1. Consider the following planar, autonomous vector field:

\[
\begin{align*}
\dot{x} &= -x + y^2, \\
\dot{y} &= -2x^2 + 2xy^2, \\
(x,y) &\in \mathbb{R}^2.
\end{align*}
\]

(a) Prove that \( y = x^2 \) is an invariant manifold for this vector field.

(b) Prove that there exists a heteroclinic connection between the equilibrium points \((x,y) = (0,0)\) and \((x,y) = (1,1)\).

2. Consider the system of equations in the plane \( \mathbb{R}^2 \) given by

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -4x - \mu y + 6x^3 - 2x^7
\end{align*}
\]

(a) Show that for \( \mu = 0 \), the system is Hamiltonian and find a Hamiltonian function.

(b) Still assuming that \( \mu = 0 \), write this system as Euler-Lagrange equations and find a Lagrangian function.

(c) Assume that \( \mu \geq 0 \). Show that solutions exist for all positive time for any initial data.

(d) Assume that \( \mu \geq 0 \). Show that \((0,1)\) is an equilibrium point and find the eigenvalues of the linearized system at this equilibrium. Explain how your answer is consistent with the possible eigenvalue configurations for a Hamiltonian system in the special case \( \mu = 0 \).

(e) What can you say about the stable, unstable and center manifolds for the equilibrium point \((0,1)\)?

(f) Assume that \( \mu > 0 \) and let \((x_0, y_0)\) be given initial data for a solution \((x(t), y(t))\). What is the possible limit of \((x(t), y(t))\) as \( t \to +\infty \)?

3. Let \( \delta, \gamma, \omega \) be constants and \( \omega \neq 0 \). Consider the following system in \( \mathbb{R}^3 \):

\[
\begin{align*}
\dot{x} &= -\delta x + \omega y - \gamma x^3 - \gamma xy^2 \\
\dot{y} &= -\omega x - \delta y - \gamma y^3 - \gamma yx^2 \\
\dot{z} &= -z + zxy.
\end{align*}
\]

(a) Discuss the stability of the origin and the corresponding invariant manifolds.
(b) Show by direct calculation that the system has a periodic orbit for some values of $\omega, \delta, \gamma$ for which both $\delta$ and $\gamma$ are nonzero.

(c) Show that there is a Hopf bifurcation as $\delta$ changes sign (keeping $\omega, \gamma$ fixed).

4. Discuss the relation between the Smale horseshoe map and the homoclinic tangle; that is, given the tangle, show how to construct the horseshoe map.

5. Consider the periodically forced pendulum equation (as in the lectures):

$$\ddot{x} + \sin x + \epsilon \dot{x} \sin t = 0$$

Suspend the system to an autonomous system on $\mathbb{R}^3$ as follows:

$$\dot{x} = v$$
$$\dot{v} = -\sin x + \epsilon v \sin \tau$$
$$\dot{\tau} = 1$$

Relate invariant manifolds of this system with invariant manifolds of the Poincaré map with LCS.

**Project Suggestions.**

1. Read and master at least one paper on LCS.

2. Get LCS running for some concrete system of your choice.