideas

1. Show that a two-link fish cannot swim in potential flow by cyclic shape changes (see Kanso et al. [2005], Kanso & Marsden [2005]).

2. Learn more about geometric phases and holonomy through the "falling cat" example (or another example of your choice) (see Montgomery [1990]).

3. Show that the dynamics of a single vortex interacting with a circular cylinder is integrable (see Ramodanov [2000]). The dynamics is no longer integrable for an elliptic cylinder of eccentricity different from zero.

4. Investigate the (linear and/or nonlinear) stability of the Föppl equilibria of a circular cylinder interacting with a vortex pair of equal and opposite strength (see Shashikanth et al. [2002], Kanso & Oskouei [2007]).

5. Investigate the equilibria and linear stability of infinite von Karman streets (see figures). Consider (1) the case of an infinite row of point vortices of equal strength; and (2) the case of two infinite rows of point vortices of equal and opposite strength (i.e., the classical von Karman street). Distinguish between staggered and un-staggered vortices (see Saffman [1992], Lamb [1932]).

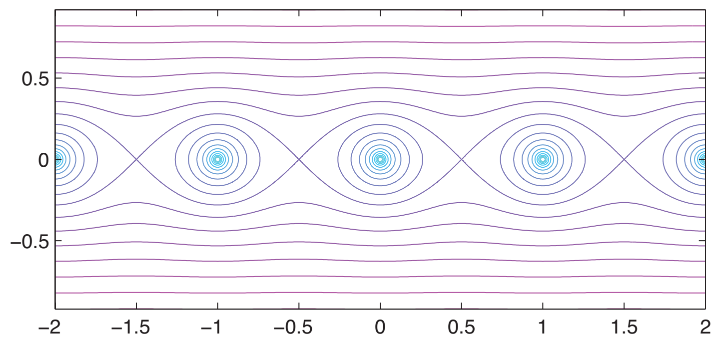

Figure 1: Streamlines corresponding to infinite vortex row. Vortex strength $\Gamma = 1$ and separation distance $a = 1$.

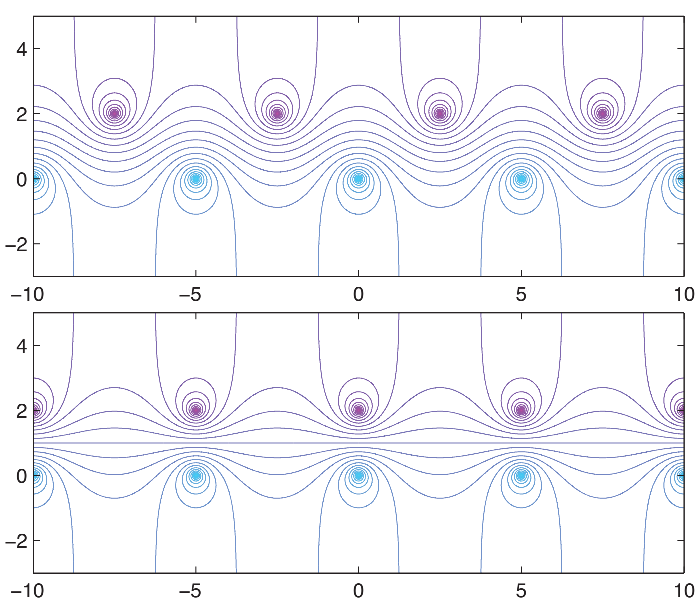

Figure 2: Streamlines corresponding to infinite vortex streets. (top) Staggered vortex street with upper row $\Gamma = 1$, lower row $\Gamma = -1$, horizontal separation distance $a = 5$ and vertical separation distance $b = 2$. (bottom) Un-staggered vortex street with same parameter values.