CDS 140b: Homework Set 3

Due by Thursday, February 7, 2008.

Problems

The Poisson bracket for the N-vortex problem is given by

$$\{f, g\} = \sum_{i=1}^{N} \frac{1}{\Gamma_i} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial g}{\partial x_i} \frac{\partial f}{\partial y_i} \right).$$

for arbitrary functions $f(x_1, y_1; \ldots; x_N, y_N)$ and $g(x_1, y_1; \ldots; x_N, y_N)$ on $\mathbb{R}^{2N}$.

1. Show that the following relations hold:

$$\{H, \mathcal{L}\} = 0, \quad \{H, \mathcal{I}_x\} = 0, \quad \{H, \mathcal{I}_y\} = 0,$$
$$\{\mathcal{I}_x, \mathcal{I}_y\} = \sum_{i=1}^{N} \Gamma_i, \quad \{\mathcal{I}_x, \mathcal{L}\} = 2\mathcal{I}_y, \quad \{\mathcal{I}_x, \mathcal{L}\} = -2\mathcal{I}_x,$$

   where H, $\mathcal{L}$, $\mathcal{I}_x$ and $\mathcal{I}_y$ are the Kirchhoff-Routh function, the angular impulse, and the x and the y component of the linear impulse, respectively. See the lecture notes for explicit expressions.

2. Use the general properties of the Poisson bracket (*i.e.* do not calculate explicitly) to show that

$$\{H, \mathcal{I}_x^2 + \mathcal{I}_y^2\} = 0 \quad \text{and} \quad \{\mathcal{L}, \mathcal{I}_x^2 + \mathcal{I}_y^2\} = 0,$$

   i.e. H, $\mathcal{L}$, and $\mathcal{I}_x^2 + \mathcal{I}_y^2$ are three (independent) integrals of the motion whose Poisson bracket vanishes.

Project ideas

1. Numerically explore chaos in the four-vortex problem (*i.e.* draw Poincaré maps, (maybe) Melnikov's method, ...).
2. Look at the existence of relative equilibria in the N-vortex problem.
3. Compute LCS for problems in chaotic advection.