A LITTLE BIT ON SYMMETRIES

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Consider a dynamical system on $\mathbb{R}^n$

$$\dot{x} = f(x) \tag{1}$$

Let $x(t)$ be a trajectory that satisfies the ODE. Consider a transformation, Φ, from trajectories to trajectories. That is to say, if Γ is the set of curves on $\mathbb{R}^n$, then $\Phi : \Gamma \to \Gamma$. For example, if $n = 2$ and γ is the curve $(x_1(t), x_2(t))$ then $\Phi(\gamma)$ could be the curve $(x_1(-t), -x_2(-t))$ or $(-x_1(t), x_2(t))$ or anything involving a space transformation and/or time reparametrization. We say that the system is Φ symmetric if given a curve $\gamma \in \Gamma$ that satisfies 1, the curve $\Phi(\gamma)$ also satisfies 1.

1. Example, a conservative system with reflection symmetry

Let

$$\ddot{x} = -x^3$$

In first order form this is

$$\dot{x} = v \tag{2}$$

$$\dot{v} = -x^3 \tag{3}$$

We observe the trajectories move along the level sets of the energy $H(x, v) = \frac{1}{2}v^2 + x^4$. This gives a good idea of the symmetries. You've all proven that this system is "reversible" in homework. Reversible is a type of symmetry where if γ is the curve represented by $(x(t), v(t))$ then $\Phi(\gamma)$ is the curve $(x(-t), -v(-t))$. This system also admits rotational symmetry of π radians. That would be the symmetry that sends $\gamma = [(x(t), v(t))]$ to $[(-x(t), -v(t))]$. This symmetry leaves t unchanged, and is purely a space transformation. To see that the system exhibits this symmetry set $[\tilde{x}(t), \tilde{y}(t))] = \Phi([x(t), y(t)])$. We then find

$$\frac{d}{dt}\tilde{x}(t) = -\frac{d}{dt}x(t) = -v(t)$$

So $\dot{\tilde{x}} = \tilde{v}$. Secondly

$$\frac{d}{dt}\tilde{v} = -\frac{d}{dt}v = -(-x^3) = -\tilde{x}^3$$

So the second equation is satisfied as well. Thus $\Phi(\gamma)$ satisfies the equation and the system is Φ symmetric. Or more precisely, the system is π rotationally symmetric.

2. Example, a dissipative system with reflection symmetry

Dissipative systems are never reversible, but they can exhibit other symmetries. Consider the last system with a dissipation term.

$$\dot{x} = v \tag{4}$$

$$\dot{v} = -x^3 - \alpha v \tag{5}$$

It also has π rotational symmetry. We see that

$$\frac{d}{dt}\tilde{x} = -\frac{d}{dt}x = -v = \tilde{v}$$

and

$$\frac{d}{dt}\tilde{v} = -\frac{d}{dt}v = x^3 + \alpha v = -\tilde{x}^3 - \alpha\tilde{v}$$

Thus this system does exhibit some symmetry. Although not quite as many as the previous system.