CALIFORNIA INSTITUTE OF TECHNOLOGY

Control and Dynamical Systems

CDS 140a Midterm Examination   Jerry Marsden   Nov. 5, 2009

This is a three hour, closed book exam

While no aids are permitted, results from the course may be used as long as they are accurately quoted.

Turn in your exam on or before 5pm, Friday, Nov. 13, 2009

Print Your Name: ____________________

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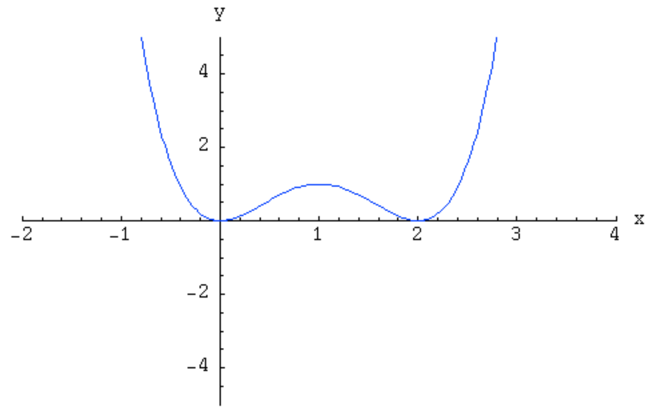
1. The graph of the function $V(x) = x^4 - 4x^3 + 4x^2$ is shown in the following figure.



Consider the planar system

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -V'(x) - \nu y\end{aligned}$$

where $\nu \geq 0$ is a constant. In what follows, distinguish the cases $\nu > 0$ and $\nu = 0$ where appropriate.

(a) Show that solution curves exist for all positive time.

(b) Find the equilibrium points of the system

(c) Compute the linearization of the system at the equilibrium points and determine the nature of the eigenvalues.

(d) What can you conclude about stability or instability of the given nonlinear system from the Liapunov eigenvalue theorem at the equilibria?

(e) Sketch the phase portrait.

2. Consider the linear systems $\dot{x} = Ax$ for each of the following matrices and determine the associated stable, unstable and center subspaces. Determine the Jordan canonical form and comment on the nature of the phase portrait, including whether or not the origin is a stable point, an unstable point, a saddle, or otherwise.

(a) $A = \begin{bmatrix} 7 & -4 \\ 1 & 11 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & -3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

(c) $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

3. Let α and β be (real) constants and consider the planar system

$$\begin{cases} \dot{x} = \alpha x \left((x^2+y^2)^2 - \beta\right) - y \\ \dot{y} = \alpha y \left((x^2+y^2)^2 - \beta\right) + x \end{cases}$$

   (a) Linearize the system at the origin and calculate the associated eigenvalues. Is the origin stable for the linearized system?

   (b) Convert the nonlinear system to polar coordinates.

   (c) For what α and β is the origin stable for the nonlinear system?

   (d) For what α and β does the system have stable periodic orbits?

4. For α a (real) constant, consider the following system of ordinary differential equations in $\mathbb{R}^3$:

$$\begin{aligned} \dot{x} &= \alpha x - y^3 \\ \dot{y} &= x \\ \dot{z} &= \alpha z - y^2 z \end{aligned}$$

   (a) For the initial condition $(x, y, z) = (1, 2, 0)$, estimate the time of existence.

   (b) Find a symmetry of the equations and explain what it means for solutions.

   (c) Linearize the system at the origin and explain what you can conclude from the Liapunov eigenvalue theorem for the full nonlinear system.

   (d) Let $\Phi(\alpha, t) = (x(\alpha, t), y(\alpha, t), z(\alpha, t)) \in \mathbb{R}^3$ denote the solution of the given system with initial condition $(1, 2, 0)$. Find a time dependent first order linear differential equation and initial conditions that determine the derivative $\partial\Phi/\partial\alpha$, regarding the solution $(x(\alpha, t), y(\alpha, t), z(\alpha, t))$ as known.

5. Consider the system

$$\begin{cases} \ddot{x}_1 &= \frac{1}{4}(x_2 - x_1)^3 - \frac{1}{2}(x_2 - x_1) \\ \ddot{x}_2 &= \frac{-1}{4}(x_2 - x_1)^3 + \frac{1}{2}(x_2 - x_1) \end{cases}$$

   (a) Let $y = (x_1 + x_2)/2$ be the average position. Prove that $\frac{dy}{dt} = \mathit{constant}$.

   (b) Let $r = (x_2 - x_1)/2$. Find $\ddot{r}$ and transform the system to (r, y) coordinates.

   (c) Using the system derived in (b), find a conserved energy for the original system.

   (d) Are there any equilibria to the system found in part (b)? If so what are they, and what are the stability properties?