CALIFORNIA INSTITUTE OF TECHNOLOGY

Control and Dynamical Systems

CDS 140a Midterm Examination   Jerry Marsden   Nov. 6, 2008

This is a three hour, closed book exam

While no aids permitted, results from the course may be used, but they must be quoted.

Turn in your exam on or before 5pm, Wednesday, Nov. 12.

Print Your Name: ______________________________

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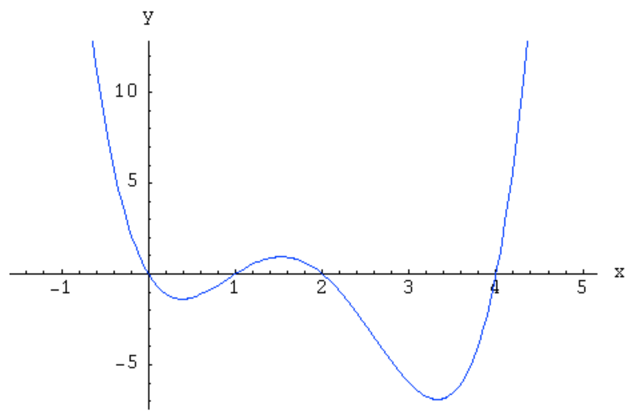
1. The graph of the function $V(x) = x(x-1)(x-2)(x-4)$ is shown in the following figure.



Consider the planar system

$$\dot{x} = y$$
$$\dot{y} = -V'(x) - \nu y$$

where $\nu \geq 0$ is a constant. In what follows, distinguish the cases $\nu > 0$ and $\nu = 0$.

(a) Show that solution curves exist for all positive time.

(b) Let $\bar{x}$ be the local maximum of V lying between 1 and 2. Compute the linearization of the system at the equilibrium point $(\bar{x}, 0)$ in terms of $V''(\bar{x})$ and determine the nature of the eigenvalues.

(c) Can you conclude anything about stability or instability of the given non-linear system from the Liapunov eigenvalue theorem at the point $(\bar{x}, 0)$?

(d) Sketch qualitatively what the phase portrait looks like.

2. Consider the linear systems $\dot{x} = Ax$ for each of the following matrices and determine the associated stable, unstable and center subspaces. Comment, to the extent you can, on the nature of the phase portrait, including whether or not the origin is a stable point, an unstable point, a saddle, or otherwise.

(a) $A = \begin{bmatrix} -3 & 4 \\ 0 & -3 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & -4 \\ 0 & 4 & -1 \end{bmatrix}$

3. Consider the system in the plane given by

$$\dot{x} = -x + x^3,$$
$$\dot{y} = x + y.$$

   (a) Find a symmetry of this system and discuss its implications for the symmetry of the phase portrait
   (b) Show that there are 3 equilibrium points of this system and find them.
   (c) Linearize the system at each of the equilibrium points.
   (d) Determine the stability or instability of each of these linearized systems.
   (e) From the linearized stability information obtained, what can you say about stability of the fixed points for the given nonlinear system?

4. Consider the following system of ordinary differential equations in $\mathbb{R}^4$:

$$\begin{aligned}\dot{x} &= r - \Omega y\\ \dot{y} &= s + \Omega x\\ \dot{r} &= x - \Omega s - 2xy^2 - 4x^3\\ \dot{s} &= y + \Omega r - 2x^2 y - 4y^3\end{aligned}$$

where points in $\mathbb{R}^4$ are denoted (x, y, r, s) and where Ω is a real constant.

   (a) Show that the time derivative of the function

$$E = \frac{1}{2}\left(r^2 + s^2 - x^2 - y^2\right) + \Omega\left(xs - yr\right) + x^2y^2 + x^4 + y^4$$

   along solutions is zero. In doing this, you might find it useful to note that the equations can be written as $\dot{x} = \partial E/\partial r$, $\dot{y} = \partial E/\partial s$, $\dot{r} = -\partial E/\partial x$, $\dot{s} = -\partial E/\partial y$.
   (b) Use (a) to argue that solutions exist for all time (both positive and negative) for arbitrary initial conditions.
   (c) Show that these solutions vary smoothly with resepect to the initial conditions and with respect to the parameter Ω.
   (d) Let $\Phi(\Omega, t) \in \mathbb{R}^4$ denote the solution of the above system with initial condition $(1, 0, 2, 0)$. Find a differential equation and initial conditions that determine the derivative $\partial\Phi/\partial\Omega$. Does your equation have a unique solution for all time?
   (e) Show that the origin is an equilibrium point and compute the linearization of the above system at the origin.
   (f) Calculate the eigenvalues of the linear system you found and discuss their dependence on the parameter Ω.
   (g) What can you conclude from the Liapunov eigenvalue theorem for the full nonlinear system?

5. Consider the planar dynamical system given by

$$\dot{x} = -x(x-1)(x-\alpha)$$
$$\dot{y} = -y + \beta x y^3$$

where α and β are real parameters.

(a) Find the equilibria in case $\beta = 0$.

(b) Still assuming that $\beta = 0$, show that solutions exist for all positive time for any initial data and any α.

(c) Still assuming that $\beta = 0$, describe the change in the phase portraits as α is varied from $1/2$ to 2.

(d) i. Using the local existence theory, estimate the time of existence of the initial condition $(x_0, y_0) = (1, 1)$ in case $\alpha = 1, \beta = 1$.

ii. Now assume that α and β are arbitrary. Argue that the lifetime of a fixed initial condition (x_0, y_0) tends to infinity as $\beta \to 0$.