CDS 140b: Homework Set 1

Due by the end of the first week after the first unit

Problems.

1. Consider the following planar, autonomous vector field:

$$\dot{x} = -x + y^2,$$
$$\dot{y} = -2x^2 + 2xy^2, \qquad (x, y) \in \mathbb{R}^2.$$

   (a) Prove that $y = x^2$ is an invariant manifold for this vector field.

   (b) Prove that there exists a heteroclinic connection between the equilibrium points $(x, y) = (0, 0)$ and $(x, y) = (1, 1)$.

2. Consider the system of equations in the plane $\mathbb{R}^2$ given by

$$\dot{x} = y$$
$$\dot{y} = -4x - \mu y + 6x^3 - 2x^7$$

   (a) Show that for $\mu = 0$, the system is Hamiltonian and find a Hamiltonian function.

   (b) Still assuming that $\mu = 0$, write this system as Euler-Lagrange equations and find a Lagrangian function.

   (c) Assume that $\mu \geq 0$. Show that solutions exist for all positive time for any initial data.

   (d) Assume that $\mu \geq 0$. Show that $(0, 1)$ is an equilibrium point and find the eigenvalues of the linearized system at this equilibrium. Explain how your answer is consistent with the possible eigenvalue configurations for a Hamiltonian system in the special case $\mu = 0$.

   (e) What can you say about the stable, unstable and center manifolds for the equilibrium point $(0, 1)$?

   (f) Assume that $\mu > 0$ and let (x_0, y_0) be given initial data for a solution $(x(t), y(t))$. What is the possible limit of $(x(t), y(t))$ as $t \to +\infty$?

3. Let δ, γ, ω be constants and $\omega \neq 0$. Consider the following system in $\mathbb{R}^3$:

$$\begin{aligned} \dot{x} &= -\delta x + \omega y - \gamma x^3 - \gamma x y^2 \\ \dot{y} &= -\omega x - \delta y - \gamma y^3 - \gamma y x^2 \\ \dot{z} &= -z + zxy. \end{aligned}$$

   (a) Discuss the stability of the origin and the corresponding invariant manifolds.

(b) Show by direct calculation that the system has a periodic orbit for some values of ω, δ, γ for which *both* δ and γ are *nonzero.*

(c) Show that there is a Hopf bifurcation as δ changes sign (keeping ω, γ fixed).

4. Discuss the relation between the Smale horseshoe map and the homoclinic tangle; that is, given the tangle, show how to construct the horseshoe map.

5. Consider the periodically forced pendulum equation (as in the lectures):

$$\ddot{x} + \sin x + \epsilon \dot{x} \sin t = 0$$

Suspend the system to an autonomous system on $\mathbb{R}^3$ as follows:

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\sin x + \epsilon v \sin \tau \\ \dot{\tau} &= 1 \end{aligned}$$

Relate invariant manifolds of this system with invariant manifolds of the Poincaré map with LCS .

Project Suggestions.

1. Read and master at least one paper on LCS .

2. Get LCS running for some concrete system of your choice .