CDS 140a Final Examination

J. Marsden, December, 2008

Attempt SEVEN of the following ten questions.

Each question is worth 20 points.

The exam time limit is three hours; no aids are permitted.

Turn in the exam by 5pm on Thursday, December 11, 2008.

Print Your Name: ______________________________

The SEVEN questions to be graded: ______________________________

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1. Let V be a one dimensional potential whose derivative is given by

$$V'(x) = (x-1)(x-2)(x-3)$$

and consider the system

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -V'(x) - \nu v \end{aligned}$$

   (a) Show that solutions of this system for any initial conditions exist for all time.

   (b) For $\nu = 0$ argue that the system has both homoclinic and periodic orbits.

   (c) What does Liapunov theory or La Salle's invariance principle say about the fate of solutions with $\nu > 0$?

2. Consider the following system in $\mathbb{R}^2$:

$$\begin{aligned} \dot{x} &= x + y^2 \\ \dot{y} &= -y. \end{aligned}$$

   (a) Determine the stable and unstable subspaces of the linearization at the origin.

   (b) Calculate the explicit solution $(x(t), y(t))$ of this system.

   (c) Find an explicit expression for the stable and unstable manifolds of this system.

3. Consider a system of the form

$$\begin{aligned} \dot{x} &= f_1(x, y) \\ \dot{y} &= f_2(x, y) \end{aligned}$$

where $f_1(0,0) = f_2(0,0) = 0$ and $Df(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Consider a candidate Lyapunov function $V(x,y) = x^2 + y^2$.

   (a) Assume that $f_1(x,y)$ and $f_2(x,y)$ are polynomials in x and y of degree at most two. Write $\dot{V}$; can you find conditions on the coefficients of f_1 and f_2 so that the origin is guaranteed to be asymptotically stable? What happens if now f_1 and f_2 are of the following form:

$$f_1(x,y) = P_1(x,y) + Ax^3, \quad f_2(x,y) = P_2(x,y) + By^3?$$

   Here, P_1 and P_2 are again polynomials of degree no higher than two. Is it possible to choose A and B so that the origin is asymptotically stable?

(b) Give conditions on a, b, c for $V(x, y) = ax^2 + bxy + cy^2$ to be positive definite in a neighborhood of the origin. What are the appropriate a, b, c to prove nonlinear (asymptotic) stability of the origin for the following system:

$$\begin{aligned}\dot{x} &= -y - x^3 \\ \dot{y} &= x - y^3.\end{aligned}$$

4. Consider the following system in $\mathbb{R}^3$:

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= x - x^5 - \delta y \\ \dot{z} &= z - z^2\end{aligned}$$

where δ is a positive constant.

(a) Which initial conditions give solutions that exist for all time? (Hint: think about Hamiltonian systems.)

(b) Show that solutions existing for all time converge to a fixed point as $t \to \infty$.

(c) Calculate the fixed points of this system and find the associated stable, unstable and center subspaces of the linearization.

5. (a) Consider the system

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= x - y - x^3 - x^2 y.\end{aligned}$$

Show that if $(x(t), y(t))$ is a solution, then so is $(-x(t), -y(t))$. What does this say about the phase portrait?

(b) Discuss the sense, if any, in which the phase portrait of the equation

$$\ddot{x} = x^2 + x\dot{x}$$

is related to that of the equation

$$\ddot{x} = x^2 - x\dot{x}$$

6. (a) Show that a gradient system $\dot{\mathbf{x}} = -\nabla V(\mathbf{x})$ cannot have any periodic orbits.

(b) Show that the system

$$\begin{aligned}\dot{x} &= \sin y \\ \dot{y} &= x \cos y\end{aligned}$$

has no periodic orbits.

7. Consider the system

$$\dot{x} = xy + y^2$$
$$\dot{y} = x + xy + x^2$$

(a) What is the nature of the equilibrium point $(0,0)$?

(b) Calculate the center and unstable manifolds at $(0,0)$ up to third order.

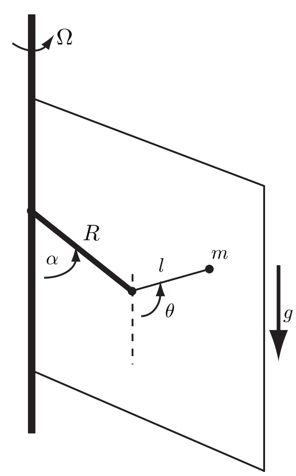
8. Consider the system

$$\dot{x} = y - y^3$$
$$\dot{y} = -x - y^2$$

(a) Show that $(-1,1)$ and $(-1,-1)$ are saddles. Show also that for the linearized system, $(0,0)$ is a center.

(b) Computer simulations show that the unstable manifold starting at $(-1,-1)$ crosses the x-axis. Assuming this to be true, use a symmetry argument to show that there is a heteroclinic orbit joining $(-1,-1)$ and $(-1,1)$.

9. A rigid bar of length R is attached to a vertical beam which rotates with constant angular velocity Ω. The angle between the bar and the rotating beam is α, measured as on the figure. To the end of the rigid bar, a pendulum of length l is attached; the mass m of the pendulum is concentrated in the end point, as indicated. The motion of the pendulum is constrained to take place in the plane spanned by the vertical beam and the rigid bar. The mass m is subject to gravity. There is no friction and the components of the pendulum are supposed to be light and rigid. For consistency, denote by θ the angle between the downward vertical and the pendulum, again as in the figure.



(a) Write down the Lagrangian for this system.

(b) Calculate the associated Euler-Lagrange equations. Find the Legendre transformation and the Hamiltonian.

(c) Show that, for small values of the angle θ, there exists an equilibrium state for the pendulum, where the angle is given by

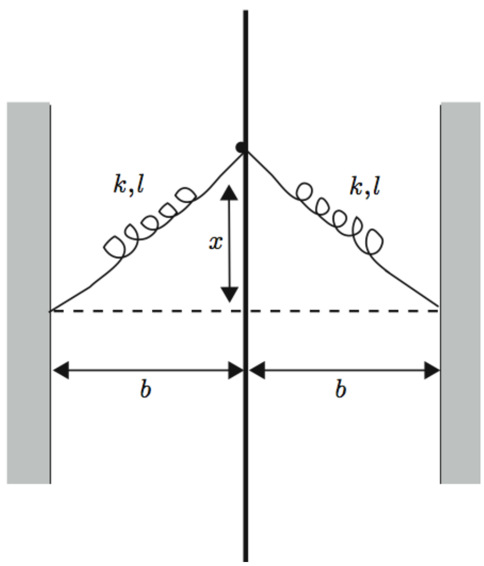
$$\theta = \frac{R\Omega^2 \sin\alpha}{g - l\Omega^2}.$$

What is the nature of this equilibrium?

10. A bead of mass m can slide along along the vertical and is attached to two walls by means of identical springs with spring constant k and natural length l. The distance between each of the walls and the vertical is given by b. There is no gravity. The distance x between the bead and the horizontal will obey the following equation of motion:

$$m\ddot{x} = -2k\left(1 - \frac{l}{\sqrt{x^2 + b^2}}\right)x.$$

You do not need to derive this equation yourself.



(a) Find the equilibrium points of this system and discuss their stability. Distinguish between the case $l < b$ and $l > b$.

(b) Given the physical realization of this system, can you argue why this should be so? That is, without doing any calculations, how do these equilibrium positions look like and what is their nature (stable/unstable)? Distinguish again between $l < b$ and $l > b$.

(c) Discuss the case $l = b$.