CDS 140a Final Examination

J. Marsden, December, 2009
The Exam is due by noon on Friday, December 11, 2009

Attempt SEVEN of the following ten questions.

Each question is worth 20 points.

The exam time limit is three hours; no aids of any kind
(including the internet, computers, books, calculators).

Print Your Name:

The SEVEN questions to be graded:

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2
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10
/140

1. Consider the system

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= \alpha x - \beta y - x^3 - \gamma x^2 y^3\end{aligned}$$

where $\alpha > 0$, $\beta \geq 0$ and $\gamma \geq 0$ are constants.

   (a) If $\beta = 0$ and $\gamma = 0$ compute a conserved energy $E(x, y)$ for the system.

   (b) Compute the time derivative of the energy $E(x, y)$ for any α, β, γ. Do solutions exist for all time?

   (c) Show that for any α, β, γ, if $(x(t), y(t))$ is a solution, so is $(-x(t), -y(t))$. Is the system time-reversible? What does this say about the phase portrait?

2. (Continuation of Problem 1).

   (a) Find all the fixed points of the system and determine their stability in terms of α, β and γ.

   (b) Using $E(x, y)$ from Problem 1 as a Liapunov function, and assuming that $\beta > 0$, what does the invariance principle say about the fate of solutions?

3. For what initial conditions in $\mathbb{R}^3$ does the solution of the system

$$\begin{aligned}\frac{dx}{dt} &= -x + y \\ \frac{dy}{dt} &= 4y + z \\ \frac{dz}{dt} &= -z\end{aligned}$$

converge to the origin as $t \to \infty$? If one writes the system as $\dot{\mathbf{x}} = A\mathbf{x}$, where $\mathbf{x}$ is the vector with components (x, y, z), what is the Jordan canonical form of A?

4. Consider the following initial value problem:

$$\begin{aligned}\dot{x} &= (1 - x)(x - 4) \\ \dot{y} &= z \\ \dot{z} &= 2y - y^3,\end{aligned}$$

where $x(0) = 3, y(0) = 4, z(0) = -2$ For what positive times does the solution exist? Is the solution unique?

5. Consider the following system:

$$\begin{aligned}\dot{x} &= -x + \left(4 - y^2 - z^2\right)^2 \\ \dot{y} &= z \\ \dot{z} &= -y\end{aligned}$$

   (a) Show that $(0, -2\cos t, 2\sin t)$ is a periodic orbit of this system.
   (b) Write down the first variation equation; that is, the equation satisfied by the derivative of the flow map, starting at the initial condition $(0, 2, 0)$.
   (c) Discuss informally: is the periodic orbit given by $(0, 2\cos t, 2\sin t)$ stable?

6. Consider the system

$$\begin{aligned}\frac{dx}{dt} &= -x^2 + x^4 \\ \frac{dy}{dt} &= -z + y(2 - y^2 - z^2) \\ \frac{dz}{dt} &= y + z(2 - y^2 - z^2)\end{aligned}$$

   (a) Determine the stable, unstable, and center subspaces of the linearization at the origin
   (b) Determine the *global* unstable manifold of the origin for this system

7. Consider the system

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= x^2 - 1 - y \\ \dot{z} &= (z - 1)(z - 2)\end{aligned}$$

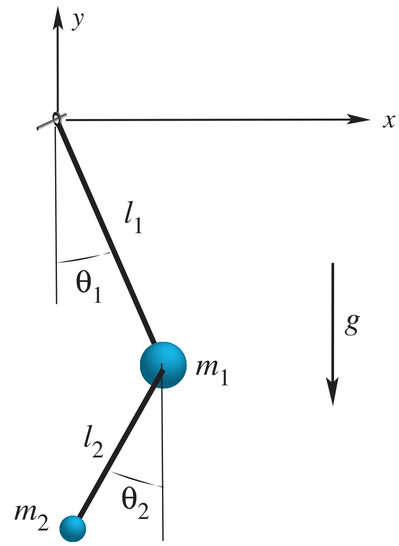
   (a) Find and classify its equilibria
   (b) Find the invariant manifold through the point $(1, 0, 1)$.

8. Consider the system

$$\begin{aligned}\dot{x} &= x + x^2 - xy^2 \\ \dot{y} &= -y - x^2 + x^2 y \\ \dot{z} &= y - z\end{aligned}$$

   (a) Calculate the linearization at the origin and its invariant subspaces.
   (b) What are the dimensions of the invariant manifolds through the origin?
   (c) Calculate a quadratic approximation to the *unstable* manifold of the origin.

9. Consider a *planar* double pendulum with notation as shown in the figure. Assume that the pendulum swings in the vertical xy-plane, all the pendulum mass is concentrated in the points as indicated in the figure, and that the pendulum arms are (light and) rigid; ignore frictional forces.



(a) Find the Lagrangian for the system.

(b) Compute the Euler-Lagrange equations.

(c) Consider the downward equilibrium solution ($\theta_1 = \theta_2 = 0$). Is the linearized solution stable? Is the equilibrium nonlinearly (Liapunov) stable? Asymptotically stable? You may use physical reasoning, but state what theorems would be used to justify your conclusions.

(d) What about the upward equilibrium solution (with $\theta_1 = \theta_2 = \pi$)?

10. Consider a pendulum hanging down from a cart that is free to move on the x-axis, as in the figure. Treat the cart as a point mass moving on the x-axis in a potential $V(x) = \frac{1}{2}x^2$, and the pendulum can swing from side to side.

(a) Find the Lagrangian for the system.

(b) Compute the Euler-Lagrange equations and transform them to Hamiltonian form.

(c) Consider the downward equilibrium solution ($\theta = 0$, cart at $x = 0$ and zero velocity). Is the linearized solution stable? Is the equilibrium nonlinearly (Liapunov) stable? Asymptotically stable? What if friction proportional to the velocity is added to the cart? You may use physical reasoning, but state what theorems would be used to justify your conclusions.