CDS 140b: Homework Set 2

Due by Thursday, January 31, 2008.

For the purpose of calibration, note down the time spent on each problem on your solution.

Problems

1. Consider the Poisson form of the rigid body equations. Show that the total angular momentum $C(\mathbf{\Pi})$, defined as

$$C(\mathbf{\Pi}) = \frac{1}{2} \|\mathbf{\Pi}\|^2$$

   Poisson commutes with any function, *i.e.* for any function $G(\mathbf{\Pi})$,

$$\{C, G\} = 0.$$

   A function Poisson commuting with any other function is called a *Casimir function.*

2. A vector field X acts as a derivation on functions as follows: $X(f) = Df \cdot X$, *i.e.* $X(f)$ is the derivative of f in the direction of X.

   (a) Show that if X_F is a vector field on phase space associated to a function $F(q, p)$, then

$$X_F(f) = \{f, F\} \tag{1}$$

   for all functions f.

   (b) Show by direct calculation that for any two functions F and G on phase space the following relation holds:

$$X_{\{F,G\}} = -[X_F, X_G] \tag{2}$$

   where the Poisson bracket is the canonical one on phase space, and the *Lie bracket* $[X, Y]$ of two vector fields is defined by

$$[X, Y](f) = X(Y(f)) - Y(X(f)). \tag{3}$$

   (c) Now consider an arbitrary Poisson structure on (a subset of) $\mathbb{R}^n$. Taking (1) to be the definition of the vector field X_F associated to a function F, and using (3) to define the Lie bracket of vector fields, show that (2) continues to hold in this general setting.

3. This exercise deals with the motion of a charged particle in a magnetic field in $\mathbb{R}^3$. You will show that the effect of having a non-zero magnetic field is to modify the Poisson bracket by a certain *magnetic term.* This is yet another example of a Poisson structure which is not canonical.

Consider the following Poisson bracket on phase space (for notational definiteness, the coordinates on phase space are $(\mathbf{q}; \mathbf{p}) = (x, y, z; p_x, p_y, p_z)$):

$$\{f, g\}_{\mathcal{B}} = \{f, g\}_{\text{can}} + \frac{e}{c} B \left(\frac{\partial f}{\partial p_y} \frac{\partial g}{\partial p_z} - \frac{\partial g}{\partial p_y} \frac{\partial f}{\partial p_z} \right), \tag{4}$$

where $\{f, g\}_{\text{can}}$ is the canonical Poisson bracket:

$$\{f, g\}_{\text{can}} = \sum_{i=1}^{3} \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} \right),$$

and the second term in (4) is referred to as the *magnetic term*, as it is proportional to the magnetic field B. Note that B is a function of (x, y, z).

Show that the equations of motion for this Poisson bracket and the kinetic energy Hamiltonian

$$H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2)$$

describe the motion of a particle of mass m and charge e in a magnetic field $\mathbf{B} = (B(x, y, z), 0, 0)$ along the x-axis. Recall that the equations of motion for such a particle are the Lorentz equations:

$$m \frac{d\mathbf{v}}{dt} = \frac{e}{c} \mathbf{v} \times \mathbf{B}.$$

4. Fill in the details for the computation of the Melnikov function for the forced pendulum. Recall that for this system

$$H = \frac{1}{2} p_\phi^2 - \cos\phi + \epsilon\phi \cos\omega t$$

where $p_\phi = \dot{\phi}$ is the momentum conjugate to ϕ.

- Find all equilibrium points of the unperturbed system and discuss their stability;
- Show that the curve given by

$$\phi(t) = \pm 2 \tan^{-1}(\sinh t)$$

  is a homoclinic orbit;
- Follow the outline in the slides to compute the Melnikov function for this perturbation. To evaluate the complex integral, note that there is a simple pole at $z = \frac{i\pi}{2}$ and evaluate the integral there. What do you conclude?

Project ideas

1. Gain further insight in perturbed Hamiltonian systems (by using the `gnicodes` Matlab numerical integrators and/or your own software).

2. Master a special topic in Hamiltonian mechanics. Examples: the formalism of generating functions and the Hamilton Jacobi equation, the theory behind symplectic numerical integrators, etc.

3. Study some of the special Poisson structures in mechanics and/or field theory.

4. Study some real-world examples of chaos. Example: the use of Melnikov's method to prove the existence of homoclinic chaos in superfluid helium.