Problem 1:

The position of M is $q_M = (x(t), x^2(t))$, the position of M is $q_m = (x(t) + l\sin\theta, x^2(t) - l\cos\theta)$. So the velocity

$$v_M = \dot{q}_M = (\dot{x}, 2x\dot{x}),$$
$$v_m = \dot{q}_m = (\dot{x} + l\cos\theta\dot{\theta}, 2x\dot{x} + l\sin\theta\dot{\theta}).$$

The kinetic energy is

$$\begin{aligned} K_E = \frac{1}{2}M\|v_M\|^2 + \frac{1}{2}m\|v_m\|^2 &= \frac{1}{2}M(\dot{x}^2 + 4x^2\dot{x}^2) + \frac{1}{2}m((\dot{x} + l\cos\theta\dot{\theta})^2 + (2x\dot{x} + l\sin\theta\dot{\theta})^2) \\ = \frac{1}{2}M(4x^2+1)\dot{x}^2 &+ \frac{1}{2}m(\dot{x}^2 + l^2\cos^2\theta\dot{\theta}^2 + 2\dot{x}l\cos\theta\dot{\theta} + 4x^2\dot{x}^2 + l^2\sin^2\theta\dot{\theta}^2 + 4x\dot{x}l\sin\theta\dot{\theta}) \\ &= \frac{1}{2}(M+m)(4x^2+1)\dot{x}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + ml\dot{x}\dot{\theta}(\cos\theta + 2x\sin\theta), \end{aligned}$$

and the potential energy is

$$P_E = Mgx^2 + mg(x^2 - l\cos\theta).$$

So the Lagrangian for the system is

$$L = K_E - P_E = \frac{1}{2}(M+m)(4x^2+1)\dot{x}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + ml\dot{x}\dot{\theta}(\cos\theta + 2x\sin\theta) - Mgx^2 - mg(x^2 - l\cos\theta).$$

Computing the E-L equations, first compute

$$\begin{aligned} \frac{\partial L}{\partial \dot{x}} &= (M+m)(4x^2+1)\dot{x} + ml\dot{\theta}(\cos\theta + 2x\sin\theta), \\ \frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) &= (M+m)(8x\dot{x}\dot{x} + (4x^2+1)\ddot{x}) + ml\ddot{\theta}(\cos\theta + 2x\sin\theta) + ml\dot{\theta}(-\sin\theta\dot{\theta} + 2\dot{x}\sin\theta + 2x\cos\theta\dot{\theta}), \\ \frac{\partial L}{\partial x} &= \frac{1}{2}(M+m)8x\dot{x}^2 + 2ml\dot{x}\dot{\theta}\sin\theta - 2(M+m)gx. \end{aligned}$$

So the E-L equation of $x, \dot{x}$ becomes

$$\begin{aligned} 8(M+m)x\dot{x}^2 + (M+m)(4x^2+1)\ddot{x} + ml\ddot{\theta}\cos\theta + 2mlx\ddot{\theta}\sin\theta - ml\dot{\theta}^2\sin\theta + 2ml\dot{\theta}\dot{x}\sin\theta + \\ 2mlx\dot{\theta}\dot{\theta}\cos\theta - 4(M+m)x\dot{x}^2 - 2ml\dot{x}\dot{\theta}\sin\theta + 2(M+m)gx = 0, \Rightarrow \\ 4(M+m)x\dot{x}^2 + (M+m)(4x^2+1)\ddot{x} + ml\ddot{\theta}(\cos\theta + 2x\sin\theta) + ml\dot{\theta}^2(2x\cos\theta - \sin\theta) + 2(M+m)gx = 0. \end{aligned}$$

Compute

$$\begin{aligned} \frac{\partial L}{\partial \dot{\theta}} &= ml^2\dot{\theta} + ml\dot{x}(\cos\theta + 2x\sin\theta), \\ \frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) &= ml^2\ddot{\theta} + ml\ddot{x}(\cos\theta + 2x\sin\theta) + ml\dot{x}(-\sin\theta\dot{\theta} + 2\dot{x}\sin\theta + 2x\cos\theta\dot{\theta}), \\ \frac{\partial L}{\partial \theta} &= ml\dot{x}\dot{\theta}(-\sin\theta + 2x\cos\theta) - mgl\sin\theta, \end{aligned}$$

so the E-L equation of $\theta, \dot{\theta}$ gives

$$\begin{aligned} ml^2\ddot{\theta} + ml\ddot{x}\cos\theta + 2mlx\ddot{x}\sin\theta - ml\dot{x}\dot{\theta}\sin\theta + 2ml\dot{x}^2\sin\theta + 2mlx\dot{x}\dot{\theta}\cos\theta \\ + ml\dot{x}\dot{\theta}\sin\theta - 2ml\dot{x}\dot{\theta}\cos\theta + mgl\sin\theta = 0, \Rightarrow \\ l\ddot{\theta} = -\ddot{x}(\cos\theta + 2x\sin\theta) - \sin\theta(2\dot{x}^2 + g). \end{aligned}$$

Problem 2: The energy function gives

$$E = \frac{\partial L}{\partial \dot{x}}\dot{x} + \frac{\partial L}{\partial \dot{\theta}}\dot{\theta} - L.$$

Taking the derivative of E with respect to t,

$$\begin{aligned}\frac{d}{dt}E &= \frac{d}{dt}(\frac{\partial L}{\partial \dot{x}})\dot{x} + \frac{\partial L}{\partial \dot{x}}\ddot{x} + \frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}})\dot{\theta} + \frac{\partial L}{\partial \dot{\theta}}\ddot{\theta} - (\frac{\partial L}{\partial x}\dot{x} + \frac{\partial L}{\partial \dot{x}}\ddot{x} + \frac{\partial L}{\partial \theta}\dot{\theta} + \frac{\partial L}{\partial \dot{\theta}}\ddot{\theta}) \\ &= \dot{x}(\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x}) + \dot{\theta}(\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta}) = 0, \text{by E-L equations,}\end{aligned}$$

so the energy is conserved, in Lagrangian formulations. On the other hand, consider the Hamiltonian of the system, using the Chain rule,

$$\begin{aligned}\frac{d}{dt}H &= \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial \theta}\dot{\theta} + \frac{\partial H}{\partial p_1}\dot{p}_1 + \frac{\partial H}{\partial p_2}\dot{p}_2 \\ &= (\frac{\partial H}{\partial x}\frac{\partial H}{\partial p_1} - \frac{\partial H}{\partial p_1}\frac{\partial H}{\partial x}) + (\frac{\partial H}{\partial \theta}\frac{\partial H}{\partial p_2} - \frac{\partial H}{\partial p_2}\frac{\partial H}{\partial \theta}) = 0, \\ &\text{where } p_1 = \frac{\partial L}{\partial \dot{x}}, p_2 = \frac{\partial L}{\partial \dot{\theta}}\end{aligned}$$

by Hamilton's equations. So the energy is also conserved from a Hamiltonian point of view.