Dynamics and Stability

application to submerged bodies and vortex-body systems

Eva Kanso
University of Southern California

CDS 140B – Introduction to Dynamics

February 5 and 7, 2008

References

1. Marsden and Ratiu, Introduction to Mechanics and Symmetry
2. Marsden, Lectures on Mechanics
3. Robinson, Dynamical Systems: Stability, Symbolic Dynamics and Chaos
4. Arnold, Mathematical Methods of Classical Mechanics
5. Lamb, Hydrodynamics
6. Saffman, Vortex Dynamics
7. Newton, The N vortex problem
8. Lenoard, N. E. [1997], Stability of a Bottom Heavy Vehicle
9. Shashikanth, B.N., J.E. Marsden, J.W. Burdick, and S.D. Kelly [2002], The Hamiltonian structure of a 2D rigid circular cylinder interacting dynamically with N Point vortices, *Phys. of Fluids*, **14**:1214–1227.
10. Kanso, E., and B. Oskouei [2007], Stability of a Coupled Body-Vortex System, *to appear in J. Fluid Mech.*
11. Kanso E., J.E. Marsden, C.W. Rowley and J. Melli-Huber [2005], Locomotion of articulated bodies in a perfect fluid, *Int. J. Nonlin. Science*, **15**, 255–289.

Outline

Tuesday

- Basic Concepts in Newtonian, Lagrangian and Hamiltonian Mechanics
- Equilibria and Stability
- Stability of Kirchhoff's equation for a Rigid Body in Potential Flow

Today

- Locomotion of an Articulated Body in Potential Flow
- Interaction of a Solid Body with Point Vortices

Motion in inviscid, incompressible fluid

Inviscid fluid: $\dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\dfrac{1}{\rho} \nabla p.$ Incompressibility: $\operatorname{div} \mathbf{u} = 0$

Fluid velocity:

$$\mathbf{u} \; = \; \nabla \phi \; + \; \nabla \times \boldsymbol{\psi}$$

Potential function: $\Delta \phi = 0,$

$$\text{B.C.} \qquad \nabla \phi \cdot \mathbf{n}|_{\partial \mathcal{B}} = \text{ normal velocity of body}, \quad \nabla \phi|_{\infty} = 0$$

Stream potential vector: $\Delta \boldsymbol{\psi} = -\boldsymbol{\omega}, \qquad \boldsymbol{\omega} = \nabla \times \mathbf{u}$

$$\text{B.C.} \qquad \nabla \times \boldsymbol{\psi} \cdot \mathbf{n}|_{\partial \mathcal{B}} = \; \mathbf{0}, \quad \nabla \times \boldsymbol{\psi}|_{\infty} = 0$$

Motion in inviscid, incompressible fluid

Inviscid fluid: $\dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\dfrac{1}{\rho} \nabla p.$ Incompressibility: $\operatorname{div} \mathbf{u} = 0$

Fluid velocity:

$$\mathbf{u} \;=\; \nabla \phi \;+\; \nabla \times \boldsymbol{\psi}$$

Potential function: $\Delta \phi = 0,$

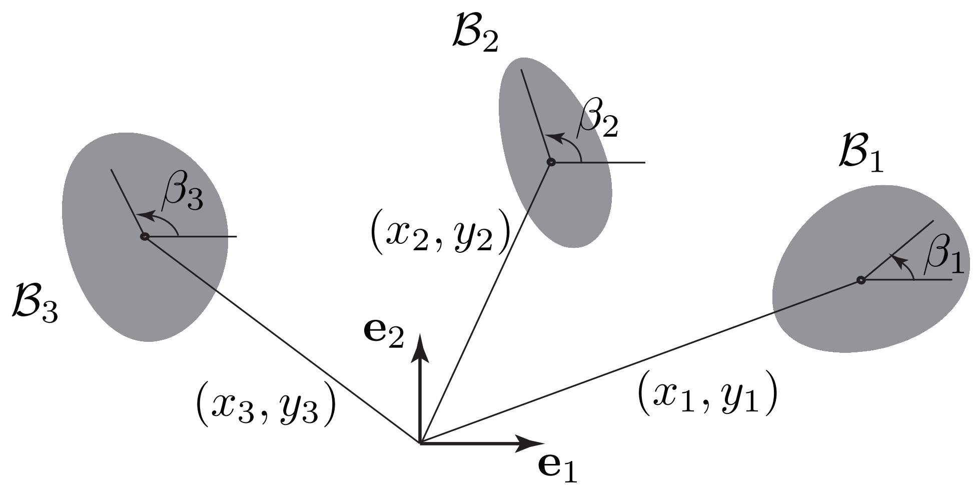
$$\text{B.C.} \qquad \nabla \phi \cdot \mathbf{n}|_{\partial \mathcal{B}} = \text{ normal velocity of body}, \quad \nabla \phi|_{\infty} = 0$$

~~Stream potential vector:~~ $\cancel{\Delta \boldsymbol{\psi} = -\boldsymbol{\omega}}, \qquad \cancel{\boldsymbol{\omega} = \nabla \times \mathbf{u}}$

~~B.C.~~ $\cancel{\nabla \times \boldsymbol{\psi} \cdot \mathbf{n}|_{\partial \mathcal{B}} = \mathbf{0}, \quad \nabla \times \boldsymbol{\psi}|_{\infty} = 0}$

Motion in potential flow

The solids and fluid are studied as one dynamical system in terms of the solid variables only. The fluid effect is encoded in the added masses.



(β_i, x_i, y_i) : orientation and position of body $\mathcal{B}_i$

$(\boldsymbol{\Omega}_i, \mathbf{V}_i)$: angular and translational velocities of $\mathcal{B}_i$. Here, $\boldsymbol{\Omega}_i = \Omega_i \, \mathbf{b}_3$, $\mathbf{V}_i = V_i^x \, \mathbf{b}_1^i + V_i^y \, \mathbf{b}_2^i$

Main idea: write, following Kirchhoff, $\phi = \sum_{i=1}^{3} \varphi_x^i \; V_i^x + \varphi_y^i \; V_i^y + \varphi_\beta^i \; \Omega_i$

- Incompressible, irrotational fluid:

$$\nabla \cdot \mathbf{u} = 0 \;\text{ and }\; \mathbf{u} = \nabla \phi \quad \Longrightarrow \quad \Delta \phi = 0$$

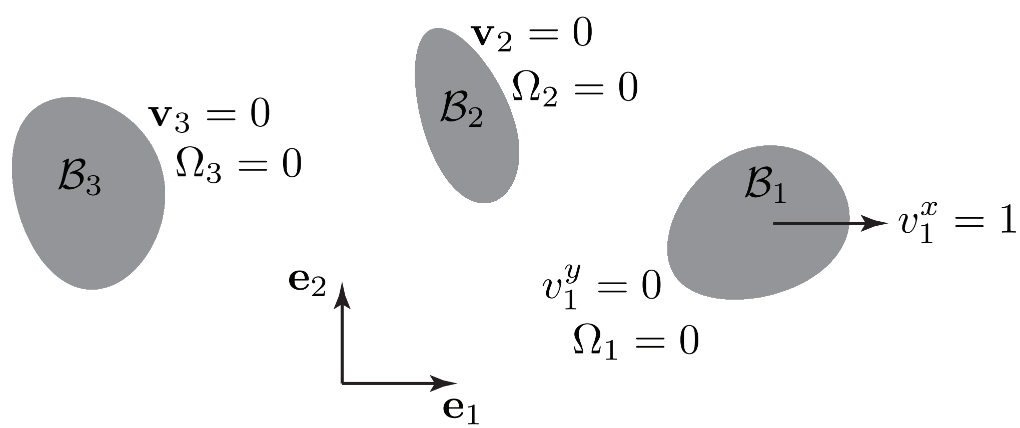
Boundary conditions:

$$\nabla \phi \cdot \mathbf{n}\big|_{\partial \mathcal{B}} = \text{ normal velocity of body} \qquad \nabla \phi\big|_{\infty} = 0$$

- Following Kirchhoff, $\phi = \sum_{i=1}^{3} \varphi_x^i \, v_i^x + \varphi_y^i \, v_i^y + \varphi_\beta^i \, \Omega_i$

- $\Delta \varphi_x^i = 0$

$$\text{B.C.} \quad \begin{cases} \nabla \varphi_x^i \cdot \mathbf{n}_i\big|_{\partial \mathcal{B}_i} = \mathbf{e}_1 \cdot \mathbf{n}_i \\ \nabla \varphi_x^i \cdot \mathbf{n}_j\big|_{\partial \mathcal{B}_j,\ j \neq i} = 0 \\ \nabla \varphi_x^i\big|_{\infty} = 0 \end{cases}$$



Kinetic energy of the fluid

- Kinetic energy: $T_f = \frac{1}{2} \int_{\mathcal{F}} \ \rho_f \, \mathbf{u} \cdot \mathbf{u} \, \mathrm{d}a$

- Write$\mathbf{u} = \nabla \phi$ and $\phi = \sum_{i=1}^{3} \varphi_x^i \, V_i^x + \varphi_y^i \, V_i^y + \varphi_\beta^i \, \Omega_i$ and rearrange:

$$T_f \ = \ \sum_i \sum_j \frac{1}{2} \begin{pmatrix} \boldsymbol{\Omega}_i^T & \mathbf{v}_i^T \end{pmatrix} \underbrace{\begin{pmatrix} \mathcal{J}_{ij} & \mathcal{D}_{ij}^T \\ \mathcal{D}_{ij} & \mathcal{M}_{ij} \end{pmatrix}}_{\mathbb{M}_{ij}^f:\ \text{Added Inertias}} \begin{pmatrix} \boldsymbol{\Omega}_j \\ \mathbf{v}_j \end{pmatrix}$$

Kinetic energy of the fluid

- Kinetic energy: $T_f = \frac{1}{2}\int_{\mathcal{F}} \ \rho_f\,\mathbf{u}\cdot\mathbf{u}\,\mathrm{d}a$

- Write$\mathbf{u} = \nabla\phi$ and $\phi = \sum_{i=1}^{3} \varphi_x^i\ V_i^x + \varphi_y^i\ V_i^y + \varphi_\beta^i\ \Omega_i$ and rearrange:

$$T_f \ = \ \sum_i\sum_j \frac{1}{2}\begin{pmatrix}\boldsymbol{\Omega}_i^T & \mathbf{v}_i^T\end{pmatrix}\underbrace{\begin{pmatrix}\mathcal{J}_{ij} & \mathcal{D}_{ij}^T \\ \mathcal{D}_{ij} & \mathcal{M}_{ij}\end{pmatrix}}_{\mathbb{M}_{ij}^f\text{: Added Inertias}}\begin{pmatrix}\boldsymbol{\Omega}_j \\ \mathbf{v}_j\end{pmatrix}$$

- Here,

$$J_{ij} \ = \ -\rho_f \int_{\partial\mathcal{B}_j} \varphi_\beta^i \frac{\partial\varphi_\beta^j}{\partial n}\,\mathrm{d}s\ ,$$

$$M_{ij} \ = \ -\rho_f \begin{pmatrix} \int_{\partial\mathcal{B}_j}\varphi_x^i\frac{\partial\varphi_x^j}{\partial n}\,\mathrm{d}s & \int_{\partial\mathcal{B}_j}\varphi_x^i\frac{\partial\varphi_y^j}{\partial n}\,\mathrm{d}s \\ \int_{\partial\mathcal{B}_j}\varphi_y^i\frac{\partial\varphi_x^j}{\partial n}\,\mathrm{d}s & \int_{\partial\mathcal{B}_j}\varphi_y^i\frac{\partial\varphi_y^j}{\partial n}\,\mathrm{d}s \end{pmatrix} \qquad D_{ij} \ = \ -\rho_f \begin{pmatrix} \int_{\partial\mathcal{B}_j}\varphi_x^i\frac{\partial\varphi_\beta^j}{\partial n}\,\mathrm{d}s \\ \int_{\partial\mathcal{B}_j}\varphi_y^i\frac{\partial\varphi_\beta^j}{\partial n}\,\mathrm{d}s \end{pmatrix}$$

Lagrangian function of (neutrally-buoyant) solid-fluid system

- The Lagrangian function:

$$\mathcal{L} \;=\; \underbrace{\sum_i \frac{1}{2}\left(\mathbf{\Omega}_i \cdot \mathbf{J}_i \mathbf{\Omega}_i \;+\; m_i \mathbf{v}_i \cdot \mathbf{v}_i\right)}_{T_s\text{: Kinetic Energy of the Solids}} \quad + \quad \underbrace{\frac{1}{2}\int_{\mathcal{F}} \rho_f\, \mathbf{u}\cdot\mathbf{u}\,\mathrm{d}a}_{T_f\text{: Kinetic Energy of Fluid}}$$

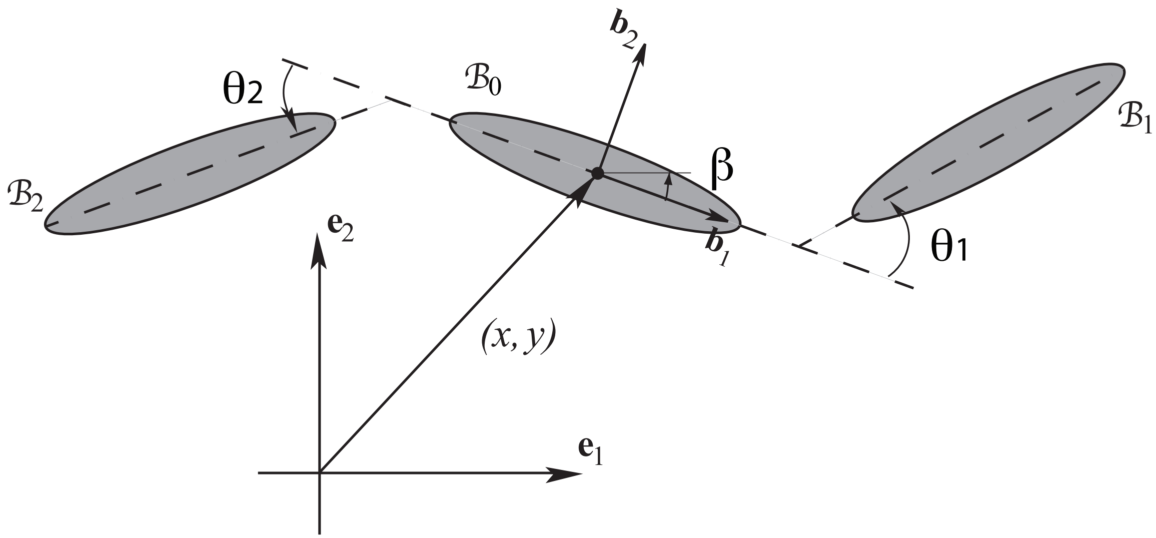
- Use derived expression for the kinetic energy of the fluid:

$$T_f \;=\; \sum_i \sum_j \frac{1}{2} \begin{pmatrix} \mathbf{\Omega}_i^T & \mathbf{V}_i^T \end{pmatrix} \underbrace{\begin{pmatrix} \mathcal{J}_{ij} & \mathcal{D}_{ij}^T \\ \mathcal{D}_{ij} & \mathcal{M}_{ij} \end{pmatrix}}_{\mathbb{M}_{ij}^f\text{: Added Inertias}} \begin{pmatrix} \mathbf{\Omega}_j \\ \mathbf{V}_j \end{pmatrix}$$

- Lagrangian $\mathcal{L}$ is written in terms of solid variables only! Equations of motion are given by:

$$\delta \int_{t_0}^{t_f} \mathcal{L}\,\mathrm{d}t \;=\; 0$$

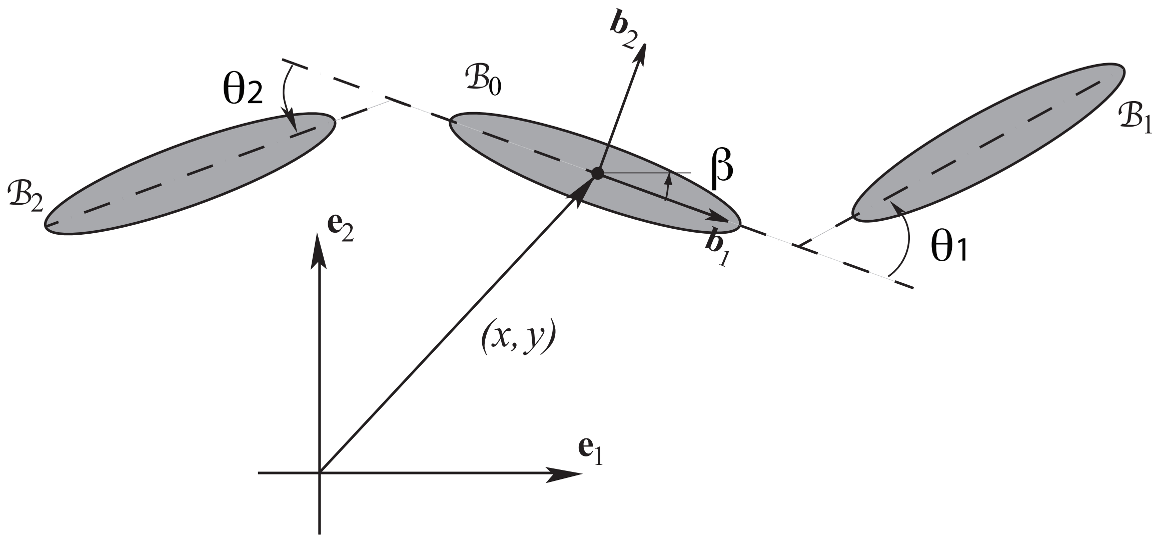
Example of a three-link fish (Kanso, Marsden, Rowley & Melli [2005])



(β, x, y) : orientation and position of three-link fish relative to $(\mathbf{e}_1, \mathbf{e}_2)$ – locomotion variables

(θ_1, θ_2) : orientation of $\mathcal{B}_1$ and $\mathcal{B}_2$ relative to $\mathcal{B}_0$ – shape variables – controlled variables

Example of a three-link fish (Kanso, Marsden, Rowley & Melli [2005])



(β, x, y) : orientation and position of three-link fish relative to $(\mathbf{e}_1, \mathbf{e}_2)$ – locomotion variables

(θ_1, θ_2) : orientation of $\mathcal{B}_1$ and $\mathcal{B}_2$ relative to $\mathcal{B}_0$ – shape variables – controlled variables

The linear and angular momenta:

$$\begin{pmatrix} \boldsymbol{\Pi}_i \\ \mathbf{P}_i \end{pmatrix} = \sum_{j=1}^{3} (\mathbb{M}_i^b + \mathbb{M}_{ij}^f) \begin{pmatrix} \boldsymbol{\Omega}_j \\ \mathbf{V}_j \end{pmatrix} , \qquad \begin{pmatrix} \boldsymbol{\Pi} \\ \mathbf{P} \end{pmatrix} = \sum_{i=1}^{3} \begin{pmatrix} \boldsymbol{\Pi}_i \\ \mathbf{P}_i \end{pmatrix}$$

Balance of linear and angular momenta of the solid-fluid system:

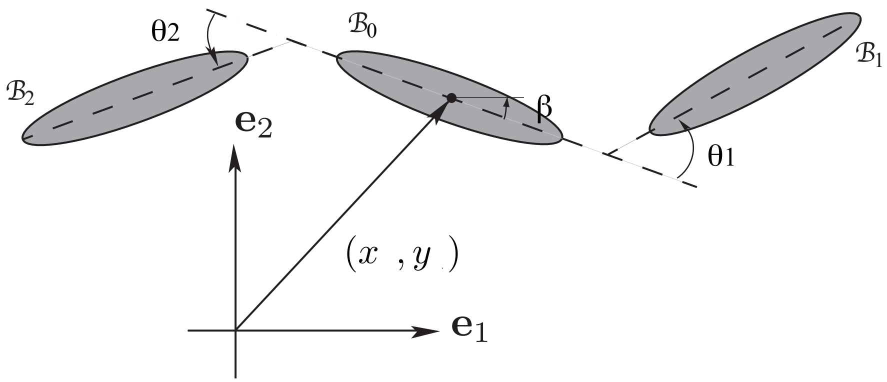
$$\dot{\mathbf{P}} = \mathbf{P} \times \boldsymbol{\Omega} , \qquad \dot{\boldsymbol{\Pi}} = \mathbf{P} \times \mathbf{V}$$

Swimming at zero momentum

forward gait

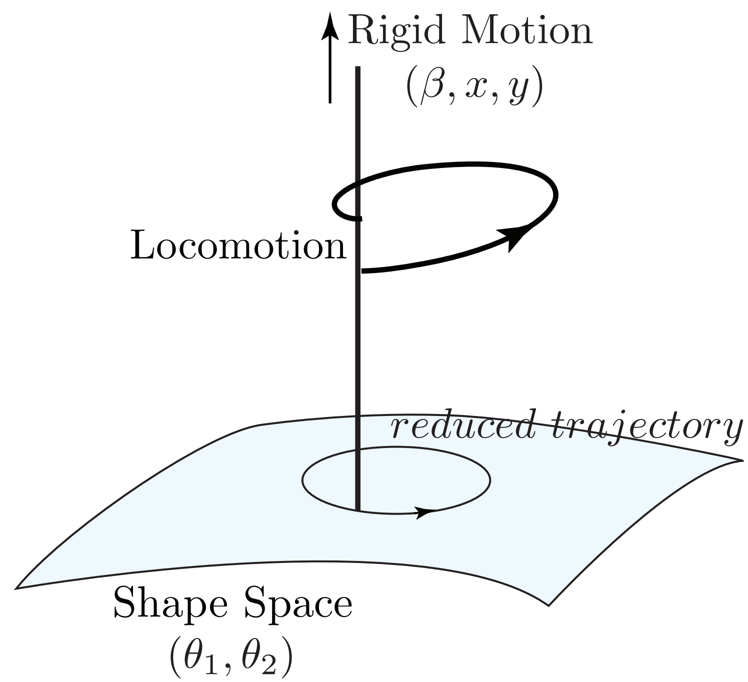
turning gait

Starting from rest, the three-link fish swims by controlling its shape changes! This is an example of locomotion due to geometric phases or holonomy.



(β, x, y) rigid motion

(θ_1, θ_2) shape variables



The configuration space forms a principal bundle over the shape space

Note: a two-link fish cannot swim in potential flow due to cyclic shape changes only!

Motion in inviscid, incompressible fluid – include vorticity

Inviscid fluid: $\dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\dfrac{1}{\rho} \nabla p.$ Incompressibility: $\operatorname{div} \mathbf{u} = 0$

Fluid velocity:

$$\mathbf{u} \; = \; \nabla \phi \; + \; \nabla \times \boldsymbol{\psi}$$

Potential function: $\Delta \phi = 0,$

B.C. $\nabla \phi \cdot \mathbf{n}|_{\partial \mathcal{B}} =$ normal velocity of body, $\nabla \phi|_{\infty} = 0$

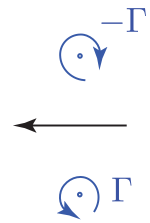
Stream potential vector: $\Delta \boldsymbol{\psi} = -\boldsymbol{\omega},$ $\boldsymbol{\omega} = \nabla \times \mathbf{u}$

B.C. $\nabla \times \boldsymbol{\psi} \cdot \mathbf{n}|_{\partial \mathcal{B}} = \mathbf{0},$ $\nabla \times \boldsymbol{\psi}|_{\infty} = 0$

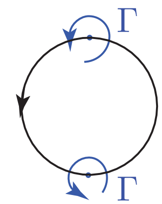
Use point vortex model

Dynamics of point vortices

Two point vortices
in infinite domain



Opposite strength

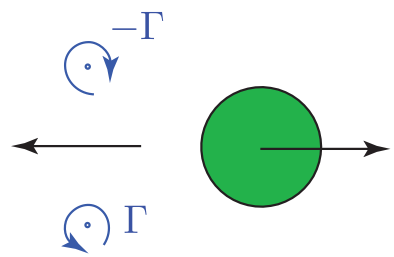


Equal strength

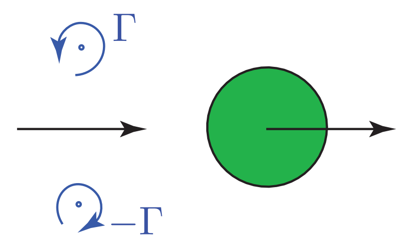
Two point vortices
interacting dynamically
with a moving cylinder

Shashikanth et al. [2002]

Borisov et al. [2003]

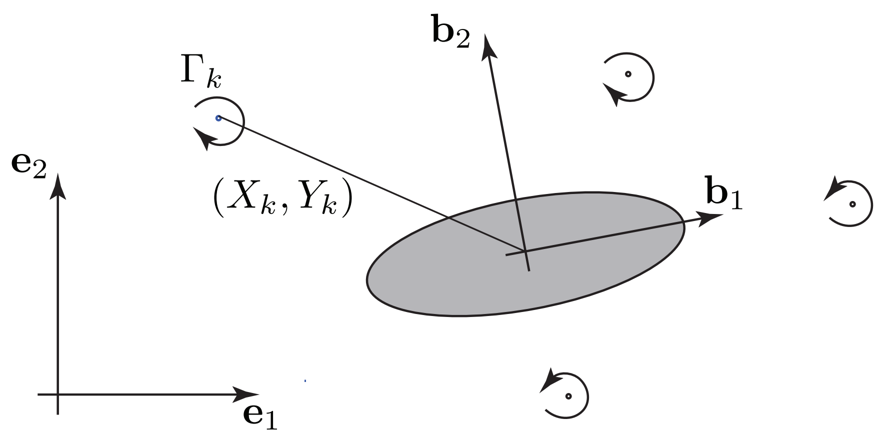


Swimming mode



Following mode

The solid and fluid are studied as one dynamical system. The configuration space consists of the solid variables and the position of the point vortices only.



(β, x, y) : orientation and position of $\mathcal{B}_0$ relative to $(\mathbf{e}_1, \mathbf{e}_2)$ – group motion

$(\mathbf{\Omega}, \mathbf{V})$: angular and linear velocity relative to $\mathcal{B}_0$- fixed frame

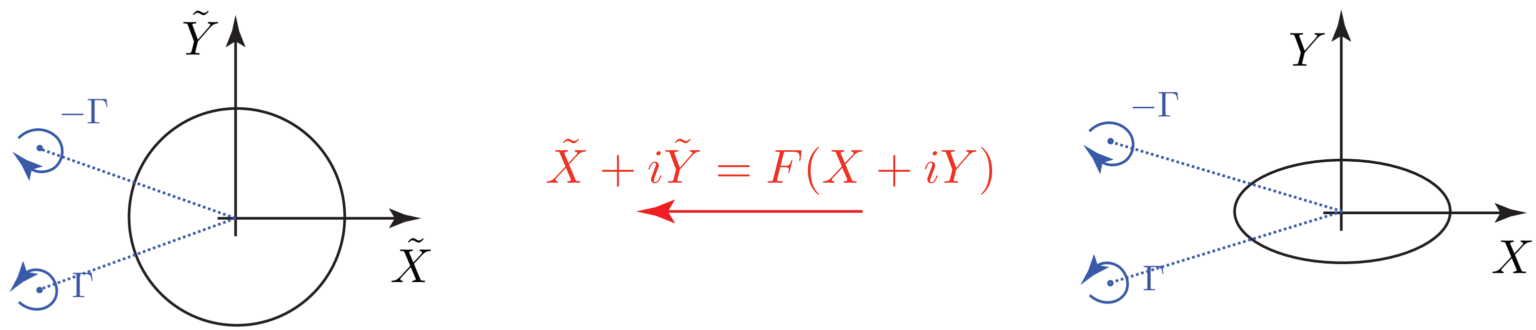
$\mathbf{X}_k = (X_k, Y_k)$: position of the k^{th}-vortex point relative to $\mathcal{B}_0$-fixed frame – wake variables

Stream function and Kirchhoff-Routh function

Fluid velocity is obtained from the stream function ψ (or potential function ϕ):

$$\psi = \underbrace{\psi_x\, V_x + \psi_y\, V_y + \psi_\beta\, \Omega}_{\psi^{\mathrm{b}}} + \sum_k \Gamma_k \psi_k$$

The stream function ψ is calculated from that of the vortex-circle using a *conformal transformation* given by



Vortex velocity is given by a Kirchhoff-Routh function $W(X_k, Y_k)$ (related to ψ) (see Joris' lecture)

Equations of Motion

Balance of linear momentum: $\dot{\mathbf{P}} \;=\; \mathbf{P} \times \boldsymbol{\Omega}$

Balance of angular momentum: $\dot{\boldsymbol{\Pi}} \;=\; \mathbf{P} \times \mathbf{V}$

Advection of the point vortices: $\Gamma_k(\dot{\mathbf{X}}_k + \mathbf{V} + \boldsymbol{\Omega} \times \mathbf{X}_k) \;=\; \dfrac{\partial W}{\partial Y_k}\mathbf{b}_1 - \dfrac{\partial W}{\partial X_k}\mathbf{b}_2$

This is a Hamiltonian system

The linear and angular momenta are:

$$\begin{pmatrix} \mathbf{P} \\ \boldsymbol{\Pi} \end{pmatrix} = (\mathbb{M}^b + \mathbb{M}^f)\begin{pmatrix} \mathbf{V} \\ \boldsymbol{\Omega} \end{pmatrix} + \begin{pmatrix} \mathbf{P}^{\mathrm{v}} \\ \boldsymbol{\Pi}^{\mathrm{v}} \end{pmatrix}$$

$\mathbf{P}^{\mathrm{v}}$ and $\boldsymbol{\Pi}^{\mathrm{v}}$ are due to the presence of point vortices

Equations of Motion

Balance of linear momentum: $\dot{\mathbf{P}} \;=\; \mathbf{P} \times \mathbf{\Omega}$

Balance of angular momentum: $\dot{\mathbf{\Pi}} \;=\; \mathbf{P} \times \mathbf{V}$

Advection of the point vortices: $\Gamma_k(\dot{\mathbf{X}}_k + \mathbf{V} + \mathbf{\Omega} \times \mathbf{X}_k) \;=\; \dfrac{\partial W}{\partial Y_k}\mathbf{b}_1 - \dfrac{\partial W}{\partial X_k}\mathbf{b}_2$

This is a Hamiltonian system

The linear and angular momenta are:

$$\begin{pmatrix} \mathbf{P} \\ \mathbf{\Pi} \end{pmatrix} = (\mathbb{M}^b + \mathbb{M}^f) \begin{pmatrix} \mathbf{V} \\ \mathbf{\Omega} \end{pmatrix} + \begin{pmatrix} \mathbf{P}^{\mathrm{v}} \\ \mathbf{\Pi}^{\mathrm{v}} \end{pmatrix}$$

$\mathbf{P}^{\mathrm{v}}$ and $\mathbf{\Pi}^{\mathrm{v}}$ are due to the presence of point vortices

$$\mathbf{P}^{\mathrm{v}} \;=\; \sum_k \Gamma_k \psi_x(\mathbf{X}_k)\, \mathbf{b}_1 + \sum_k \Gamma_k \psi_y(\mathbf{X}_k)\, \mathbf{b}_2 \;+\; \sum_k \Gamma_k \mathbf{X}_k \times \mathbf{b}_3$$

$$\mathbf{\Pi}^{\mathrm{v}} \;=\; \left(\sum_k \Gamma_k \psi_\beta(\mathbf{X}_k) \;-\; \frac{1}{2} \sum_k \Gamma_k \|\mathbf{X}_k\|^2 \right) \mathbf{b}_3$$

Hamiltonian Function and Non-canonical bracket

(Note: here I dropped the boldface notation to refer only to the components of the vectors in body-frame).

$$
\begin{aligned}
\dot{\Pi} + V \times P &= 0 \\
\dot{P} + \Omega \times P &= 0 \\
\Gamma_k \left(\frac{d\mathrm{X}_k}{dt} + \Omega \times \mathrm{X}_k + V \right) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left(\frac{\partial W}{\partial \mathrm{X}_k} \right).
\end{aligned} \tag{1}
$$

- The Hamiltonian $H(\Pi, P, \mathrm{X}_k)$ is the *kinetic energy of the body-fluid system* [1]

  $H = \frac{1}{2}(\Pi, P)^T \mathbb{I}^{-1}(\Pi, P) - (\Pi^{\mathrm{v}}, P^{\mathrm{v}})^T \mathbb{I}^{-1}(\Pi, P) + \frac{1}{2}(\Pi^{\mathrm{v}}, P^{\mathrm{v}})^T \mathbb{I}^{-1}(\Pi^{\mathrm{v}}, P^{\mathrm{v}}) + W^{\mathrm{v}}(\mathrm{X}_k),$

  where $\mathbb{I} = \mathbb{M}^b + \mathbb{M}^f$ and $W^{\mathrm{v}} = W - \sum_k \Gamma_k \psi^{\mathrm{b}}$.

- The bracket: $\{F, G\}(\Pi, P, \mathrm{X}_k) = \{F, G\}_{\text{submerged body}} + \{F, G\}_{\text{point vortices}}$

[1] This Hamiltonian function is actually the *kinetic energy of the body-fluid system minus the infinite contributions* that arise from the singular nature of the velocity field at the Point vortices and the fact that the flow domain is unbounded.

- The submerged body bracket is the same as that for Kirchhoff's equations in 2d:

$$\{F, G\}_{\text{submerged body}} = (\nabla_{\binom{\Pi}{P}} F)^T \Lambda \, \nabla_{\binom{\Pi}{P}} G$$

where

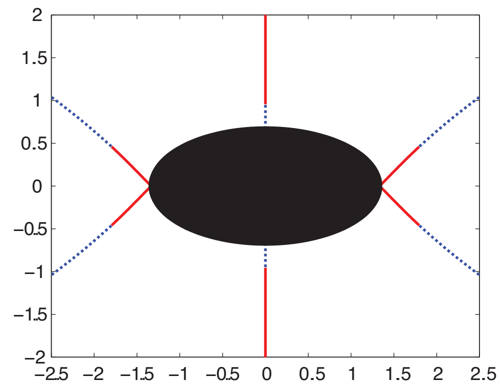
$$\Lambda(\Pi, P) = \begin{pmatrix} 0 & -P_2 & P_1 \\ P_2 & 0 & 0 \\ -P_1 & 0 & 0 \end{pmatrix}$$

- The bracket for point vortices is the same as that presented in Joris' lecture:

$$\{F, G\}_{\text{point vortices}} = \sum_{k=1}^{n} \left(\frac{\partial F}{\partial X_k} \frac{\partial G}{\partial Y_k} - \frac{\partial F}{\partial Y_k} \frac{\partial G}{\partial X_k} \right).$$

Ellipse interacting dynamically with a vortex pair

Relative equilibria: ellipse and vortex pairs move rigidly (at the same velocity)

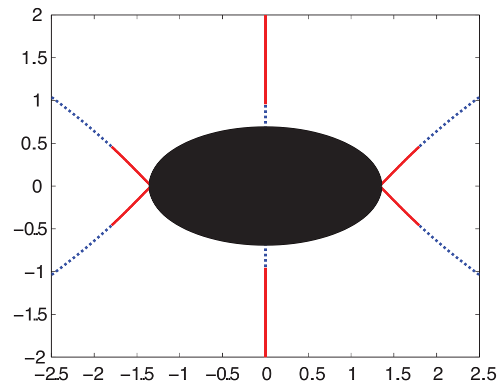


Two families of relative equilibria:

1. moving Föppl equilibria
2. equilibria along the ellipse axis of symmetry ($\perp$ to direction of motion)

Ellipse interacting dynamically with a vortex pair

Relative equilibria: ellipse and vortex pairs move rigidly (at the same velocity)



Two families of relative equilibria:

1. moving Föppl equilibria
2. equilibria along the ellipse axis of symmetry ($\perp$ to direction of motion)

Linear stability analysis:

1. moving Föppl equilibria: mostly unstable s.t. symmetric perturbations
2. equilibria along the axis of symmetry: mostly stable s.t. symmetric perturbations

Symmetric perturbations:

$$\delta X_1 = \delta X_2 = \delta X_s$$
$$\delta Y_1 = -\delta Y_2 = \delta Y_s$$
$$\delta P_x, \qquad \delta P_y = 0, \qquad \delta \Pi = 0$$

Anti-symmetric perturbations:

$$\delta X_1 = -\delta X_2 = \delta X_a$$
$$\delta Y_1 = \delta Y_2 = \delta Y_a$$
$$\delta P_x = 0, \qquad \delta P_y, \qquad \delta \Pi$$

Any perturbation is written as the sum of symmetric and antisymmetric perturbations

Symmetric perturbations:

$$\delta X_1 = \delta X_2 = \delta X_s$$
$$\delta Y_1 = -\delta Y_2 = \delta Y_s$$
$$\delta P_x, \qquad \delta P_y = 0, \qquad \delta \Pi = 0$$

Anti-symmetric perturbations:

$$\delta X_1 = -\delta X_2 = \delta X_a$$
$$\delta Y_1 = \delta Y_2 = \delta Y_a$$
$$\delta P_x = 0, \qquad \delta P_y, \qquad \delta \Pi$$

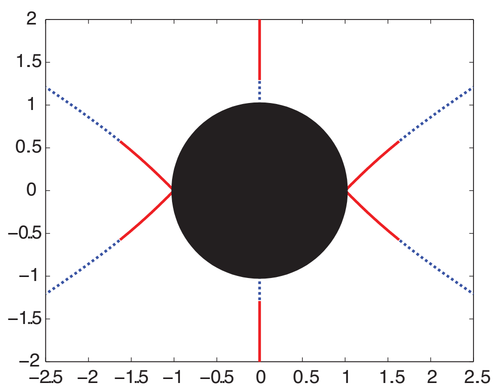
Any perturbation is written as the sum of symmetric and antisymmetric perturbations

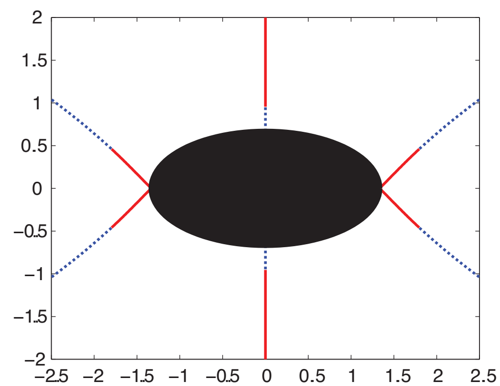
Linearized equations decouple under symmetric and anti-symmetric perturbations:

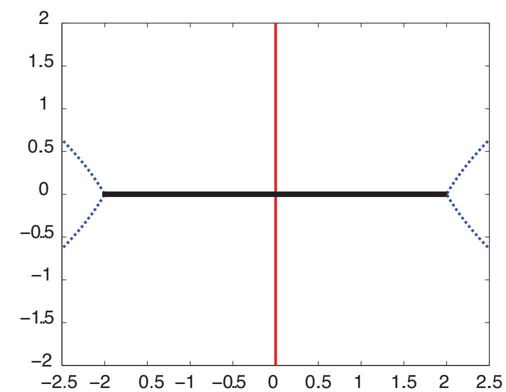
$$\frac{d}{dt}\begin{pmatrix} \delta X_s \\ \delta Y_s \\ \delta P_x \\ \hline \delta X_a \\ \delta Y_a \\ \delta P_y \\ \delta \Pi \end{pmatrix} = \left(\begin{array}{c|c} D_{s_{3\times 3}} & 0_{3\times 4} \\ \hline 0_{4\times 3} & D_{a_{4\times 4}} \end{array}\right) \begin{pmatrix} \delta X_s \\ \delta Y_s \\ \delta P_x \\ \hline \delta X_a \\ \delta Y_a \\ \delta P_y \\ \delta \Pi \end{pmatrix} .$$

Both families of equilibria are unstable subject to antisymmetric perturbations

The stability region depends on the aspect ratio b/a of the ellipse



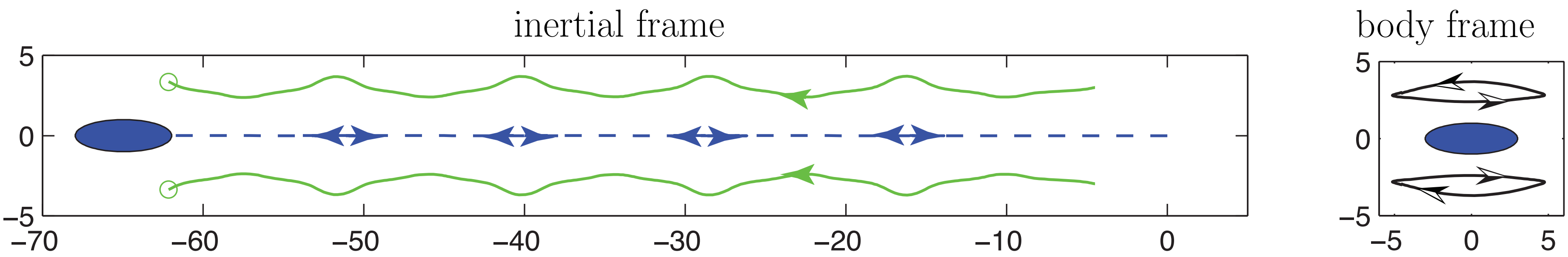


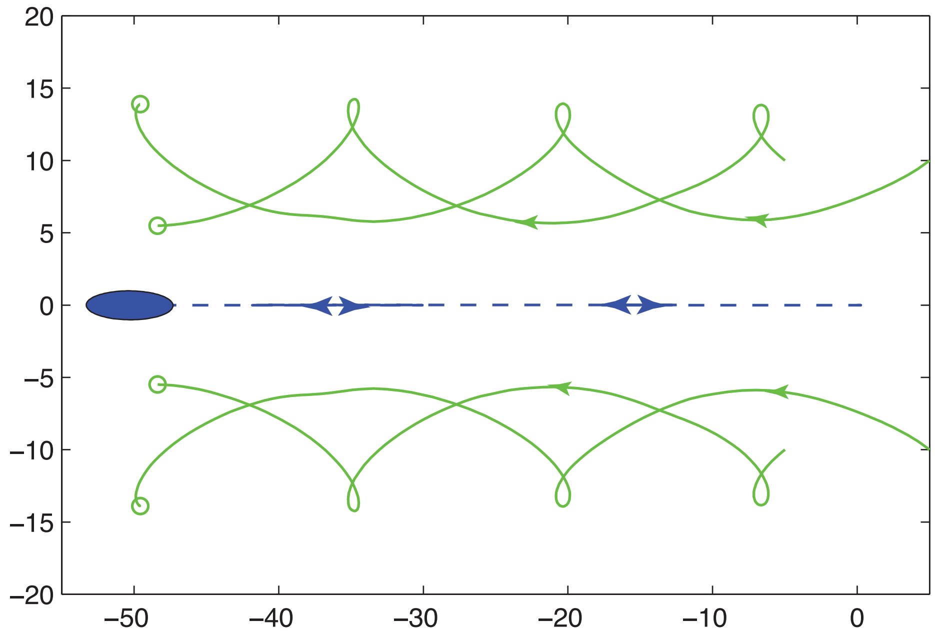


Dependence on aspect ratio:

1. stability region along the Föppl curves decreases as the aspect ratio decreases
2. stability region along the axis of symmetry increases as the aspect ratio decreases

Numerical evidence that these marginally stable equilibria are indeed nonlinearly stable




20
15
10
5
0
−5
−10
−15
−20
−50
−40
−30
−20
−10
0

Can these equilibria and their stability be exploited to design swimming motions?

Ellipse swimming in an externally generated vortex street