CDS 140a Final

1) a) $\begin{cases} \dot{x} = y \\ \dot{y} = \alpha x - x^3 = -V'(x) \end{cases}$

with $V(x) = -\frac{1}{2}\alpha x^2 + \frac{1}{4}x^4$

$E = \frac{1}{2}y^2 + V(x)$

$\quad = \frac{1}{2}y^2 - \frac{1}{2}\alpha x^2 + \frac{1}{4}x^4$

5/5

is conserved.

✓ Indeed, $\dot{E} = y\dot{y} + V'(x)\dot{x} = -yV'(x) + V'(x)y = 0$

b) $\dot{E} = y\dot{y} + V'(x)\dot{x}$

$\quad = y(-V'(x) - \beta y - \gamma x^2 y^3) + V'(x)y$

$\quad = -\beta y^2 - \gamma x^2 y^4$

8/8 $\quad \leq 0$

$\Rightarrow$ solutions are confined to sublevel sets of E as t increases, which are compact since $E \sim \frac{1}{2}y^2 + \frac{1}{4}x^4$ for x, y large

$\Rightarrow$ solutions are bounded on bounded t-intervals

$\Rightarrow$ solutions exist for all t.

c) ✓ $\begin{cases} \tilde{x} = -x \\ \tilde{y} = -y \end{cases}$

$\dot{\tilde{x}} = -\dot{x} = -y = \tilde{y}$

$\dot{\tilde{y}} = -\dot{y} = -\alpha x + \beta y + x^3 + \gamma x^2 y^3 = \alpha\tilde{x} - \beta\tilde{y} + \tilde{x}^3 - \gamma\tilde{x}^2\tilde{y}^3$

$\Rightarrow (\tilde{x}, \tilde{y})$ is a solution if (x, y) is.

The system is not time-reversible, since if

$\begin{cases} \tilde{x}(t) = x(-t) \\ \tilde{y}(t) = -y(-t) \end{cases}$

Then

$$\dot{\tilde{x}}(t) = -\dot{x}(-t) = -y(-t) = \tilde{y}(t)$$
$$\begin{aligned}\dot{\tilde{y}}(t) = \dot{y}(-t) &= \alpha x(-t) - \beta y(-t) - x^3(-t) - \gamma x^2(-t)\, y^3(-t) \\ &= \alpha \tilde{x}(t) + \beta \tilde{y}(t) - \tilde{x}^3(t) + \gamma \tilde{x}^2(t)\, \tilde{y}^3(t) \\ &\neq \alpha \tilde{x}(t) - \beta \tilde{y}(t) - \tilde{x}^3(t) - \gamma \tilde{x}^2(t)\, \tilde{y}^3(t)\end{aligned}$$

7/7

unless $\beta = \gamma = 0$

$\therefore$ The phase portrait is symmetric under rotation by π radians but not under reflection about the y-axis with time running backward.

2) a) Eq. pts: $\dot{x} = 0 \Rightarrow y = 0$

$\dot{y} = 0 \Rightarrow x(\alpha - x^2) = 0 \Rightarrow x = 0, \pm\sqrt{\alpha}$

$\Rightarrow (0,0), (-\sqrt{\alpha}, 0), (\sqrt{\alpha}, 0)$

Linearization:

$$A = \begin{bmatrix} 0 & 1 \\ \alpha - 3x^2 - 2\gamma x y^3 & -\beta - 3\gamma x^2 y^2 \end{bmatrix}$$

$$A\big|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ \alpha & 0 \end{bmatrix}, \quad \lambda = \pm\sqrt{\alpha} \Rightarrow (0,0) \text{ is unstable (saddle)}$$

$$A\big|_{(\pm\sqrt{\alpha},0)} = \begin{bmatrix} 0 & 1 \\ \alpha - 3\alpha & -\beta \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2\alpha & -\beta \end{bmatrix}$$

$$-\lambda(-\lambda - \beta) + 2\alpha = \lambda^2 + \lambda\beta + 2\alpha = 0$$

$$\Rightarrow \lambda_\pm = \frac{-\beta \pm \sqrt{\beta^2 - 8\alpha}}{2}$$

If $\beta > 0$, $\operatorname{Re}\lambda_\pm < 0$ for any $\alpha > 0 \Rightarrow (\pm\sqrt{\alpha}, 0)$ (asymptotically) stable

If $\beta = 0$, $\operatorname{Re}\lambda_\pm = 0 \Rightarrow$ linearization is a center at $(\pm\sqrt{\alpha}, 0)$

We know, however, that $(\pm\sqrt{\alpha}, 0)$ are minima of E

(2a p. 2) since $V'(\pm\sqrt{\alpha}) = -\alpha x + x^3\big|_{x=\pm\sqrt{\alpha}} = 0$
and $V''(\pm\sqrt{\alpha}) = -\alpha + 3x^2\big|_{x=\pm\sqrt{\alpha}} = 2\alpha > 0$.
Moreover, from part 1(b),
$$\dot{E} = -\gamma x^2 y^4$$
$$\leq 0$$
✓ when $\beta = 0$,
Thus E is a Lyapunov function in a neighborhood of $(\pm\sqrt{\alpha}, 0)$
$\Rightarrow (\pm\sqrt{\alpha}, 0)$ are stable for $\beta = 0$.

10

b) Let S be any sublevel set of E containing $(0,0)$ and $(\pm\sqrt{\alpha}, 0)$. Observe that
S is positively invariant under the flow by 1(b),
$\dot{E}(x,y) \leq 0$ for all $x, y \in S$ by 1(b),
and the largest invariant set contained in S for which $\dot{E} = 0$ are the points $(0,0)$ and $(\pm\sqrt{\alpha}, 0)$, ($\dot{E} = 0$ also for $y = 0$ but solutions are transverse to the x-axis since $\dot{x} = 0$ along $y = 0$.)

10

Lastly, $E \geq 0$ in S (adding a constant if necessary), so by La-Salle's invariance principle, all solutions in S tend to $(0,0)$ or $(\pm\sqrt{\alpha}, 0)$ as $t \to \infty$.

It follows that all solutions tend to $(0,0)$ or $(\pm\sqrt{\alpha}, 0)$ as $t \to \infty$.

3) $\dot{\vec{x}} = A\vec{x}$ $\qquad A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

A upper tri $\Rightarrow \lambda = -1, -1, 4$

$\underline{\lambda = 4}$

$$\begin{bmatrix} -5 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -5 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \Rightarrow \begin{matrix} -5a+b=0 \\ c=0 \end{matrix} \Rightarrow v_1 = \begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix}$$

$\underline{\lambda = -1}$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \Rightarrow \begin{matrix} b=0 \\ 5b+c=0 \end{matrix} \Rightarrow v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Eigenspace deficiency. Generalized eigenvector:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} b=1 \\ 5b+c=0 \end{matrix} \xRightarrow{\text{Choose } a=0} v_3 = \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}$$

$$E^s = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}\right\}$$

$$E^u = \text{span}\left\{\begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix}\right\}$$

$$E^c = \emptyset$$

For initial conditions in E^s, solutions will converge to the origin as $t \to \infty$.

Due to the eigenspace deficiency of $\lambda = -1$, the JCF of A is

$$J = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

Ans 4:- $\dot{x} = (1-x)(x-4)$

$\left.\begin{aligned} \dot{y} &= z \\ \dot{z} &= 2y - y^3 \end{aligned}\right\}$ uncoupled from x and this is like mech system with m=1 where pos^n = y vel. = z.

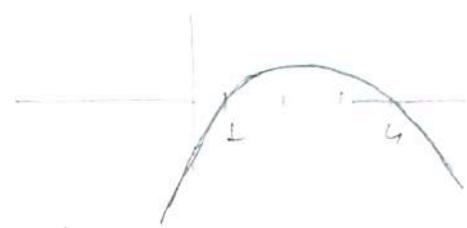
Let force be $F = -\nabla V$.

$$-\frac{\partial V}{\partial y} = 2y - y^3.$$

$$\therefore V(y) = -y^2 + \frac{y^4}{4} + \underbrace{V(0)}_{\text{arbitrary const. that does not affect eq}^n}.$$

With energy defined only in terms of y, z, we see that $\frac{dE}{dt} = 0$. Thus for this system, solutions exist as long as the solution for the independently driven $x(t)$ exists.

$\dot{x} = (1-x)(x-4)$.



For large +ve x, $\dot{x} < 0$

" large −ve x, $\dot{x} < 0$

" $x \in (1,4)$, $\dot{x} > 0$

$\cancel{\frac{A}{1-x} + \frac{B}{x-4} = \frac{1}{(1-x)(x-4)}}$

We note that $x = 1, 4$ are eq pts for this one-dim sys. The signs of $\dot{x}$ tell us that sol^{n}s exist for all +ve times for $x(0) > 1$ and it tries to converge to the pt. $x = 4$.

Yes. The solution is unique for positive time.

Reason :-

~~[illegible]~~

The vector field is polynomial and thus smooth and therefore locally Lipschitz, and ~~Moreover~~ is bounded in a compact domain. Thus we can choose a closed ball B small enough around the starting pt. to find a locally unique solution. Now this solution extends globally due to the fact that :-

1. y, z coordinates behave like a conserved energy mechanical system.
2. On x-coordinate .. $\dot{x}$ ~~[illegible]~~ such that it moves x closer to the stable pt. $(x = 4)$

Thus ~~[illegible]~~ soln exists for all +ve times and is unique by extending $F_t(x(0), y(0), z(0))$ from the locally unique soln.

5) a) If $(x(t), y(t), z(t)) = (0, -2\cos t, 2\sin t)$, Then

$$-x + (4 - y^2 - z^2)^2 = 0 + (4 - 4\cos^2 t - 4\sin^2 t)^2$$
$$= (4-4)^2$$
$$= 0$$
$$= \dot{x}$$

$$z = 2\sin t = \dot{y}$$
$$-y = 2\cos t = \dot{z}$$

and $(x(2\pi), y(2\pi), z(2\pi)) = (x(0), y(0), z(0))$

7/7 $\Rightarrow$ $(0, -2\cos t, 2\sin t)$ is a periodic orbit

b) Let F be The flow of this system and let $v_0 = (x_0, y_0, z_0)$. Then along The
7/7 periodic orbit,

$$\frac{d}{dt}\frac{\partial F}{\partial v_0} = \frac{\partial}{\partial(x,y,z)}\begin{bmatrix} -x + (4 - y^2 - z^2)^2 \\ z \\ -y \end{bmatrix} \frac{\partial F}{\partial v_0}$$

$$= \left.\begin{bmatrix} -1 & 2(4-y^2-z^2)(-2y) & 2(4-y^2-z^2)(-2z) \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}\right|_{(0,-2\cos t, 2\sin t)} \cdot \frac{\partial F}{\partial v_0}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial y_0} & \frac{\partial x}{\partial z_0} \\ \frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} & \frac{\partial y}{\partial z_0} \\ \frac{\partial z}{\partial x_0} & \frac{\partial z}{\partial y_0} & \frac{\partial z}{\partial z_0} \end{bmatrix}$$

$$\left.\frac{\partial F}{\partial v_0}\right|_{t=0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5) c) Let $r=\sqrt{y^2+z^2}$, $\theta = \arctan(z/y)$.

Then $r\dot{r} = y\dot{y}+z\dot{z} = 0 \Rightarrow \dot{r}=0$

$r^2\dot{\theta} = y\dot{z}-z\dot{y} = -r^2 \Rightarrow \dot{\theta}=-1$

$\dot{x} = -x + (4-r^2)^2$

6/6

This system can be solved explicitly:

$$\begin{cases} r(t) = r_0 \\ \theta(t) = \theta_0 - t \\ x(t) = e^{-t}x_0 + (4-r_0^2)^2(1-e^{-t}) \end{cases}$$

For any r_0, θ_0, x_0, $x(t) \to (4-r_0^2)^2$ as $t \to \infty$ and $|x(t)| \le |x_0| + (4-r_0^2)^2$ for all $t \ge 0$, which shows that any initial condition near the periodic orbit remains near it for all positive time.

$\Rightarrow$ The periodic orbit $(0, 2\cos t, 2\sin t)$ is stable.

6) a) Linearization:

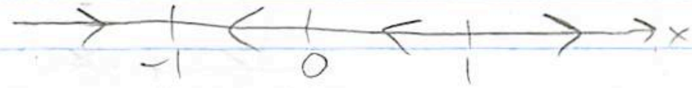
$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

A block diagonal $\Rightarrow \lambda = 0,\ 2 \pm i$

By inspection,

$$E^c = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\} \qquad E^u = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\} \qquad E^s = \emptyset$$

b) $\dot{x} = x^2(x^2-1) \Rightarrow$

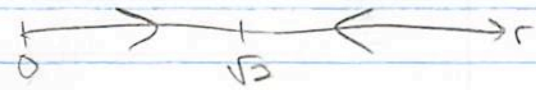


Let $r^2 = y^2 + z^2$, $\theta = \arctan(z/y)$. Then

$$\begin{aligned} r\dot{r} &= y\dot{y} + z\dot{z} \\ &= -yz + y^2(2-r^2) + zy + z^2(2-r^2) \\ &= r^2(2-r^2) \end{aligned}$$

$$\begin{aligned} r^2\dot{\theta} &= y\dot{z} - z\dot{y} \\ &= y^2 + z^2 \\ &= r^2 \end{aligned}$$

$$\Rightarrow \begin{cases} \dot{r} = r(2-r^2) \\ \dot{\theta} = 1 \end{cases}$$



By inspection, the y-z plane is invariant under the flow and tangent to E^u. However, as shown above, only points in the y-z plane of distance $< \sqrt{2}$ tend to the origin as $t \to -\infty$.

$$\therefore W^u = \left\{\begin{bmatrix} 0 \\ y \\ z \end{bmatrix} : y^2 + z^2 < 2\right\}$$

(6b p.2) <u>Remark</u> In fact, any solution with initial condition

$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \in \mathbb{R}^3, \quad y_0^2 + z_0^2 < 2, \quad -1 < x_0 < 0$$

tends to the origin as $t \to -\infty$; however, if I remember correctly, we require that the unstable manifold have dimension equal to $\dim E^u = 2$ and be tangent to E^u at the origin.

7. (a). The equilibrium points satisfy $\dot{x} = \dot{y} = \dot{z} = 0$.

$\therefore$ the equilibrium points are:

~~(0, 1, 1), (0, 1,~~

$(1, 0, 1)$, $(1, 0, 2)$, $(-1, 0, 1)$ $(-1, 0, 2)$.

The linearization of the system equations is:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 2x & -1 & 0 \\ 0 & 0 & 2z-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

At $(1, 0, 1)$ $A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ eigenvalues are $\lambda_1 = -2, \lambda_2 = 1, \lambda_3 = -1$

$\therefore$ $(1, 0, 1)$ is a saddle

At $(1, 0, 2)$ $A = \begin{bmatrix} 0 & 1 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ eigenvalues are $\lambda_1 = -2, \lambda_2 = 1$ $\lambda_3 = 1$

$(1, 0, 2)$ is a saddle.

At $(-1, 0, 1)$ $A = \begin{bmatrix} 0 & 1 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ eigenvalues are $\lambda_1 = -1+i$ $\lambda_2 = -1-i$ $\lambda_3 = -1$

$(-1, 0, 1)$ is a sink.

At $(-1, 0, 2)$ $A = \begin{bmatrix} 0 & 1 & 0 \\ -2 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ eigenvalues are $\lambda_1 = -1+i, \lambda_2 = -1-i, \lambda_3 = 1$

$(-1, 0, 2)$ is a saddle.

(b). Since the system equations are decoupled, we can see $\{x = 1, y = 0, z \in \mathbb{R}\}$ is an invariant manifold passing through $(1, 0, 1)$, it's stable.

On the x, y-plane through $(1, 0, 1)$, there is a one-dimensional stable manifold and a one-dimensional unstable manifold passing through $(1, 0, 1)$.

8) a) Linearization:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

7/7 A lower tri $\Rightarrow \lambda = 1, -1, -1$

Since A is in JCF, we can read off the invariant subspaces:

$$E^u = \text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\} \quad E^s = \text{span}\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\} \quad E^c = \emptyset$$

5/5 b) $\dim W^u = \dim E^u = 1$

$\dim W^s = \dim E^s = 2$

$\dim W^c = \dim E^c = 0$

c) Assume W^u may be parametrized as

$$\begin{cases} y = g(x) = ax^2 + bx^3 + \dots \\ z = h(x) = cx^2 + dx^3 + \dots \end{cases}$$

$$\dot{z} = h'(x)\dot{x} \Rightarrow y - z = (2cx + 3dx^2 + \dots)(x + x^2 - x(ax^2 + bx^3 + \dots)^2)$$

$$\Rightarrow (a-c)x^2 + (b-d)x^3 + \dots = 2cx^2 + \dots$$

$$\Rightarrow a - c = 2c$$

$$a = 3c$$

$$\dot{y} = g'(x)\dot{x} \Rightarrow -y - x^2 + x^2 y = (2ax + 3bx^2 + \dots)(x + x^2 - x(ax^2 + \dots)^2)$$

$$\Rightarrow -ax^2 - bx^3 - x^2 + x^2(ax^2 + \dots) = 2ax^2 + \dots$$

$$\Rightarrow -a - 1 = 2a$$

$$\Rightarrow a = -1/3$$

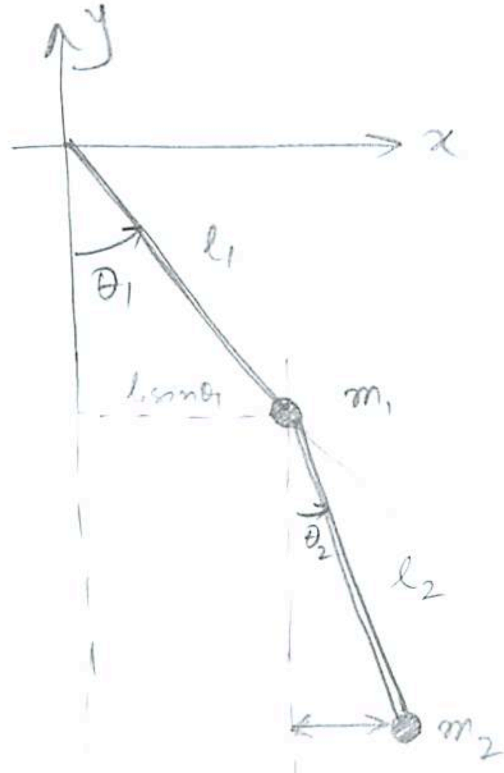
$$c = -1/9$$ 8/8

(8c p.2) $\therefore W^u$ is given approximately by the graph of

$$y = -\frac{1}{3}x^2$$
$$z = -\frac{1}{9}x^2$$ ✓

8/8

Ans 9:-



We invert θ_2 to maintain that anti-clockwise is +ve.

$$x_{m_1} = l_1 \sin\theta_1 \quad , \quad y_{m_1} = -l_1 \cos\theta_1 .$$

$$x_{m_2} = l_1 \sin\theta_1 + l_2 \sin\theta_2 \, , \quad y_{m_2}^{*} = -l_1 \cos\theta_1 - l_2 \cos\theta_2 .$$

$$\therefore \; \dot{x}_{m_1} = l_1 \cos\theta_1 \, \dot{\theta}_1 \, ; \quad \dot{y}_{m_1} = l_1 \sin\theta_1 \, \dot{\theta}_1$$

$$\dot{x}_{m_2} = l_1 \cos\theta_1 \, \dot{\theta}_1 + l_2 \cos\theta_2 \, \dot{\theta}_2 \, ;$$

$$\dot{y}_{m_2} = l_1 \sin\theta_1 \, \dot{\theta}_1 + l_2 \sin\theta_2 \, \dot{\theta}_2$$

$$v_{m_1}^2 = \left(l_1 \cos\theta_1 \, \dot{\theta}_1\right)^2 + \left(l_1 \sin\theta_1 \, \dot{\theta}_1\right)^2$$

$$= l_1^2 \dot{\theta}_1^2$$

$$v_{m_2}^2 = l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \left[\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2\right]$$

$$= l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) .$$

$$KE = \frac{1}{2} m_1 v_{m_1}^2 + \frac{1}{2} m_2 v_{m_2}^2$$

$$PE = -m_1 g l_1 \cos\theta_1 - m_2 g \left(l_1 \cos\theta_1 + l_2 \cos\theta_2 \right)$$

$$\therefore \mathcal{L}(\theta_1, \dot{\theta}_1, \theta_2, \dot{\theta}_2)$$

$$= \frac{1}{2} m_1 v_{m_1}^2 + \frac{1}{2} m_2 v_{m_2}^2 + (m_1 + m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2$$

$$= \frac{m_1}{2} l_1^2 \dot{\theta}_1^2 + \frac{m_2}{2} \left[l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right]$$

$$+ (m_1 + m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2$$

---

b. $$\frac{\partial L}{\partial \theta_1} = \frac{m_2}{\cancel{2}} \cdot \cancel{2} l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \left[-\sin(\theta_1 - \theta_2) \right]$$

$$- (m_1 + m_2) g l_1 \sin\theta_1 .$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 + m_2 l_1^2 \dot{\theta}_1 + \frac{m_2}{\cancel{2}} \cancel{2} l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2).$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_1} = \frac{\partial L}{\partial \theta_1}$$

$$m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \Big[\ddot{\theta}_2 \cos(\theta_1 - \theta_2) - \sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2)(\dot{\theta}_2) \Big]$$

$$= m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \big[-\sin(\theta_1 - \theta_2) \big] - (m_1 + m_2) g l_1 \sin\theta_1$$

or,

$$m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) \cancel{- m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2)} + m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)$$

$$= \cancel{- m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2)} - (m_1 + m_2) g l_1 \sin\theta_1$$

or,

$$\ddot{\theta}_1 \left(m_1 l_1^2 + m_2 l_1^2 \right) + m_2 l_1 l_2 \Big[\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) \Big] = -(m_1 + m_2) g l_1 \sin\theta_1 . \qquad (1)$$

$$\frac{\partial L}{\partial \theta_2} = \frac{m_2}{\cancel{2}} \cdot \cancel{2}\, l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \left[+ \sin(\theta_1 - \theta_2) \right]$$

$$- m_2 g l_2 \sin\theta_2$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + \frac{m_2}{\cancel{2}} \cdot \cancel{2}\, l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2)$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \Big[\ddot{\theta}_1 \cos(\theta_1 - \theta_2)$$

$$+ \dot{\theta}_1 \left\{ -\sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2) \right\} \Big]$$

$$\therefore \frac{\partial L}{\partial \theta_2} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}_2}$$

$$\Rightarrow \cancel{m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2)} - m_2 g l_2 \sin\theta_2$$

$$= m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2)$$

$$- m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2)$$

$$+ \cancel{m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2)}$$

$$\Rightarrow m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2)$$

$$- m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) = -m_2 g l_2 \sin\theta_2 .$$

—— (2)

c. For θ_1 and θ_2 small around $(0,0)$.

$$\cos(\theta_1-\theta_2) \approx 1$$
$$\sin(\theta_1-\theta_2) \approx \theta_1-\theta_2$$
$$\sin\theta_1 \approx \theta_1$$
$$\sin\theta_2 \approx \theta_2 .$$

Using these we get, (neglecting higher order simpl.)

$$(m_1 l_1^2 + m_2 l_1^2)\,\ddot{\theta}_1 + m_2 l_1 l_2 \left[\ddot{\theta}_2\right] = -(m_1+m_2) g l_1 \theta_1$$

and $m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 = -m_2 g l_2 \theta_2$

$$\therefore\ l_1 \ddot{\theta}_1 + \frac{m_2}{m_1+m_2} l_2 \ddot{\theta}_2 = -g\theta_1$$

$$l_2 \ddot{\theta}_2 + l_1 \ddot{\theta}_1 = -g\theta_2$$

Let $\frac{m_2}{m_1+m_2} = \alpha$ and $\bar{\alpha} = 1-\alpha$.

$$\therefore\ \alpha l_2 \ddot{\theta}_2 - l_2 \ddot{\theta}_2 = g\theta_2 - g\theta_1$$

$$\therefore\ \ddot{\theta}_2 = \frac{g}{l_2(\alpha-1)}(\theta_2-\theta_1) = \frac{g}{l_2(1-\alpha)}(\theta_1-\theta_2).$$

$$\therefore\ l_1\ddot{\theta}_1 = -g\theta_2 - l_2\ddot{\theta}_2$$
$$= -g\theta_2 - \frac{g}{1-\alpha}(\theta_1-\theta_2)$$
$$= -g\left[\theta_2 + \frac{\theta_1}{1-\alpha} - \frac{\theta_2}{1-\alpha}\right] = -g\left[\frac{-\alpha\theta_2+\theta_1}{1-\alpha}\right]$$

$$\therefore \ddot{\theta}_1 = \frac{-g}{l_1(1-\alpha)} \left[\theta_1 - \alpha\theta_2 \right]$$

Let $\dfrac{g}{l_1(1-\alpha)} = \beta$.

$$\therefore \frac{g}{l_2(1-\alpha)} = \frac{\beta l_1}{l_2}.$$

$$\therefore \ddot{\theta}_1 = -\beta(\theta_1 - \alpha\theta_2)$$

$$\ddot{\theta}_2 = -\frac{\beta l_1}{l_2}\left(\frac{\theta_2}{\alpha} - \theta_1\right)$$

Let $\dot{\theta}_1 = a$, $\dot{\theta}_2 = b$

$$\therefore \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{a} \\ \dot{b} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\beta & +\alpha\beta & 0 & 0 \\ \frac{\beta l_1}{l_2} & -\frac{\beta l_1}{l_2} & 0 & 0 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ a \\ b \end{pmatrix}.$$

We might be interested in finding out the eigenvalues of this but it is too much of a trouble. We give physical intuition instead of using Lyapunov spectral thm.

~~Linearized so~~

Linearized sol^n is stable. For small perturbations the torque of gravity brings the system towards the eq pt. but it oscillates around the pt. due to finite energy. ✓

Note that KE + PE = E is conserved since there are no dissipative ~~the~~ forces.

$$\therefore \frac{dE}{dt} = 0$$

Now note the terms in E

$$E = \frac{1}{2} m_1 v_{m_1}^2 + \frac{1}{2} m_2 v_{m_2}^2 - m_1 g l_1 \cos\theta_1$$

$$- m_2 g \left[l_1 \cos\theta_1 + l_2 \cos\theta_2 \right].$$

Note that ~~near eq~~ at $v_{m_1} = v_{m_2} = 0$,

E varies as $\underbrace{-\cos\theta_1}_{\text{min at } \theta_1 = 0}$ and $\underbrace{-\cos\theta_2}_{\text{min at } \theta_2 = 0.}$

absolute ✓

$\therefore$ E has a minimum at $\theta_1 = \theta_2 = v_{m_1} = v_{m_2} = 0$

Note:- $\frac{dE}{dt} = 0$ with E having strict minimum at this eq. pt.

implies stability from Lyapunov stability theorem using $E + [m_1 g l_1 + m_2 g(l_1 + l_2)]$ as the Lyapunov function. [Also Dirichlet-Lagrange stability thm predicts the same result.]

— Not asymptotically stable since $\frac{dE}{dt} = 0$ means it remains on iso-energy curve and by continuity of E ~~near the eq pt. implies that~~ we have that it is not asymp. stable.

d. At $\theta_1 = \theta_2 = \pi$ (note our sign change does not affect this.)

and $v_{m_1} = v_{m_2} = 0$,

we see it has a saddle.

Along θ_1, θ_2 it has maximum and along $\dot{\theta}_1$ and $\dot{\theta}_2$ it has minimum. Lyapunov ~~f~~ instability thm cannot be applied but linearizing will ~~will~~ yield the result. Physically when we shift a little from the top, a torque due to gravity pulls the system down and is thus unstable. [FREAKINGLY LONG QUESTION!]

Not asymptotically stable. ~~with $\dot{\theta}_1, \dot{\theta}_2$ ...~~

(10) a)

$V = \frac{1}{2}x^2 - mgl\cos\theta$

7/7

$\begin{cases} \xi = x + l\sin\theta \\ \eta = -l\cos\theta \end{cases}$ $\qquad \begin{cases} \dot\xi = \dot x + l\dot\theta\cos\theta \\ \dot\eta = l\dot\theta\sin\theta \end{cases}$

$$L = \frac{1}{2}M\dot x^2 + \frac{1}{2}m(\dot\xi^2 + \dot\eta^2) - V$$
$$= \frac{1}{2}M\dot x^2 + \frac{1}{2}m(\dot x^2 + 2l\dot\theta\dot x\cos\theta + l^2\dot\theta^2) - \frac{1}{2}x^2 + mgl\cos\theta$$

b) $\frac{d}{dt}\frac{\partial L}{\partial \dot x} - \frac{\partial L}{\partial x} = \frac{d}{dt}(M\dot x + m\dot x + ml\dot\theta\cos\theta) + x$

$$= (M+m)\ddot x + ml\ddot\theta\cos\theta - ml\dot\theta^2\sin\theta + x$$
$$= 0$$

7/7

$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} - \frac{\partial L}{\partial \theta} = \frac{d}{dt}(ml\dot x\cos\theta + ml^2\dot\theta) - (-ml\dot\theta\dot x\sin\theta - mgl\sin\theta)$

$$= ml\ddot x\cos\theta - ml\dot x\dot\theta\sin\theta + ml^2\ddot\theta + ml\dot\theta\dot x\sin\theta + mgl\sin\theta$$
$$= ml\ddot x\cos\theta + ml^2\ddot\theta + mgl\sin\theta$$
$$= 0$$

$\begin{cases} p_x = \frac{\partial L}{\partial \dot x} = (M+m)\dot x + ml\dot\theta\cos\theta \\ p_\theta = \frac{\partial L}{\partial \dot\theta} = ml\dot x\cos\theta + ml^2\dot\theta \end{cases} \Rightarrow \begin{bmatrix} p_x \\ p_\theta \end{bmatrix} = \begin{bmatrix} M+m & ml\cos\theta \\ ml\cos\theta & ml^2 \end{bmatrix}\begin{bmatrix} \dot x \\ \dot\theta \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \dot x \\ \dot\theta \end{bmatrix} = \frac{1}{d}\begin{bmatrix} ml^2 & -ml\cos\theta \\ -ml\cos\theta & M+m \end{bmatrix}\begin{bmatrix} p_x \\ p_\theta \end{bmatrix}$ $\qquad d = ml^2(M+m) - m^2l^2\cos^2\theta$
$\qquad = mMl^2 + m^2l^2\sin^2\theta$

$H = p_x\dot x + p_\theta\dot\theta - L$
$= p_x\frac{1}{d}(ml^2p_x - ml\cos\theta p_\theta) + p_\theta\frac{1}{d}(-ml\cos\theta p_x + (M+m)p_\theta) - L$

(10b p.2) $\cancel{= \frac{1}{d}\left(ml^2 p_x^2 + (M+m) p_\theta^2\right) - \frac{1}{2} M}$

$$H = \dot{x}\cdot\frac{\partial L}{\partial \dot{x}} + \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L$$

$$= \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + ml\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}x^2 - mgl\cos\theta$$

$$= \frac{1}{2}(M+m)\frac{1}{d^2}\left(ml^2 p_x - ml\cos\theta\, p_\theta\right)^2$$

$$+ \frac{1}{2}ml^2\frac{1}{d^2}\left(-ml\cos\theta\, p_x + (M+m)p_\theta\right)^2$$

$$+ ml\frac{1}{d^2}\left(ml^2 p_x - ml\cos\theta\, p_\theta\right)\left(-ml\cos\theta\, p_x + (M+m)p_\theta\right)$$

$$+ \frac{1}{2}x^2 - mgl\cos\theta$$

Hamilton's eqns are then

$$\begin{cases} \dot{x} = \frac{\partial H}{\partial p_x} \\ \dot{\theta} = \frac{\partial H}{\partial p_\theta} \\ \dot{p}_x = -\frac{\partial H}{\partial x} \\ \dot{p}_\theta = -\frac{\partial H}{\partial \theta} \end{cases}$$

10) c) For small x, θ, $\begin{cases} (M+m)\ddot{x} + ml\ddot{\theta} - x = 0 \\ ml\ddot{x} + ml^2\ddot{\theta} + mgl\theta = 0 \end{cases}$

6/6

$$\cancel{\Rightarrow \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \frac{1}{d}\begin{bmatrix} ml^2 & -ml \\ -ml & M+m \end{bmatrix}\begin{bmatrix} x \\ -mgl\theta \end{bmatrix}}$$

Upon writing this in 1st order form, one will find that the linearization has all eigenvalues on the imag axis $\Rightarrow$ linearization is stable

((0 < p. 2)) but not asymptotically stable.

The eq. is nonlinearly stable. To show this, use the Lyapunov fctn E $(=H)$. E is p.d. in a neighborhood of the origin and $\dot{E}=0$, as needed.

The nonlinear system is not asy. stable since solutions lie on the level sets of E, which are homeomorphic to ellipsoids near the origin.

If friction is added, the origin becomes asy. stable. To prove this, let S = a level set of E containing the origin, Ω = largest invariant set $\subseteq S$ for which $\dot{E}=0$.
Then S is positively invariant under the flow, $\dot{E} \leq 0$, and Ω = the origin (provided S is sufficiently small so that no other eq. are contained in S)
$\therefore$ La Salle $\Rightarrow$ All solutions in S tend to origin as $t \to \infty$.