Dynamics and Stability

application to submerged bodies, vortex streets and vortex-body systems

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CDS 140B – Introduction to Dynamics

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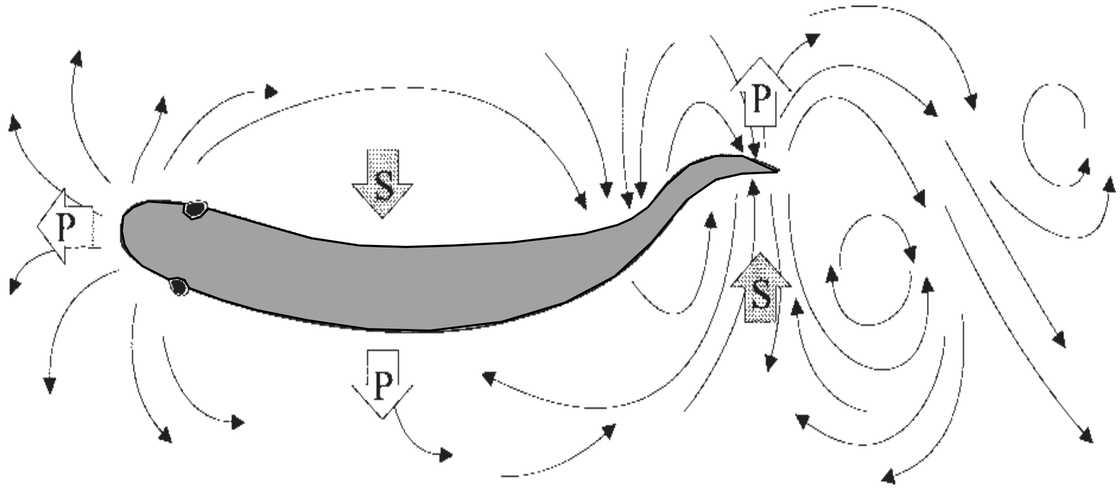
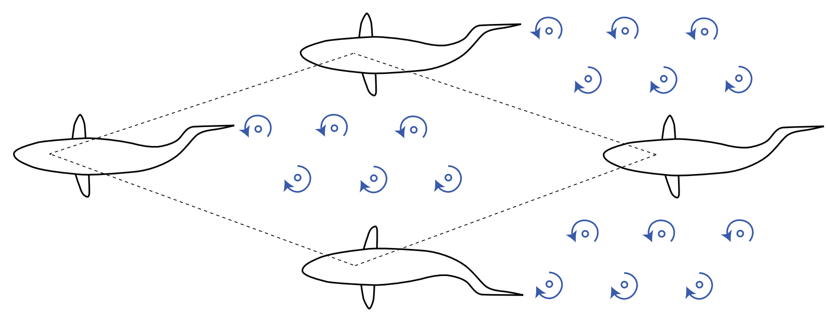
Fish swim by coupling of their shape changes with the surrounding fluid



Flow field around a Carangiform fish based on PIV data (Muller et al. 1997)



Trout swimming in an experimentally generated vortex street (Liao et al. 2003)

Fish swimming in school (Stakiotakis et al. 1999)

References

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Outline

Today

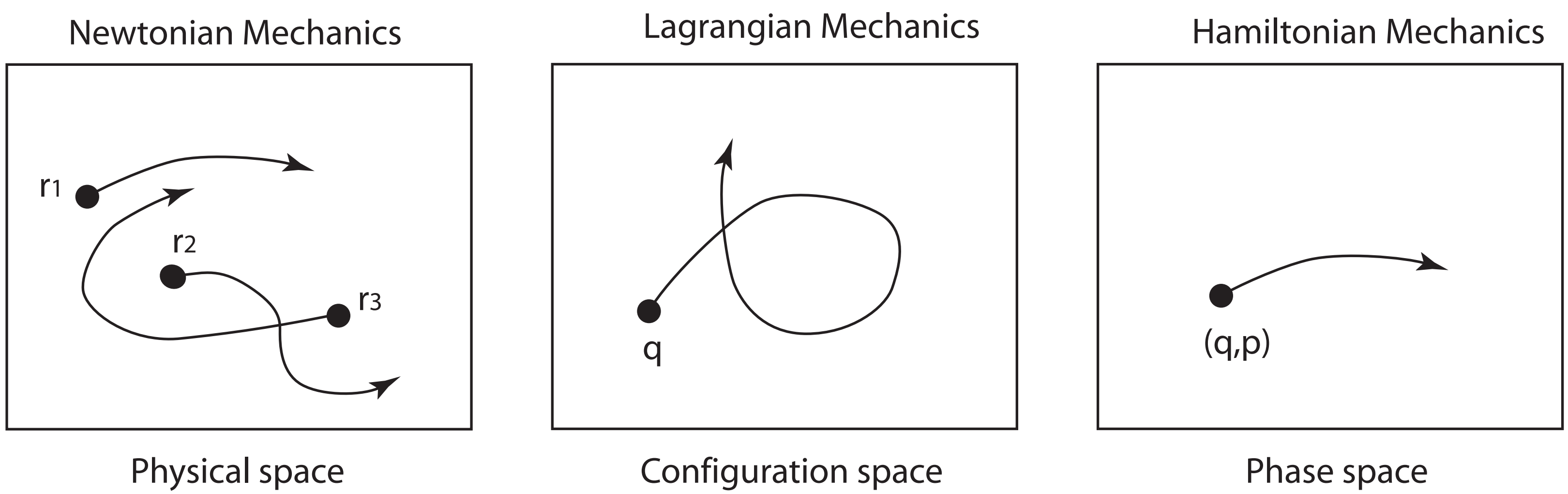
- Basic Concepts in Newtonian, Lagrangian and Hamiltonian Mechanics
- Equilibria and Stability
- Stability of Kirchhoff's equation for a Rigid Body in Potential Flow

Thursday

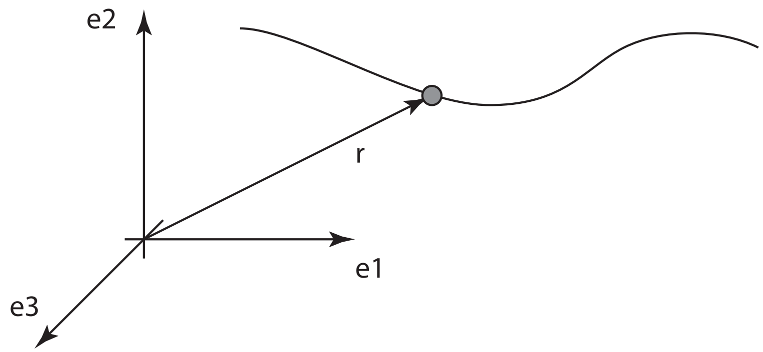
- Locomotion of an Articulated Body in Potential Flow
- Infinite Vortex Street
- Interaction of a Solid Body with Point Vortices

Basic Concepts

Newtonian, Lagrangian and Hamiltonian Mechanics



Newtonian Mechanics – single particle



Balance of Linear Momentum. The motion of the particle is governed by Newton's second law: force = mass × acceleration,

$$\mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t) = \dot{\mathbf{p}}$$

where the dot denotes the derivative with respect to time t and $\mathbf{p} = m\dot{\mathbf{r}}$ is the **linear momentum** of the particle.

Balance of Angular Momentum. Define the **angular momentum** $\boldsymbol{\pi}$ of a particle and the moment $\mathbf{M}$ acting upon it as: $\boldsymbol{\pi} = \mathbf{r} \times \mathbf{p}$, and $\mathbf{M} = \mathbf{r} \times \mathbf{F}$,

$$\mathbf{M} = \dot{\boldsymbol{\pi}}$$

Energy. The **kinetic energy** T is defined as $T = \frac{1}{2} m \dot{\mathbf{r}} \cdot \dot{\mathbf{r}}$. For m constant,

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \mathbf{F} \cdot \dot{\mathbf{r}}$$

Newtonian Mechanics – single particle

Conservative Force. A force is said to be conservative if it depends only on the position $\mathbf{r}$ and is such that the work it does is independent of the path taken. For *a closed path*, the work done vanishes.

$$\oint \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0 \iff \nabla \times \mathbf{F} = \mathbf{0}$$

It is a deep property of flat space[1] $\mathbb{R}^3$ that $\nabla \times \mathbf{F} = \mathbf{0}$ implies we may write the force as

$$\mathbf{F} = -\nabla V(\mathbf{r})$$

for some **potential function** $V(\mathbf{r})$ (also referred to as **potential energy**).

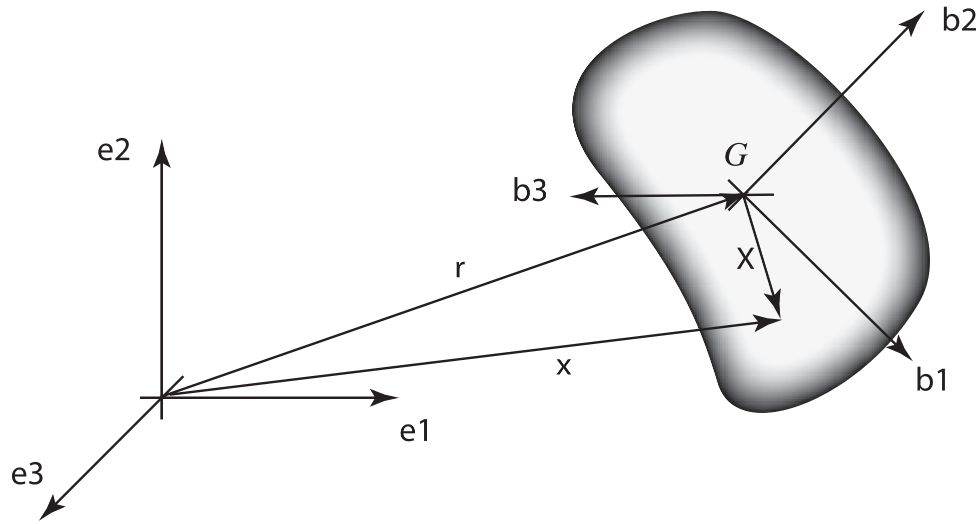
Conservation of Energy. For conservative systems, total energy $E = T + V$ is *a constant* or *integral of motion* (a quantity that is conserved by the dynamics).

Examples. (1) Simple Harmonic Oscillator. (2) Pendulum. (3) Particle Moving Under Gravity

Systems of Particles. The results above can be easily generalized to the case of a system of N particles: we simply add an index to everything!

[1] you will see this property later when you study differential geometry

Rigid Body



Rigid Body Rotation. It is described by proper-orthogonal tensor $R(t)$ where $R^T R = RR^T = I$. R has 9 components, proper-orthoganility imposes 6 constraints $\Rightarrow$ a minimum of 3 generalized coordinates are needed to parameterize R (e.g., Euler angles).

Angular Velocity Tensors. In inertial frame, the angular velocity tensor is $\dot{R}R^T$. In body-fixed frame, the angular velocity tensor is $R^T\dot{R}$. Both tensors are skew-symmetric.

Angular Velocity Vectors. $\exists$ a unique angular velocity vector $\boldsymbol{\omega}$ associated with $\dot{R}R^T$ and a unique angular velocity vector Ω associated with $R^T\dot{R}$ such that

$$(\dot{R}R^T)\mathbf{u} = \boldsymbol{\omega} \times \mathbf{u}, \quad \forall\, \mathbf{u} \qquad \text{and} \qquad (R^T\dot{R})U = \boldsymbol{\Omega} \times U, \quad \forall\, U$$

Define the $\hat{}$ map: $\hat{\boldsymbol{\omega}} = \dot{R}R^T$ and $\hat{\Omega} = R^T\dot{R}$. In component form: $\omega_i = -\frac{1}{2}\epsilon_{ijk}(\dot{R}R^T)_{jk}$ and $(\dot{R}R^T)_{ij} = -\epsilon_{ijk}\omega_k$ (similarly for Ω.)

Rigid Body

Rotational Kinetic Energy. $T_{rot} = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{I}\boldsymbol{\omega}$, where the components of the moment of interia tensor $\mathbf{I}$ in inertial frame are given by:

$$\mathbf{I}_{ij} = \int_{\mathcal{B}} \rho(x_k x_k)\delta_{ij} - x_i x_j)\mathrm{dv}$$

Also, one has $T_{rot} = \frac{1}{2}\Omega^T \mathbb{I}\Omega$, where the components of the moment of interia tensor $\mathbb{I}$ in body-fixed frame are given by:

$$\mathbb{I}_{ij} = \int_{\mathcal{B}} \rho(X_k X_k)\delta_{ij} - X_i X_j)\mathrm{dv}$$

Balance of Angular Momentum. In inertial frame, the **angular momentum** $\boldsymbol{\pi} = \mathbb{I}\boldsymbol{\omega}$

$$\mathbf{M} = \dot{\boldsymbol{\pi}}$$

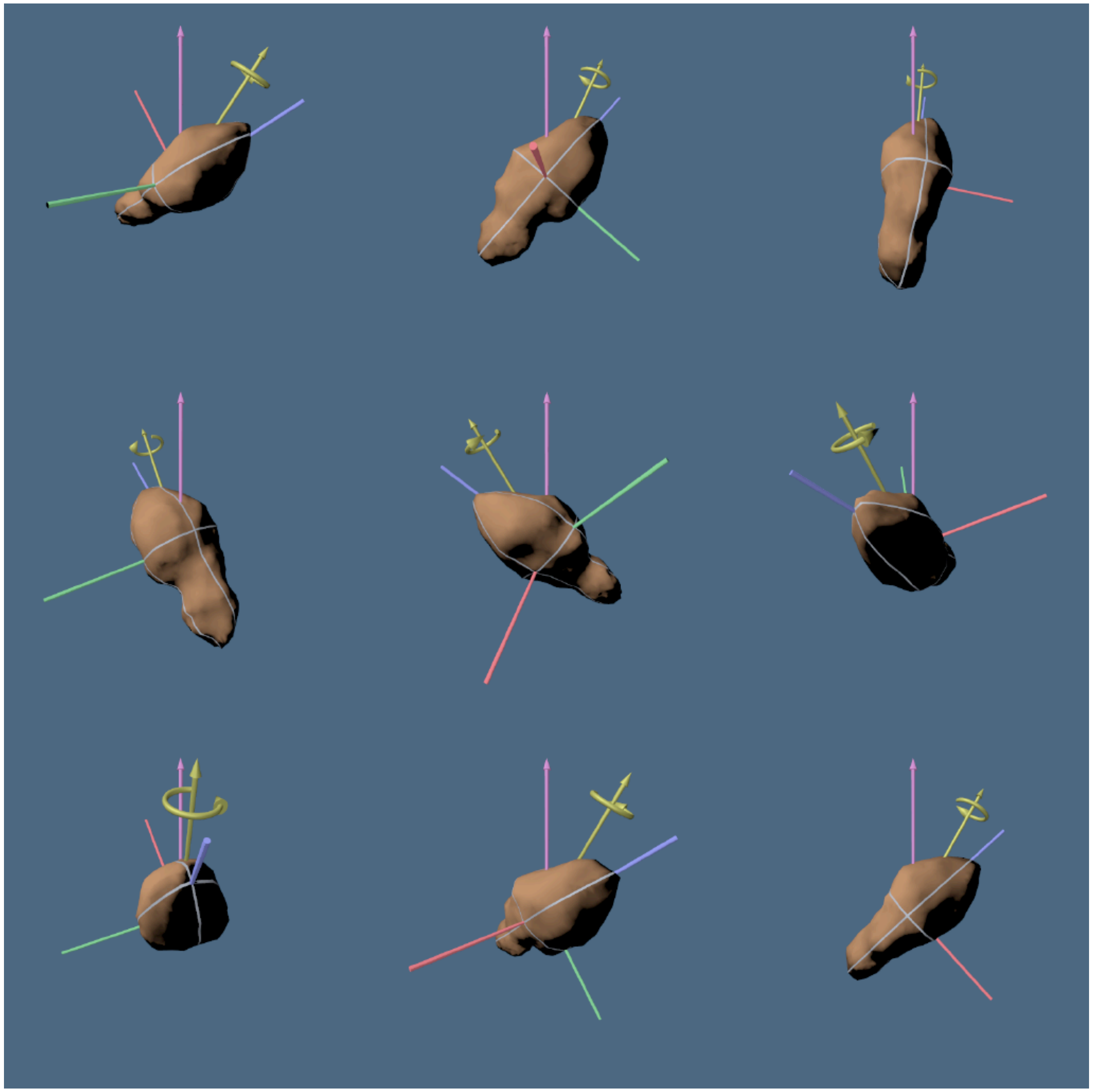
In body-fixed frame, the angular momentum $\Pi = \mathbb{I}\Omega$,

$$\boldsymbol{\pi} = R\Pi$$

If $\mathbf{M} = 0$, then (Euler's equations)

$$0 = \dot{\Pi} + \Omega \times \Pi \quad \Rightarrow \quad \dot{\Pi} = \Pi \times \Omega$$

Example: Asteriod Toutatis



Chaotic motion: instead of the spinning about a single axis as do the planets and a vast majority of asteroids, Toutatis "tumbles" somewhat like a football after a botched pass.

Lagrangian Mechanics

The Lagrangian. Define the *Lagrangian* function L to be a function of position q and velocity $\dot{q}$ given by

$$L(q, \dot{q}) = T(\dot{q}) - V(q) \tag{1}$$

where $T = \frac{1}{2} a_{ij} \dot{q}_i \dot{q}_j$ is the kinetic energy and $V(q_i)$ is the potential energy.

The Action. Consider all smooth paths $q(t)$ with fixed end points $q(t_0) = q_0$ and $q(t_1) = q_1$. To each path, we assign a number called the *action* S defined as

$$S[q(t)] = \int_{t_0}^{t_1} L(q, \dot{q}) \mathrm{d}t.$$

The action is a functional, i.e., a function of the path which is itself a function.

Hamilton's Principle. The actual physical path taken by the system is a stationary point of S.

Lagrange's Equations. $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial L}{\partial \dot{q}}\right) - \dfrac{\partial L}{\partial q} = 0$

Remarks

- Lagrange's equations are invariance under point transformation. This fact follows directly from the action principle which is a statement about the paths and not about coordinates.
- The Lagrangian is not unique and one can obtain equations of motion from Lagrangian which are not of the form $L = T - V$ but the latter always gives the right evolution of the system.
- Hamilton's principle is also called the principle of least action which is a slight misnomer since the true path is often a stationary point.
- All the fundamental laws of physics can be written in terms of an action principle. This includes electromagnetism, general relativity, the standard model of particle physics, and attempts to go beyong the known laws of physics such as string theory. Also, there is a generalisation of the action principle to quantum mechanics due to Feynman in which the particle takes all paths with some probability determined by the action S.

Example of a Particle in Polar Coordinates

- Consider a particle of mass m moving in 2D in a potential field $V(r)$.

$$L(r, \dot{r}, \theta, \dot{\theta}) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

- Lagrange's equations: $m\ddot{r} - mr\dot{\theta}^2 + \partial V/\partial r = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}(mr^2\dot{\theta}) = 0 \Longrightarrow mr^2\dot{\theta} = J$
- $J = mr^2\dot{\theta}$ is an integral of motion – it is the angular momentum of the particle. The value of J is determined from initial conditions: $J = mr_0^2\dot{\theta}_0$.

- Write $\dot{\theta} = J/mr^2$ and substitute back into Lagrange's equation for r to get

$$m\ddot{r} + \frac{\partial}{\partial r}\left(\frac{J^2}{2mr^2}\right) + \frac{\partial V}{\partial r} = 0 \quad \Longrightarrow \quad m\ddot{r} = -\frac{\partial V_{\text{eff}}}{\partial r}, \quad \text{where} \quad V_{\text{eff}} = \frac{J^2}{2mr^2} + V$$

- Clearly, this problem has another integral of motion

$$E = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}} = \text{constant}$$

Generalized Momenta and Ignorable Coordinates

Generalized Momentum. Define $p = \dfrac{\partial L}{\partial \dot{q}}$ to be the conjugate momentum to q. This *generalized momentum* coincides with the real momentum only in Cartesian coordinates.

Ignorable or Cyclic Coordinates. Given $L(q_i, \dot{q}_i, t)$, $i = 1, \ldots, n$ and suppose that $\partial L/\partial q_j = 0$ for some q_j. Then q_j is said to be *ignorable* or *cyclic* and the conjugate momentum $p_j = \partial L/\partial q_j$ is conserved

$$\frac{\mathrm{d}p_j}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j} = 0$$

hence $p_j =$ constant. The fact that L is independent of a coordinate is associated with the presence of symmetries.

Symmetries and Noether's Theorem

Symmetry of the Lagrangian under Transformations. Consider a one-parameter family of transformations:

$$q(t) \longrightarrow Q(s,t), \quad s \in \mathbb{R}, \quad \text{s.t.} \quad Q(0,t) = q(t).$$

This transformation is said to be a continuous *symmetry* of the Lagrangian L if

$$\frac{\partial}{\partial s}\big(L(Q,\dot{Q},t)\big) = 0 \ \text{ or, equivalently, } \ L(q,\dot{q},t) = L(Q,\dot{Q},t)$$

That is, the Lagrangian is invariant under such transformation.

Noether's Theorem. For each such symmetry there exists a conserved quantity (i.e., an integral of motion).

Noether's Theorem

Noether's Theorem. For each such symmetry there exists a conserved quantity (i.e., an integral of motion).

Proof. Note that

$$\frac{\partial L}{\partial s} = \frac{\partial L}{\partial Q}\frac{\partial Q}{\partial s} + \frac{\partial L}{\partial \dot{Q}}\frac{\partial \dot{Q}}{\partial s}$$

Hence,

$$0 = \left.\frac{\partial L}{\partial s}\right|_{s=0} \Longrightarrow 0 = \frac{\partial L}{\partial s} = \left.\frac{\partial L}{\partial q}\frac{\partial Q}{\partial s}\right|_{s=0} + \left.\frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{Q}}{\partial s}\right|_{s=0}$$

By virtue of Lagrange's equation, the above equation becomes

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)\left.\frac{\partial Q}{\partial s}\right|_{s=0} + \left.\frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{Q}}{\partial s}\right|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\left.\frac{\partial L}{\partial \dot{q}}\frac{\partial Q}{\partial s}\right|_{s=0}\right)$$

which means that the quantity $\frac{\partial L}{\partial \dot{q}}\frac{\partial Q}{\partial s}$ evaluated at $s = 0$ is constant for all time, i.e., it is an integral of motion.

Example: Homogeneity of Space

Consider a system of N particles with Lagrangian

$$L = \frac{1}{2}\sum_i m_i \dot{\mathbf{r}}_i - V(|\mathbf{r}_i - \mathbf{r}_j|)$$

This Lagrangian is invariant under translations of the form $\mathbf{r}_i \longrightarrow \mathbf{r}_i + s\mathbf{e}$ where $\mathbf{e}$ is any vector and s is any real number (both $\mathbf{e}$ and s are time independent). That is,

$$L(\mathbf{r}_i, \dot{\mathbf{r}}_i, t) = L(\mathbf{r}_i + s\mathbf{e}, \dot{\mathbf{r}}_i, t).$$

This symmetry means that the space is homogeneous and a translation of the system by $s\mathbf{e}$ does nothing to the equations of motion. From Noether's theorem, the conserved quantity associated with translations is:

$$\sum_i m_i \dot{\mathbf{r}}_i \cdot \mathbf{e} = \sum_i \mathbf{p}_i \cdot \mathbf{e}$$

which is the total linear momentum $\sum_i \mathbf{p}_i$ in the direction $\mathbf{e}$. Since this holds for all $\mathbf{e}$, we conclude that the total linear momentum is conserved.

Example: Isotropy of Space

Consider a system of N particles with Lagrangian

$$L = \frac{1}{2}\sum_i m_i \dot{\mathbf{r}}_i - V(|\mathbf{r}_i - \mathbf{r}_j|)$$

This Lagrangian is invariant under rotations about any fixed direction $\mathbf{e}$ that transform all vectors: $\mathbf{r}_i \longrightarrow \mathbf{r}'_i = \mathbf{R}_e \mathbf{r}_i$ (here $\mathbf{R}_e$ is an orthogonal rotation matrix associated with the rotation about $\mathbf{e}$). That is,

$$L(\mathbf{r}_i, \dot{\mathbf{r}}_i, t) = L(\mathbf{r}'_i, \dot{\mathbf{r}}'_i, t).$$

To calculate the conserved quantity associated with this symmetry, it suffices to work with the infinitesimal form of the rotations: $\mathbf{r}_i \longrightarrow \mathbf{r}'_i = \mathbf{r}_i + s\mathbf{e} \times \mathbf{r}_i$ where s here is infinitesimal. To this end, the symmetry of the Lagrangian under rotations reads as:

$$L(\mathbf{r}_i, \dot{\mathbf{r}}_i, t) = L(\mathbf{r}_i + s\mathbf{e} \times \mathbf{r}_i, \dot{\mathbf{r}}_i + s\mathbf{e} \times \dot{\mathbf{r}}_i, t).$$

The conserved quantity associated with this symmetry is:

$$\sum_i \frac{\partial L}{\dot{\mathbf{r}}_i} \cdot (\mathbf{e} \times \mathbf{r}_i) = \sum_i \mathbf{e} \cdot (\mathbf{r}_i \times \mathbf{p}_i) = \mathbf{e} \cdot \sum_i \boldsymbol{\pi}_i$$

That is, the component of the total angular momentum $\sum_i \boldsymbol{\pi}_i$ in the direction $\mathbf{e}$ is conserved. Since $\mathbf{e}$ is arbitrary, we get conservation of total angular momentum.

Example: Homogeneity of time

Homogeneity of time means that the Lagrangian $L(q^i, \dot{q}^i, t)$ is invariant under transformation of the form $t \longrightarrow t + s$, i.e., $\partial L/\partial t = 0$. This implies that the *energy function* defined as: $H = \sum_i \dot{q}^i(\partial L/\partial \dot{q}^i) - L$ is conserved. This can be verified by appealing to Noether's theorem or by direct computation as follows

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \sum_i \left(\ddot{q}^i \frac{\partial L}{\partial \dot{q}^i} + \dot{q}^i \frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial L}{\partial \dot{q}^i}) - \frac{\partial L}{\partial q^i}\dot{q}^i - \frac{\partial L}{\partial \dot{q}^i}\ddot{q}^i \right)$$

which vanishes whenever Lagrange's equations hold.

Summary

- Invariance under translations of space $\Longrightarrow$ Conservation of total linear momentum

- Invariance under rotations of space $\Longrightarrow$ Conservation of total angular momentum

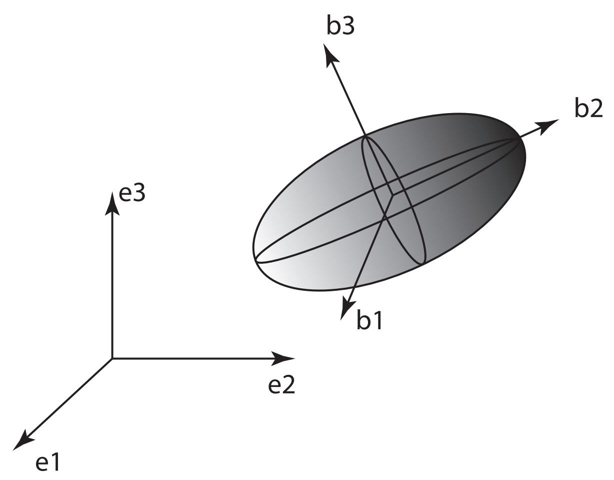
- Invariance under time translation $\Longrightarrow$ Conservation of total energy

Hamiltonian Mechanics

Canonical Hamiltonian Systems. See Joris' lectures

Noncanonical Hamiltonian Systems. Examples:

1. Euler's equations for rigid body rotation (Joris' lectures)
2. Kirchhoff's equations for rigid body motion in potential flow



Kirchhoff's equations

Consider a neutrally-buoyant (ellipsoidal) rigid body moving in potential flow: The solid and fluid are studied as one dynamical system in terms of the solid variables only. The fluid effect is encoded in the added masses.

- The total kinetic energy of the body-fluid system $T = T_f + T_b$

$$T = \frac{1}{2}\left(\Omega^T \mathcal{J} \Omega + V^T \mathcal{M} V\right)$$

- Let $\mathbf{p}$ and $\boldsymbol{\pi}$ be the linear and angular momenta of the solid-fluid system in inertial frame and P and Π be those in body-fixed frame

$$\mathbf{p} = RP, \qquad \boldsymbol{\pi} = R\Pi + \mathbf{r} \times \mathbf{p}$$

- No applied forces and torques on the body-fluid system:

$$\dot{\boldsymbol{\pi}} = 0, \qquad \dot{\mathbf{p}} = 0$$

- In body-fixed frame, the above equations become

$$\dot{\Pi} \;=\; \Pi \times \Omega + P \times V, \qquad \dot{P} \;=\; P \times \Omega$$

Example: Rigid Body Moving in Potential Flow

Kirchhoff's Equations. The balance of angular and linear momenta of solid-fluid system

$$\dot{\Pi} \;=\; \Pi \times \Omega + P \times V, \qquad \dot{P} \;=\; P \times \Omega$$

- Hamiltonian: $H(\Pi, P) = \frac{1}{2}(\Pi^T A \Pi + P^T B P)$, where $A = \mathcal{J}^{-1}$ and $B = \mathcal{M}^{-1}$

- Define the Poisson bracket on functions $F(\Pi, P)$ and $G(\Pi, P)$ as

$$\{F, G\} = (\nabla F)^T \Lambda \nabla G, \qquad \text{where} \quad \Lambda = \begin{pmatrix} \hat{\Pi} & \hat{P} \\ \hat{P} & 0 \end{pmatrix}$$

- Kirchhoff's equations are equivalent to

$$\begin{pmatrix} \dot{\Pi} \\ \dot{P} \end{pmatrix} \;=\; \Lambda \nabla H = \begin{pmatrix} \Pi \times \Omega + P \times V \\ P \times \Omega \end{pmatrix}$$

Casimir Functions

Casimir Functions. A Casimir function C is defined such that $\{C, K\} = 0$ for any function K. That is, Casimir functions are conserved quantities along the flow defined by the Poisson bracket $\{,\}$ for any H.

Casimir Functions of Kirchhoff's Equations. A Casimir function C is one s.t. ∇C is in the nullspace of $\Lambda = \begin{pmatrix} \hat{\Pi} & \hat{P} \\ \hat{P} & 0 \end{pmatrix}$. The nullspace of Λ has rank 2. Two independent Casimirs are given by

$$C_1(\Pi, P) = \Pi \cdot P \ , \qquad C_2(\Pi, P) = \|P\|^2$$

Any Casimir can be expressed as a smooth function of these two.

Equilibria and Stability

Equilibria and Stability

Consider a dynamical system: $\dot{y} = \mathbf{f}(y, t)$. If $f(y)$ only, then system is autonomous.

Equilibrium. y_e is an equilibrium iff $\dot{y} = 0$.

Stability. y_e is stable if given any neighborhood ϵ of y_e, there is another neighborhood of y_e $\delta(\epsilon) \subset \epsilon$ such that any motion starting in δ remains in ϵ

$$\forall\epsilon \quad \exists\delta(\epsilon) \quad \text{s.t.} \|y(0) - y_e\| < \delta(\epsilon) \Rightarrow \|y(t) - y_e\| < \epsilon$$

y_e is unstable whenever given any neighborhood of y_e, each neighborhood of y_e in ϵ contains a point M s.t. a motion starting at M leave ϵ.

Lyapunov Function and Lyapunov Stability

Lyapunov function. A Lyapunov function $U = U(y)$ is typically used in conjunction with an equilibrium y_e and is normalized such that $U(y_e) = 0$. In general, given a function $U(y)$, we say

- $U(y)$ is positive definite if $U(y) > 0 \; \forall \; y \neq y_e$ and $U(y_0) = 0$
- $U(y)$ is negative definite if $U(y) < 0 \; \forall \; y \neq y_e$ and $U(y_0) = 0$
- $U(y)$ is definite it it is either positive or negative definite
- $U(y)$ is positive semi-definite if $U(y) \geq 0 \; \forall \; y \neq y_e$ and $U(y_e) = 0$

A *necessary condition* for $U(y)$ to be positive definite is that its *Hessian* matrix $\frac{\partial^2 U}{\partial y_i \partial y_j}$ at y_e is positive definite. Recall that a matrix is positive definite if and only if all eigenvalues are strictly positive.

Theorem (Lyapunov 1892). Given a positive definite function U, then if $\dot{U} \leq 0$, the equilibrium is (nonlinearly) stable.

Lyapunov Function and Lyapunov Stability

Theorem (Lyapunov 1892). Given a positive definite function U, then if $\dot{U} \leq 0$, the equilibrium is (nonlinearly) stable.

Proof. In order to prove this result: (1) use the function U as a *norm* and (2) note that the gradient ∇U is normal to $U =$ constant. By hypothesis,

$$\dot{U} = \frac{\partial U}{\partial y}\dot{y} = \frac{\partial U}{\partial y} f \;\leq 0 \qquad \text{for } \dot{y} = f(y, t)$$

hence, starting at a point $y(t_0)$ near the equilibrium y_e, one has

$$U(y(t)) - U(y(t_0)) = \int_{t_0}^{t} \dot{U} \mathrm{d}t \;\leq\; 0 \implies U(y(t)) \;\leq\; U(y(t_0))$$

that is, $y(t)$ stays near y_e for all time t and y_e is stable.

Equilibria and Stability of Canonical Hamiltonian Systems

Canonical Hamiltonian System. $H(q,p)$ where $(q,p) \in \mathbb{R}^n \times \mathbb{R}^n$.

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

- Introduce $z = (q,p)$, then $\dot{z} = J\nabla H$, where $J = \begin{pmatrix} 0_{n\times n} & I_{n\times n} \\ -I_{n\times n} & 0_{n\times n} \end{pmatrix}$
- A point z_e where $\dot{z}_e = 0$ is called an equilibrium or a stationary point of the flow. Equilibria are precisely the critical points of H where $\nabla H|_{z_e} = 0$
- H is conserved: $\dot{H} = (\nabla H)^T \dot{z} = (\nabla H)^T J \nabla H = 0$
- If z_e is a critical point at which the Hessian matrix, i.e., the second derivative of $\frac{\partial^2 H}{\partial z_i \partial z_j}$, is definite, then the level surfaces of H form locally a foliation by slightly deformed ellipsoids.

Sufficient condition for stability of an equilibrium of a canonical Hamiltonian system:

Dirichlet Criterion: If z_e is an equilibrium at which the $\dfrac{\partial^2 H}{\partial z_i \partial z_j}$ is definite, then z_e is stable.

Lagrange-Dirichlet Criterion

Apply Dirichlet criterion to a classical mechanical system $H = T + V$ with kinetic energy $T = \frac{1}{2} a_{ij} p_i p_j$ and potential energy $V(q)$.

$$z_e = (q_e, p_e) \text{ is an equilibrium} \Rightarrow \left.\frac{\partial H}{\partial z}\right|_{z_e} = 0 \Rightarrow \left.\begin{pmatrix} \frac{\partial V}{\partial q} \\ a_{ij} p_j \end{pmatrix}\right|_{(q_e, p_e)} = 0 \Rightarrow p_e = 0$$

Thus, the equilibrium $z_e = (q_e, 0)$. Now, compute $\dfrac{\partial^2 H}{\partial z_i \partial z_j} = \begin{pmatrix} \frac{\partial^2 V}{\partial q_i \partial q_j} & 0_{n\times n} \\ 0_{n\times n} & a_{ij} \end{pmatrix}$

This means that if $\dfrac{\partial^2 H}{\partial z_i \partial z_j}$ is definite, it has to be positive definite (since a_{ij} is the kinetic energy quadratic form and it's always positive definite).

$$\Rightarrow \left.\frac{\partial^2 H}{\partial z_i \partial z_j}\right|_{(q_e, p_e)} \text{ is positive definite if and only if } \frac{\partial^2 V}{\partial q_i \partial q_j} \text{ is positive definite at } q_e.$$

Dirichlet Criterion is equivalent to Lagrange Criterion for stability. It is sometimes referred to as Lagrange-Dirichlet criterion.

Lagrange Criterion

Lagrange Criterion. For a mechanical system with Lagrangian $L = T - V$, $(q_e, 0)$ is a stable equilibrium provided that the matrix $\dfrac{\partial^2 V}{\partial q_i \partial q_j}$ is positive definite at q_e (that is, if q_e is a strict local minimum of V). If $\left.\dfrac{\partial^2 V}{\partial q_i \partial q_j}\right|_{q_e}$ has a negative definite direction, then q_e is an unstable equilibrium.

Non-Canonical Hamiltonian Systems

Energy-Casimir Method. It is a generalization of the classical Dirichlet criterion to Hamiltonian systems with non canonical bracket: $\dot{z} = \{z, H\}$, where z need not be of even dimension.

Energy-Casimir method provides a step by step procedure for constructing a Lyapunov function to prove stability. The Lyapunov function is a function of the Hamiltonian, the Casimir functions and any other conserved quantities.

Main idea of energy-Casimir method:

1. Choose a function C (of the Casimirs and other conserved quantities) such that $H + C$ has a critical point at the equilibrium z_e: $\left.\dfrac{\partial(H+C)}{\partial z}\right|_{z_e} = 0$
2. compute the second variation $\left.\dfrac{\partial^2(H+C)}{\partial z_i \partial z_j}\right|_{z_e}$
3. If this matrix is positive definite, then z_e is stable.

Remarks

1. These energy techniques (Dirichlet-Lagrange and Energy-Casimir) give stability information only. As such, one cannot use them to infer instability without further investigation.
2. The energy-Casimir method is restricted to certain types of systems since its implementation relies on Casimir functions (which in some cases are difficult to find and may not even exist).
3. To overcome this difficutly, one could use the energy-momentum method, which is closely linked to the method of reduction.

Reduction and Relative Equilibria

Relative Equilibria. When the phase space is obtained by reduction, the equilibrium z_e is called a relative equilibrium of the original Hamiltonian system.

Stability of Relative Equilibria. Lyapunov stability of z_e in the reduced phase space using the energy-Casimir method implies stability of the corresponding relative equilibrium in the original phase space modulo the symmetry *group*.

Example: Stability of an Ellipsoidal Submerged Body

- Recall Kirchhoff's equations for the solid-fluid system

$$\dot{\Pi} = \Pi \times \Omega + P \times V, \qquad \dot{P} = P \times \Omega$$

- Assume ellipsoidal body s.t. actual and added mass have coincident principal axes:

$$\mathcal{J} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}, \qquad \mathcal{M} = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}$$

- Angular and linear momenta: $\Pi = \mathcal{J}\Omega$ and $P = \mathcal{M}V$

Example: Stability of an Ellipsoidal Submerged Body

- Kirchhoff's equations in component form:

$$\dot{\Pi}_1 = \frac{I_2 - I_3}{I_2 I_3}\Pi_2\Pi_3 + \frac{m_2 - m_3}{m_2 m_3}P_2 P_3 \,, \qquad \dot{P}_1 = \frac{P_2\Pi_3}{I_3} - \frac{P_3\Pi_2}{I_2}$$

$$\dot{\Pi}_2 = \frac{I_3 - I_1}{I_3 I_1}\Pi_3\Pi_1 + \frac{m_3 - m_1}{m_3 m_1}P_3 P_1 \,, \qquad \dot{P}_2 = \frac{P_3\Pi_1}{I_1} - \frac{P_1\Pi_3}{I_3}$$

$$\dot{\Pi}_3 = \frac{I_1 - I_2}{I_1 I_2}\Pi_1\Pi_2 + \frac{m_1 - m_2}{m_1 m_2}P_1 P_2 \,, \qquad \dot{P}_3 = \frac{P_1\Pi_2}{I_2} - \frac{P_2\Pi_1}{I_1}$$

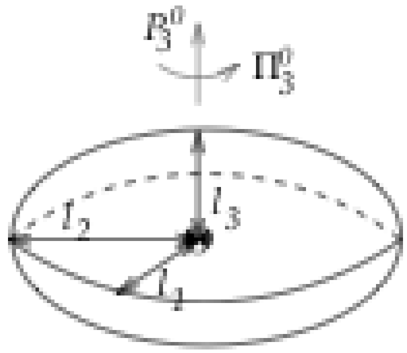
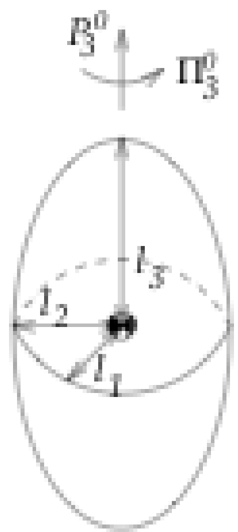
- Three sets of two-parameter families of equilibrium solutions: each family correspond to constant translation along an rotation about one of the principal axes.

- Becuase of symmetry, one only needs to study one of these equilibria, say,

$$\Pi_e = (0, 0, \Pi_3^o), \qquad P_e = (0, 0, P_3^o)$$

- Leonard (2007) used the energy-Casimir method to study the stability of such equilibria

Example: Stability of an Ellipsoidal Submerged Body

 $I_3 < I_2 = I_1$	*Stable.* *(Vehicle can be in any orientation with respect to the inertial frame since gravity plays no role).*
 $I_3 > I_2 = I_1$	*Stable if angular/translational momentum is sufficiently high, i.e.,* $(\Pi_3^0 / P_3^0)^2 > 4 I_2 (1/m_3 - 1/m_2) > 0$ *Unstable if* $(\Pi_3^0 / P_3^0)^2 < 4 I_2 (1/m_3 - 1/m_2)$ *(Vehicle can be in any orientation with respect to the inertial frame since gravity plays no role).*