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Final CDS 140a Fall 2008

1) a) Hamiltonian system with damping (Example C, p. 34 in the notes)

b) $\dot{x} = v$ fixpoints $v = 0$, $x = 1, 2, 3$

$\dot{v} = -V'(x)$

linearization $A = \begin{pmatrix} 0 & 1 \\ -T(x) & 0 \end{pmatrix}$

$T(x) = (x-2)(x-3) + (x-1)(x-3) + (x-1)(x-2)$

$\underline{x=1}$ $A = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$ $\lambda = \pm\sqrt{2}\,i$ $\boxed{\text{hyperbolic}}$

$\underline{x=2}$ $A = \begin{pmatrix} 0 & 1 \\ +1 & 0 \end{pmatrix}$ $\lambda = \pm 1$

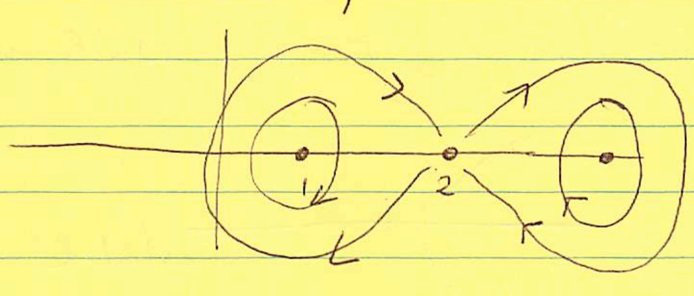
Look at H around $(1,0)$ to prove existence of periodic orbits.

$H(1+\delta, y) = y^2/2 + \delta^2$

level sets look like ellipses

$\underline{x=3}$ $A = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}$ ~~hyperbolic~~ elliptic.

phase:



c) $\dot{H} = -\sigma v^2 \leqslant 0$

will turn centers into sinks.

2) $\begin{cases} \dot{u} = u + v^2 \\ \dot{v} = -v \\ \cancel{\dot{w} = -w + v^2} \end{cases}$

$v = v_0 e^{-t}$

$\dot{u} = u + v_0^2 e^{-2t}$

$u(t) = -\frac{v_0^2}{3} e^{-2t} + \left(u_0 + \frac{v_0^2}{3}\right) e^{t}$

(1) ~~fund~~ determine the stable and unstable subspaces of the linearization at the origin.

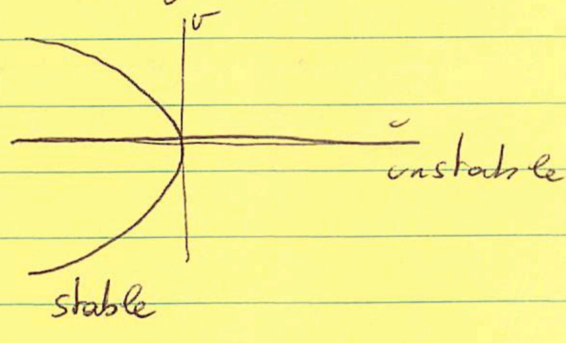
$E_U = \langle e_x \rangle \quad E_S = \langle e_y \rangle$

(2) Calculate the explicit solution $(u(t), v(t))$

(3) Using (2) find an implicit expression for the stable and unstable manifolds.

-> All points for which $u_0 + \frac{v_0^2}{3} = 0$ will tend towards the origin: stable.

Same reasoning: $v_0 = 0$ unstable manifold.



3) $\dot{x} = f_1(x,y) = -y + Ax^2 + Bxy + Cy^2$

$\dot{y} = f_2(x,y) = \underbrace{x}_{} + Ex^2 + Fxy + Gy^2$

a) because $Df(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Asy. stable $V = x^2 + y^2$

$\dot{V} = 2(x f_1 + y f_2)$

$= 2\left(A x^3 + x^2 y (B+E) + x y^2 (C+F) + G y^3 \right)$

put $y = 0$ for $\dot{V} \leq 0$ we need $A = 0$

$x = 0$ $\quad$ $G = 0$

Similar reasoning: $B + E = 0$, $C + F = 0$ and then only $\dot{V}$

Add degree 3 coefficients: $A \geq 0$, $B \geq 0$, $AB \neq 0$

$\Rightarrow \dot{V} = A x^4 + B y^4 < 0$

b) $V = (x \ y) \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$\downarrow$

- pos. def. if $D^2 V$ is pos. definite: $\det \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} > 0$

- Find a, b, c s.t. $(0,0)$ is minimum.

$\dot{V} < 0$

$$\dot{V} = 2ax\dot{x} + b(\dot{x}y + x\dot{y}) + 2cy\dot{y}$$
$$= 2ax(-y-x^3) + b(y(-y-x^3) + x(x-y^3)) + 2cy(x-y^3)$$

$$= -2axy - by^2 + bx^2 + 2cxy$$
$$-2ax^4 - byx^3 - bxy^3 - 2cy^4$$

Has to be $\dot{V} < 0$ for all (x,y)

$x=0$ $\quad \dot{V} = -by^2 - 2cy^4$

$$= -y^2(\underbrace{b + 2cy^2}_{\text{Has to } > 0})$$

Has to > 0 for ~~all~~ y small ↳ neighb. of the origin

$\Rightarrow \boxed{b > 0}$

$y=0$ $\quad \dot{V} = b\cancel{x^2} - 2ax^4$

$$= -x^2(\underbrace{2ax^2 - b}_{>0})$$

>0 if $b<0$

$\Rightarrow b = 0$ and $c, a \geq 0$

$ac \neq 0$

$$\Rightarrow \dot{V} = -\underbrace{2(a-c)xy}_{} - 2ax^4 - 2cy^4$$

level off this dominant term: $\boxed{a=c}$

4) (a)

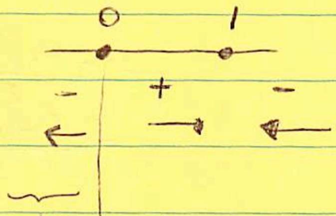
$$\dot{x} = y$$
$$\dot{y} = x - x^5 - \delta y$$
$$\dot{z} = z - z^2$$

$\Big\}$ → Hamiltonian with dissipation

$$H = y^2/2 - \frac{x^2}{2} + \frac{x^6}{6}$$

$H \leq c$ compact. ⇒ ok.

$$\dot{z} = z - z^2 = z(1-z)$$



problems for $z_0 < 0$

(b) Solutions existing for all time: $z \to 1$
$(x,y) \to \dots$ (damping)

(c) FP: $y = 0$, $z = 0, 1$, $x = 0, 1$

linearization around $(0,0,0)$

$$Df(0,0,0) = \cancel{\begin{pmatrix} 0 & 1 & 0 \\ -4-\delta & -\delta & 0 \\ 0 & 0 & 1 \end{pmatrix}} \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & -\delta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\cancel{\begin{vmatrix} -\lambda & 1 \\ -4-\delta & -\delta-\lambda \end{vmatrix} = \lambda(\delta+\lambda) + 4 + \delta = 0}$$

$$\cancel{\lambda^2 - \delta\lambda - (\delta+1) = 0 \quad D = \delta^2 + 4(\delta+4) > 0}$$

$$\cancel{\lambda = \frac{\delta \pm \sqrt{D}}{2}}$$

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\delta-\lambda \end{vmatrix}$$
$$\parallel$$
$$\lambda(\delta+\lambda) - 1 = 0 \qquad \lambda^2 + \delta\lambda - 1 = 0$$

$$D = \delta^2 + 4 > 0 \qquad \lambda = \frac{-\delta \pm \sqrt{\delta^2+4}}{2}$$

one pos, one neg root

eigenvectors $\lambda_\pm = \dfrac{-\delta \pm \sqrt{\delta^2+4}}{2}$

$$\begin{pmatrix} 0 & 1 \\ 1 & -\delta \end{pmatrix} v = \lambda_\pm v$$

$$v_y = \lambda_\pm v_x$$

Unstable eigenspace $y = \dfrac{-\delta + \sqrt{\delta^2+4}}{2} x$: V

Stable " " $y = \dfrac{-\delta - \sqrt{\delta^2+4}}{2} x$: W

$E_u = V \oplus \langle e_z \rangle$

$E_s = W$

$E_c = \{0\}$

etc.

5) (a) this is also a question on the practice formal.

(b) Reverse time: fixpoints in same location

stable $\Leftrightarrow$ unstable

center stays center.

6) (a) $\dfrac{d}{dt} V = \underline{\nabla} V \cdot \underline{\dot{x}} = -\|\nabla V\|^2 \leq 0$

If $\underline{\nabla} V = 0$: fixpoint, so $\dot{V} < 0$: precludes having periodic orbit.

(b) $\boxed{V = -x \sin y}$

7) Check with Moler for the algorithm to do this.

8) $\begin{cases} \dot{x} = y - y^3 \\ \dot{y} = -x - y^2 \end{cases}$

(a) $\cdot$ $Df(-1,1) = \begin{pmatrix} 0 & -2 \\ -1 & -2 \end{pmatrix}$ ~~$\neq 2\, Df(1,1)$~~

$$\begin{vmatrix} -\lambda & -2 \\ -1 & -2-\lambda \end{vmatrix} = \lambda(2+\lambda) - 2 = 0$$

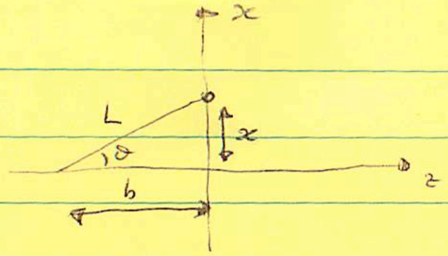
$D = 10$, roots of opposite sign: saddle.
etc.

$\cdot$ $Df(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ center in the linearization

(b) Symmetry: $\begin{cases} y \mapsto -y \\ t \mapsto -t \end{cases}$
takes unstable manifold of $(1,1)$ into stable manifold of $(-1,-1)$.

9) See next page.

10)



$L = \sqrt{b^2+x^2}$ $\qquad$ $F = -k\left(\sqrt{b^2+x^2} - \ell\right)$

$$F_x = F \sin\theta = F \frac{x}{\sqrt{b^2+x^2}}$$

$$= -k\left(1 - \frac{\ell}{\sqrt{b^2+x^2}}\right)x.$$

$\Rightarrow$ E.o.M: $\quad m\ddot{x} = -2k\left(1 - \frac{\ell}{\sqrt{b^2+x^2}}\right)x$

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\frac{2k}{m}\left(1 - \frac{\ell}{\sqrt{b^2+x^2}}\right)x \end{cases}$$

Equilibria: $v=0$, $x=0$, $\ell = \sqrt{b^2+x^2}$

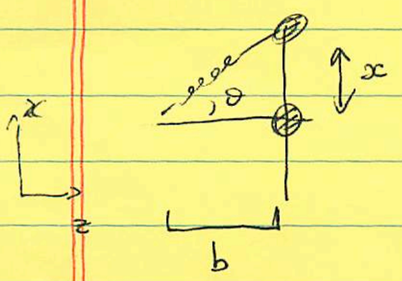
real solutions if $b<\ell$:
$x = \pm\sqrt{\ell^2-b^2}$

3 regimes: $b>\ell$: springs under tension:
1 equilibrium

$b<\ell$: springs in compression:
2 stable equilibria: $x = \pm\sqrt{\dots}$
Origin becomes unstable.

$b=\ell$: Origin stable

Spring $F = -k\left(\sqrt{x^2+b^2} - \ell\right) \underset{\sim}{e}_\ell$



$L = \sqrt{x^2+b^2}$

$\underset{\sim}{e}_\ell = \cos\theta \, e_z + \sin\theta \, e_x$

$\sin\theta = \frac{x}{\sqrt{x^2+b^2}}$

$m\ddot{x} =$

$$F_1 = -k\left(\sqrt{x^2+b^2} - \ell\right)\frac{x}{\sqrt{x^2+b^2}}$$

$$= -k\left(1 - \frac{\ell}{\sqrt{x^2+b^2}}\right)x$$

$$m\ddot{x} = -2k\left(1 - \frac{\ell}{\sqrt{x^2+b^2}}\right)x$$

Equilibria $\left(1 - \frac{\ell}{\sqrt{x^2+b^2}}\right)x = 0$

$x = 0$ or $x^2 + b^2 = \ell^2$

$x = \pm\sqrt{\ell^2 - b^2}$

$\uparrow$

if $\ell > b$

$$\frac{df}{dx} = -2k\left[\left(1 - \frac{\ell}{\sqrt{x^2+b^2}}\right) + \frac{1}{2}\frac{\ell \, 2x^2}{(x^2+b^2)^{3/2}}\right]$$

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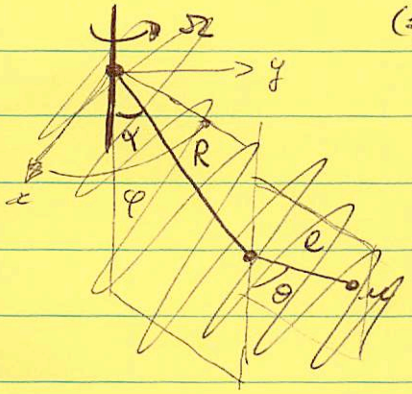
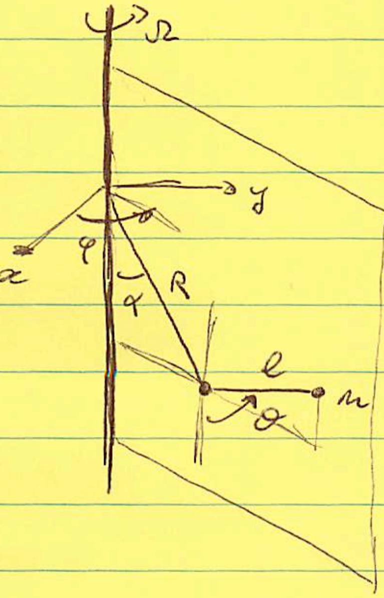
Pendulum on a beam



(1) find the Lagrangian

oh!

$$x = (R\sin\alpha + l\sin\theta)\cos\varphi$$
$$y = (R\sin\alpha + l\sin\theta)\sin\varphi$$
$$z = -(R\cos\alpha + l\cos\theta)$$



$$\dot{x} = -R\sin\alpha\sin\varphi\,\dot{\varphi} + l\dot{\theta}\cos\theta\cos\varphi - l\sin\theta\sin\varphi\,\dot{\varphi}$$
$$\dot{y} = R\sin\alpha\cos\varphi\,\dot{\varphi} + l\dot{\theta}\cos\theta\sin\varphi + l\sin\theta\cos\varphi\,\dot{\varphi}$$
$$\dot{z} = +l\sin\theta\,\dot{\theta}$$
$$\boxed{\dot{\varphi} = \Omega}$$

$$\frac{2K}{m} = R^2\sin^2\alpha\,\dot{\varphi}^2 + l^2\dot{\theta}^2\cos^2\theta + l^2\sin^2\theta\,\dot{\varphi}^2 + l^2\dot{\theta}^2\sin^2\theta + 2Rl\,\dot{\varphi}^2\sin\alpha\sin\theta$$
$$= l^2\dot{\theta}^2 + (R\sin\alpha + l\sin\theta)^2\,\Omega^2$$

$$V = mgz = -mg(R\cos\alpha + l\cos\theta)$$
$$\simeq -mgl\cos\theta$$

$$\Rightarrow L = \frac{m}{2}\left(l^2\dot{\theta}^2 + (R\sin\alpha + l\sin\theta)^2\,\Omega^2\right) + mgl\cos\theta$$

Euler–Lagrange eqn.

$$\frac{\partial L}{\partial \dot\theta} = m l^2 \dot\theta$$

$$\frac{\partial L}{\partial \theta} = m\left(R\sin\alpha + l\sin\theta\right)\Omega^2 l\cos\theta - mgl\sin\theta$$

$$\Rightarrow \quad \ddot\theta = \left(R\sin\alpha + l\sin\theta\right)\frac{\Omega^2}{l}\cos\theta - g/l\,\sin\theta$$

~~Legen~~ Legendre trafo

$$p = \frac{\partial L}{\partial \dot\theta} = m l^2 \dot\theta$$

$$H = \left(p\dot\theta - L\right)_{\dot\theta = \frac{p}{ml^2}}$$

$$= p^2/ml^2 - \frac{m}{2}\left(l^2\left(\frac{p}{ml^2}\right)^2 + \left(R\sin\alpha + l\sin\theta\right)^2\Omega^2\right) - mgl\cos\theta$$

$$= \frac{p^2}{2ml^2} - \frac{m\cancel{l}}{2}\left(R\sin\alpha + l\sin\theta\right)^2\Omega^2 - mgl\cos\theta$$

Equilibrium points.

$$\begin{cases} \dot\theta = \dfrac{p}{ml^2} \\ \dot p = m\left(R\sin\alpha + l\sin\theta\right)\Omega^2 l\cos\theta - mgl\sin\theta \end{cases}$$

Equilibria?

$p = 0$, $\quad (R \sin\alpha + \ell \sin\theta)\Omega^2 \cos\theta = g \sin\theta$

$$\downarrow$$

Small θ: $\quad R \sin\alpha\, \Omega^2 + \ell\, \Omega^2 \theta - g\theta = 0$

$$\text{or} \quad \theta = \frac{R \sin\alpha\, \Omega^2}{g - \ell\Omega^2}$$

(Ω should definitely not approach the bif. value $\Omega_{crit} = \sqrt{\frac{g}{\ell}}$)