Smooth versus Analytic functions

Henry Jacobs

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Functions of the form
$$f(x) = \sum_{i \geq 0} a_i x^i$$
that converge everywhere are called analytic. We see that analytic functions are equal to there Taylor expansions. Obviously all analytic functions are smooth or C^∞ but not all smooth functions are analytic. For example

$$g(x) = e^{-1/x^2}$$

Has derivatives of all orders, so $g \in C^\infty$. This function also has a Taylor series expansion about any point. In particular the Taylor expansion about 0 is

$$g(x) \approx 0 + 0x + 0x^2 + \dots$$

So that the Taylor series expansion does in fact converge to the function

$$\tilde{g}(x) = 0$$

We see that g and $\tilde{g}$ are competely different and only equal each other at a single point. So we've shown that g is not analytic.

This is relevent in this class when finding approximations of invariant manifolds. Generally when we ask you to find a 2nd order approximation of the center manifold we just want you to express it as the graph of some function on an affine subspace of $\mathbb{R}^n$. For example say we're in $\mathbb{R}^2$ with an equilibrium point at the origin, and a center subspace along the y-axis. Than if you're asked to find the center manifold to 2nd order you assume the manifold is locally (i.e. near (0,0)) defined by the graph $(h(y), y)$. Where $h(y) = 0, h'(y) = 0$. Thus the taylor approximation is $h(y) = ay^2 + hot.$ and you must solve for a using the invariance of the manifold and the dynamics. I'll call this the **analytic approximation method**. On the other hand if you're asked to solve for the manifold explicitly (not approximately) then you can **not** just do a power expansion with ∞ terms. This is because the analytic approximation method only holds in a neighborhood of the equilibrium and may converge to a completely different function (as shown by the example of a smooth but not analytic function), in which case the radius about which your approximation converges to the true solution may be 0.

Bottomline: You can't really use the analytic approximation method to find manifolds exactly, only approximately. With this said, if you're ever asked to find an invariant manifold exactly it will probably be a very simple one (like an axis). In homework 7, one of the problems asks you to prove $W = \{(x, y) : y =$

$x^3\}$ is an invariant manifold. Later you're asked to find the center subspace, which is tangeant to the direction $(1, 0)$ at the origin. Noting that W is tangeant to $(1, 0)$ at the origin paired with the fact that W is invariant is enough to show that an segment of W which contains the origin is a suitable center manifold. Or simply $W_s = W$. Usually it's difficult to find these manifolds, but we'd never intentionally ask you to find something we wouldn't think we could find ourselves.

LESSON: Only use the analytic approximation method to approximate manifolds, never to solve for them exactly.