Liapunov Functions

Besides the Liapunov spectral theorem, there is another basic method of proving stability that is a generalization of the energy method we have seen in the introductory mechanical examples.

Definition (Liapunov Function). *. Let X be a C^r vector field on $\mathbb{R}^n$, $r \geq 1$, and let x_e be an equilibrium point for X, that is, $X(x_e) = 0$. A **Liapunov function** for X at x_e is a continuous function $L : U \to \mathbb{R}$ defined on a neighborhood U of x_e, differentiable on $U \backslash \{x_e\}$, and satisfying the following conditions:*

(i) $L(x_e) = 0$ *and* $L(x) > 0$ *if* $x \neq x_e$;

(ii) *The directional derivative of L along X, denoted $X[L] \leq 0$ on $U \backslash \{x_e\}$; this means that $\frac{d}{dt} L \leq 0$ along solution curves of X.*

*The Liapunov function L is said to be **strict**, if* (ii) *is replaced by the condition* (ii)′*: $X[L] < 0$ in $U \backslash \{x_e\}$.*

Condition (i) is sometimes called the ***potential well hypothesis***.[7] By the Chain Rule for the time derivative of V along integral curves, condition (ii) is equivalent to the statement: L is nonincreasing along integral curves of X.

Theorem (Liapunov Stability Theorem.). *Let X be a C^r vector field on $\mathbb{R}^n$, $r \geq 1$, and let x_e be an equilibrium point for X, that is, $X(x_e) = 0$. If there exists a Liapunov function for X at x_e, then x_e is stable.*

Proof. Since the statement is local, we can assume that $x_e = 0$. By the local existence theory, there is a neighborhood U of 0 such that all solutions starting in U exist for time $t \in [-\delta, \delta]$, with δ depending only on X and U, but not on the solution. Now fix $\varepsilon > 0$ and an open ball $D_\varepsilon(0)$ that is included in U. Let $\rho(\varepsilon) > 0$ be the minimum value of L on the sphere of radius ε, and define the open set $U' = \{ x \in D_\varepsilon(0) \mid L(x) < \rho(\varepsilon) \}$. By (i), $U' \neq \varnothing$, $0 \in U'$, and by (ii), no solution starting in U' can meet the sphere of radius ε (since L is decreasing on integral curves of X). Thus all solutions starting in U' never leave $D_\varepsilon(0) \subset U$ and therefore by uniformity of time of existence, these solutions can be extended indefinitely in time. This shows that 0 is stable. ■

There is a more global concept that is related to this circle of ideas that we discuss somewhat informally. Namely, a region $R \subset \mathbb{R}^n$ with a (smooth)

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[7] In infinite dimensions, one needs to augment (i) by the additional condition that there is an $\varepsilon > 0$ such that for all $0 < \varepsilon' \leq \varepsilon$,

$$\inf\{ L(\varphi^{-1}(x)) \mid \|x - x_e\| = \varepsilon' \} > 0.$$

In finite dimensions, this condition is automatic by the compactness of spheres.

boundary with a well defined inside and outside, is called a ***positively trapping region*** if for initial conditions that start in R, the solution stays in R for $t \geq 0$. If the given vector field is pointing inwards (or is tangent) to the boundary of R, this is a sufficient condition for R to be a trapping region. If R is bounded, necessarily the dynamical system is positively complete on R. Often sublevel sets $R = \{x \in \mathbb{R}^n \mid L(x) \leq C\}$ of Liapunov functions provide such trapping regions because the vector field is inwards corresponds to the condition that L is decreasing along solutions at the boundary.

Theorem (Liapunov Asymptotic Stability Theorem.). *Let X be a C^r vector field on $\mathbb{R}^n$, $r \geq 1$, and let x_e be an equilibrium point for X, that is, $X(x_e) = 0$. Suppose that L a strict Liapunov function for X at x_e. Then x_e is asymptotically stable.*

Proof. We can assume $x_e = 0$. By the preceding Theorem, 0 is stable, so if t_n is an increasing sequence, $t_n \to \infty$, and $x(t)$ is an integral curve of X starting in U', it lies in a bounded set and so the sequence $\{x(t_n)\} \in \mathbb{R}^n$ has a convergent subsequence. Thus, there is a sequence $t_n \to +\infty$ such that $x(t_n) \to x_0 \in D_\varepsilon(0)$, some ϵ disk. We shall prove that $x_0 = 0$. Since $L(x(t))$ is a *strictly* decreasing function of t by (ii)$'$, $L(x(t)) > L(x_0)$ for all $t > 0$. If $x_0 \neq 0$, let $c(t)$ be the solution of X starting at x_0, so that $L(c(t)) < L(x_0)$, again since $t \mapsto L(x(t))$ is strictly decreasing. Thus, for any solution $\tilde{c}(t)$ starting close to x_0, $L(\tilde{c}(t)) < L(x_0)$ by continuity of L. Now take $\tilde{c}(0) = x(t_n)$ for n large to get the contradiction $L(x(t_n + t)) < L(x_0)$. Therefore $x_0 = 0$ is the only limit point of $\{\, x(t) \mid t \geq 0 \,\}$ if $x(0) \in U'$, that is, 0 is asymptotically stable. ■

The same method can be used to detect the instability of equilibrium solutions, as in the following result.

Theorem (Liapunov Instability Theorem). *Let X be a C^r vector field on $\mathbb{R}^n$, $r \geq 1$, and let x_e be an equilibrium point for X, that is, $X(x_e) = 0$. Assume there is a continuous function $L : U \to \mathbb{R}$ defined in a neighborhood of U of x_e, which is differentiable on $U \backslash \{x_e\}$, and satisfies $L(x_e) = 0$, $X[L] > 0$ (respectively, $\leq a < 0$) on $U \backslash \{x_e\}$. If there exists a sequence $x_k \to x_e$ such that $L(x_k) > 0$ (respectively, < 0), then x_e is unstable.*

Proof. We need to show that there is a neighborhood W of x_e such that for any neighborhood V of x_e, $V \subset U$, there is a point x_V whose integral curve leaves W. By local compactness, we can assume that $X[L] \geq a > 0$. Since x_e is an equilibrium, there is a neighborhood $W_1 \subset U$ of x_e such that each integral curve starting in W_1 exists for time at least $1/a$. Let $W = \{\, x_e \in W_1 \mid L(m) < 1/2 \,\}$.

We can assume as usual that $x_e = 0$. Let $c_n(t)$ denote the integral curve of X with initial condition $x_n \in W$. Then

$$L(c_n(t)) - L(x_n) = \int_0^t X[L](c_n(\lambda))\, d\lambda \geq at$$

so that

$$L(c_n(1/a)) \geq 1 + L(m_n) > 1,$$

that is, $c_n(1/a) \notin W$. Thus all integral curves starting at the points $x_n \in W$ leave W after time at most $1/a$. Since $x_n \to 0$, the origin is unstable. ■

Example 1. Consider the simple pendulum $\ddot{x} + \sin x = 0$. As usual, we write this as a dynamical system:

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\sin x \end{aligned}$$

Let us test the origin $(x, v) = (0, 0)$ for stability using the energy function as Liapunov function: $E(x, v) = \frac{1}{2}v^2 - \cos x$ on the set U where, say, $-\pi < x < \pi$. Since $\frac{d}{dt}E = 0$, condition (ii) of the definition of a Liapunov function holds. The origin $(x, v) = (0, 0)$ is a strict local minimum of E and so condition (i) holds. Thus, the origin is stable by the Liapunov stability theorem.

Example 2. Let us test the upright solution $(x, v) = (\pi, 0)$ for instability. However, the Liapunov instability theorem does not apply with the same choice of Liapunov function since that result requires that E has a nonzero time derivative. On the other hand, the Liapunov spectral theorem, or invariant manifold theory shows that indeed this equilibrium is unstable.

Example 3. Consider the simple pendulum with friction $\ddot{x} + \sin x = -\nu\dot{x}$. Again, write this as a dynamical system:

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\sin x - \nu v \end{aligned}$$

This time let us test the origin $(x, v) = (0, 0)$ for asymptotic stability using the same energy function as Liapunov function: $E(x, v) = \frac{1}{2}v^2 - \cos x$ on the set U where, say, $-\pi < x < \pi$. This time

$$\frac{d}{dt}E(x, v) = -\nu v^2,$$

which is non-positive. However, it is not negative everywhere on the set U because it vanishes when $v = 0$. Thus, E is not a strict Liapunov function and the Liapunov asymptotic stability theorem fails to apply. On the other hand, the Liapunov spectral theorem does apply to show asymptotic stability because the linearization of the system at the origin is given by the matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & -\nu \end{bmatrix}$$

which has eigenvalues $\lambda = \left(-\nu \pm \sqrt{\nu^2 - 4}\right)$ which, for $\nu > 0$ are in the left half plane (note that the eigenvalues can be imaginary or real, corresponding to whether or not the origin is a spiral sink or a node (underdamped or overdamped). Thus, by the Liapunov spectral theorem, the origin is asymptotically stable.

While the Liapunov asymptotic stability theorem does not apply, a more general result, the LaSalle Invariance principle does apply. We turn to this topic in the next section.

Example 4. The vector field on the plane with components

$$X(x, y) = (-y - x^5, x - 2y^3)$$

has the origin as an isolated equilibrium. The eigenvalues of the linearization of X at $(0, 0)$ are $\pm i$ and so Liapunov's Spectral Stability Criterion does not give any information regarding the stability of the origin. If we suspect that $(0, 0)$ is asymptotically stable, we can try searching for a Liapunov function of the form $L(x, y) = ax^2 + by^2$, so we need to determine the coefficients $a, b \neq 0$ in such a way that $X[L] < 0$. We have

$$X[L] = 2ax(-y - x^5) + 2by(x - 2y^3) = 2xy(b - a) - 2ax^6 - 4by^4,$$

so that choosing $a = b = 1$, we get $X[L] = -2(x^6 + 2y^4)$ which is strictly negative if $(x, y) \neq (0, 0)$. Thus the origin is asymptotically stable by the Liapunov asymptotic stability theorem.

Example 5. Consider the vector field in the plane with components

$$X(x, y) = (-y + x^5, x + 2y^3)$$

with the origin as an isolated critical point and characteristic exponents $\pm i$. Again Liapunov's Stability Criterion cannot be applied, so that we search for a function $L(x, y) = ax^2 + by^2,\ a, b \neq 0$ in such a way that $X[L]$ has a definite sign. As above we get

$$X[L] = 2ax(-y + x^5) + 2by(x + 2y^3) = 2xy(b - a) + 2ax^6 + 4by^4,$$

so that choosing $a = b = 1$, it follows that $X[L] = 2(x^6 + y^4) > 0$ if $(x, y) \neq (0, 0)$. Thus, the origin is unstable.

The preceding two examples show that if the spectrum of X lies on the imaginary axis, the stability nature of the equilibrium is determined by the nonlinear terms.

Example 6. Now we give another proof of the Dirichlet Stability Theorem. Consider Newton's equations in $\mathbb{R}^3$, $\ddot{\mathbf{q}} = -(1/m)\nabla V(\mathbf{q})$ written as a first order system $\dot{\mathbf{q}} = \mathbf{v}, \dot{\mathbf{v}} = -(1/m)\nabla V(\mathbf{q})$ and so define a vector field X on $\mathbb{R}^3 \times \mathbb{R}^3$. Let $(\mathbf{q}_0, \mathbf{v}_0)$ be an equilibrium of this system, so that $\mathbf{v}_0 = \mathbf{0}$ and $\nabla V(\mathbf{q}_0) = \mathbf{0}$. In previous lectures we have seen that the total energy

$$E(\mathbf{q}, \mathbf{v}) = \frac{1}{2} m \|\mathbf{v}\|^2 + V(\mathbf{q})$$

is conserved, so we use E to construct a Liapunov function L. Since $L(\mathbf{q}_0, \mathbf{0}) = 0$, define

$$L(\mathbf{q}, \mathbf{v}) = E(\mathbf{q}, \mathbf{v}) - E(\mathbf{q}_0, \mathbf{0}) = \frac{1}{2} m \|\mathbf{v}\|^2 + V(\mathbf{q} - V(\mathbf{q}_0),$$

which satisfies $X[L] = 0$ by conservation of energy. If $V(\mathbf{q}) > V(\mathbf{q}_0)$ for $\mathbf{q} \neq \mathbf{q}_0$, then L is a Liapunov function. Thus we have proved

The Dirichlet-Lagrange Stability Theorem: *An equilibrium point* $(\mathbf{q}_0, \mathbf{0})$ *of Newton's equations for a particle of mass* m*, moving under the influence of a potential* V*, which has a local absolute minimum at* $\mathbf{q}_0$*, is stable.*

The Invariance Principle

A key ingredient in proving more general global asymptotic stability results is what is often referred to as the LaSalle invariance principle.[8] It allows one to prove attractivity of more general invariant sets than equilibrium points.

Theorem (The Invariance Principle)**.** *Consider a smooth dynamical system on* $\mathbb{R}^n$ *given by* $\dot{x} = X(x)$ *and let* Ω *be a compact set that is (positively) invariant under the flow of* X*. Let* $V : \Omega \to \mathbb{R}$, $V \geq 0$*, be a* C^1 *function such that*

$$\dot{V}(x) = \frac{\partial V}{\partial x} \cdot X \leq 0$$

in Ω*. Let* S *be the largest invariant set in* Ω *where* $\dot{V}(x) = 0$*. Then every solution with initial point in* Ω *tends asymptotically to* S *as* $t \to \infty$*. In particular, if* S *is an isolated equilibrium, it is asymptotically stable.*

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[8] The theorem is actually due to Barbashin, E. A. and N. N. Krasovskii [1952], Stability of motion in the large, *Doklady Mathematics, (Translations of Proceedings of Russian Academy of Sciences)* **86**, 453–456, and apparently independently appears in the book LaSalle, J. P. and S. Lefschetz [1961], *Stability by Lyapunov's direct method with applications.* Academic Press, New York, Krasovskii, N. [1963], *Stability of Motion.* The book Khalil, H. K. [1992], *Nonlinear systems.* Macmillan Publishing Company, New York. Stanford University Press, (originally published 1959) has a nice treatment. We refer to one of these sources for the proof of the theorem.

In the statement of the theorem, $V(x)$ need not be positive definite, but rather only semidefinite, and that if in particular S is an equilibrium point, the theorem proves that the equilibrium is asymptotically stable. The set Ω in the LaSalle theorem also gives us an estimate of the region of attraction of an equilibrium. This is one of the reasons that this is a more attractive methodology than that of spectral stability tests, which could in principle give a very small region of attraction.

Example. Now we study an example similar to Example 3 in the preceding section, where the Liapunov asymptotic stability theorem failed, but we assert that the Invariance Principle succeeds. So again consider the simple pendulum with friction $\ddot{x} + \sin x = -\nu\dot{x}$ and write this as a dynamical system:

$$\begin{aligned}\dot{x} &= v \\ \dot{v} &= -\sin x - \nu v\end{aligned}$$

This time let us test the origin $(x, v) = (0, 0)$ for asymptotic stability using the same energy function as Liapunov function: $E(x, v) = \frac{1}{2}v^2 - \cos x$ on the set Ω where, say, $E(x, v) < 2$. As before,

$$\frac{d}{dt}E(x, v) = -\nu v^2,$$

Since E is non-increasing on trajectories, the open set Ω is positively invariant. The only invariant set S containing the set where $\dot{E} = 0$, that is, where $v = 0$, is the set of equilibrium points, since the vector field is transverse to the x-axis. In our case, there is just one equilibrium point, namely the origin. Thus, by the Invariance Principle, the origin is asymptotically stable.