1.6 Mechanical Systems

Mechanics provides an excellent class of systems for both motivating the ideas of dynamical systems and to which the ideas of dynamical systems apply.

We saw some simple examples of Newtonian mechanical systems in the introductory section. To illustrate and motivate the introduction of additional structures governing mechanical systems, we will develop two examples as we go along.

Example 1. Consider a particle moving in Euclidean 3 space, $\mathbb{R}^3$, subject to potential forces. As a variant of this example, also consider an object whose mass is time-varying and is subject to external forces (e.g. a rocket burning fuel and generating propulsion).

Example 2. Consider a circular hoop, rotating along the $z-$axis with a certain fixed angular velocity ω, as shown in Figure 1.6.1. Consider a particle with mass m moving in this hoop.

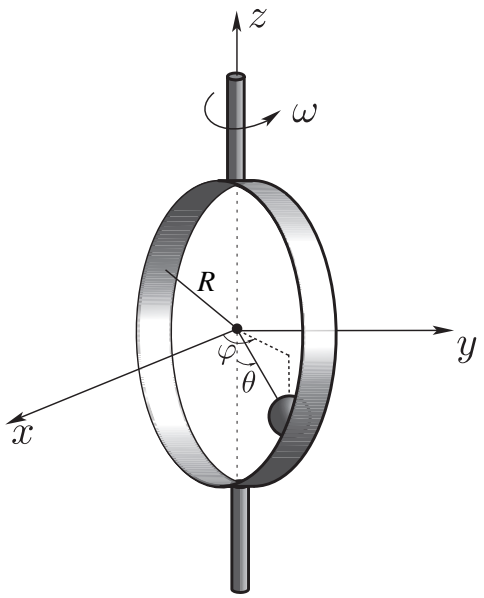

FIGURE 1.6.1. A particle moving in a rotating hoop.

Configuration Space Q. The configuration space of a mechanical system is a space whose points determine the spatial positions of the system. Although this space is generally parametrized by generalized coordinates, denoted $(q^1, ..., q^n)$, the space Q itself need not be an Euclidean space. It can be rather thought as a configuration *manifold*. Since we will be working

with coordinates for this space, it is ok to think of Q as Euclidean space for purposes of this section. However, the distinction turns out to be an important general issue.

Example 1. Here $Q = \mathbb{R}^3$ since a point in space determines where our system is; the coordinates are simply standard Euclidean coordinates: $(x, y, z) = (q^1, q^2, q^3)$.

Example 2. Here $Q = S^1$, the circle of radius R since the position of the particle is completely determined by where it is in the hoop. Note that the hoop's position in space is already determined as it has a *prescribed* angular velocity.

The Lagrangian $L(q, v)$. The Lagrangian is a function of $2n$ variables, if n is the dimension of the configuration space. These variables are the positions and velocities of the mechanical system. We write this as follows

$$L(q^1, ..., q^n, v^1, ..., v^n)) = L(q^1, ..., q^n, \dot{q}^1, ..., \dot{q}^n). \tag{1.6.1}$$

At this stage, the $\dot{q}^i$s are not time derivatives yet (since L is just a function of $2n$ variables, but as soon as we introduce time dependence so that the q^is are functions of time, then we will require that the v^is to be the time derivatives of the q^is.

In many (but not all) of our examples we will set $L = K_E - P_E$, i.e. as the difference between the kinetic and potential energies.

The sign in front of the potential energy in this definition of L is very important. For instance, consider a particle with constant mass, moving in a potential field V, which generates a force $F = -\nabla V$. We will see shortly that the equation $F = ma$ is a particular case of the basic equations associated to L, namely the Euler Lagrange equations, therefore necessarily we need the minus sign before the P_E. There are other deeper reasons why the minus sign that involve relativistic invariance.[4]

Recall also that the ***kinetic energy*** of a particle with mass m moving in $\mathbb{R}^3$ is given by $K_E = \frac{1}{2}m\|v\|^2$. This definition comes about because the kinetic energy is related to the work done by the force in a simple way as follows: if the particle governed by $F = ma$, then

$$\frac{d}{dt}\frac{1}{2}m\|v\|^2 = mv \cdot F$$

and so the fundamental theorem of calculus shows that change in its kinetic energy from point a to point b is given by:

$$\Delta K_E = \int_a^b F(t) \cdot v \, dt,$$

[4]For instance, the wave equation $\phi_{tt} - \phi_{xx} = 0$ has a Lagrangian also given by the kinetic minus the potential energy: $L(\phi, \phi_t) = \frac{1}{2}\int (\phi_t^2 - \phi_x^2)\, dx$. Note that this expression is invariant under the Poincaré group (the basic group of special relativity), but with the plus sign (giving the total energy) it would not be invariant.

that is, the line integral of F along the path taken by the particle, or the *work* done by F along the path of the particle. In particular, if the force is given by $F = -\nabla V$ for a potential V, then

$$\Delta K_E = \int_a^b F(s) \cdot ds = V(a) - V(b),$$

or

$$\Delta \left(K_E + P_E \right) = 0,$$

which gives conservation of energy. This will be also verified below.

Example 1. From the above discussion, we see that in Example 1, we should have

$$L(q, v) = \frac{1}{2} m \|v\|^2 - V(q)$$

Example 2. With reference to Figure 1.6.2, in an inertial (or laboratory) frame, the velocity is:

$$\mathbf{v}(t) = R\dot{\theta}\mathbf{e}_\theta + \omega R \sin\theta \mathbf{e}_\phi.$$

Where the vectors $\mathbf{e}_\theta$, $\mathbf{e}_\phi$ and $\mathbf{e}_R$ are an orthonormal basis of vectors in $\mathbb{R}^3$, associated with spherical coordinates. Obviously, since the mass is moving inside the hoop, its component along $\mathbf{e}_R$ is zero. The component $R\dot{\theta}\mathbf{e}_\theta$ is the velocity along $\mathbf{e}_\theta$ since the distance of the mass m from the straight down position along the circle is given by $R\theta$.

The kinetic energy of this system is

$$K_E = \frac{1}{2} m \|\mathbf{v}\|^2 = \frac{1}{2} m (R^2 \dot{\theta}^2 + \omega^2 R^2 \sin^2\theta),$$

while the potential energy is given by $P_E = mgh = -mgR\cos\theta$ (with the zero of potential energy taken to be the plane $z = 0$ and with θ measured, as above, from the downward position of the mass). Therefore the Lagrangian for this system is

$$L(\theta, \dot{\theta}) = \frac{1}{2} m (R^2 \dot{\theta}^2 + \omega^2 R^2 \sin^2\theta) + mgR\cos\theta.$$

The Euler-Lagrange Equations. The first step in the description of a Lagrangian system was giving the configuration space and the second was giving the Lagrangian. Now we come to the third step, which is writing down the *Euler-Lagrange equations*:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0 \tag{1.6.2}$$

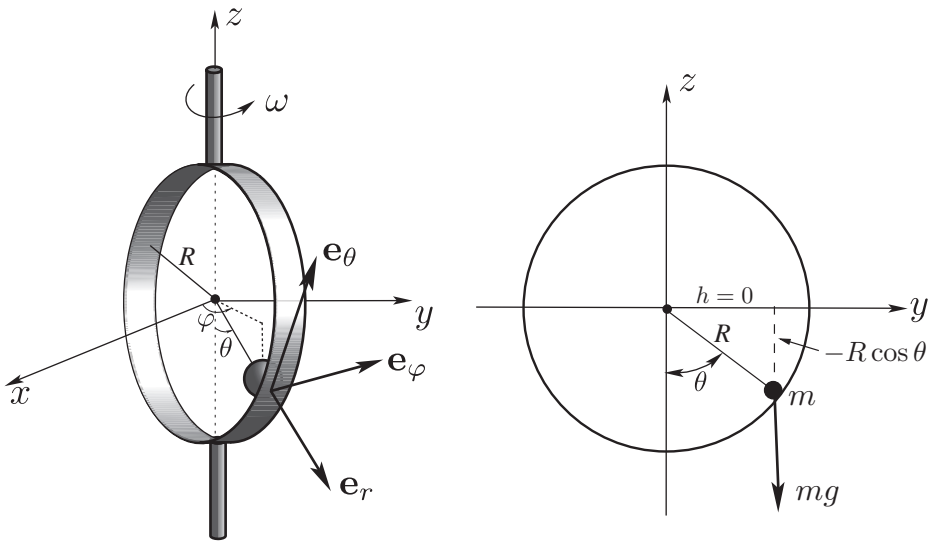

FIGURE 1.6.2. A particle moving in a rotating hoop with the attached orthonormal frame and a side view.

Historical Note. This equation was introduced by Lagrange (25 Jan 1736, Turin Italy—10 April 1813, Paris, France), and it corresponds to Newton's Second Law $F = ma$. Lagrange writes this equation in generalized coordinates $(q^i, \dot{q}^i)$, the point being that a change of coordinates does not alter the form of the Euler–Lagrange equations, whereas changing coordinates in Newton's law is tricky as one has to transform accelerations. To this end, the definition of $L = K_E - P_E$ is crucial, and makes the Euler–Lagrange equations independent of the chosen coordinate system). It is interesting also to note how Lagrange originally named what we call L today as $H = K_E - P_E$, after the dutch scientist Huygens, famous and admired at that time for his works on geometric optics.

Let us go back to our examples:

Example 1. Here the Euler–Lagrange equations become

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = \frac{d}{dt}(mv) + \nabla V = 0;$$

that is,

$$\frac{d}{dt}(mv) = -\nabla V,$$

which is the same as $F = ma$.

Example 2. Here we first compute

$$\frac{\partial L}{\partial \dot{\theta}} = mR^2\dot{\theta}, \quad \frac{\partial L}{\partial \theta} = mR^2\omega^2 \sin\theta\cos\theta - mgR\sin\theta,$$

and so the Euler–Lagrange equations become

$$\frac{d}{dt}(mR^2\dot{\theta}) - (mR^2\omega^2 \sin\theta\cos\theta - mgR\sin\theta) = 0. \tag{1.6.3}$$

Derivation from $F = ma$ (for those who might have doubts). We will now write the equations of the motion of this mass in Euclidean coordinates, and use Newton's second law to derive these equations (1.6.3). A key point is to realize that the hoop will exert forces of constraint on the particle. It is also important to transform between the accelerations written in Euclidean accelerations to spherical coordinates (not a pleasant, but a straightforward task). We will show that this procedure results in the same equations of motion.

The position of the particle in space is specified by the angles θ and φ, as shown in Figure 1.6.2. We can take $\varphi = \omega t$, so the position of the particle becomes determined by θ alone. Let the orthonormal frame along the coordinate directions $\mathbf{e}_\theta$, $\mathbf{e}_\varphi$, and $\mathbf{e}_r$ be as shown.

The forces acting on the particle are:

1. Friction, proportional to the velocity of the particle relative to the hoop: $-\nu R\dot{\theta}\mathbf{e}_\theta$, where $\nu \geq 0$ is a constant.[5]
2. Gravity: $-mg\mathbf{k}$.
3. Constraint forces in the directions $\mathbf{e}_r$ and $\mathbf{e}_\varphi$ to keep the particle in the hoop.

The equations of motion are derived from Newton's second law $\mathbf{F} = m\mathbf{a}$. To get them, we need to calculate the acceleration $\mathbf{a}$; here $\mathbf{a}$ means the acceleration relative to the *fixed inertial frame* xyz in space; it does not mean $\ddot{\theta}$. Relative to this xyz coordinate system, we have

$$\begin{aligned} x &= R\sin\theta\cos\varphi, \\ y &= R\sin\theta\sin\varphi, \\ z &= -R\cos\theta. \end{aligned} \tag{1.6.4}$$

Calculating the second derivatives using $\varphi = \omega t$ and the chain rule gives

$$\begin{aligned} \ddot{x} &= -\omega^2 x - \dot{\theta}^2 x + (R\cos\theta\cos\varphi)\ddot{\theta} - 2R\omega\dot{\theta}\cos\theta\sin\varphi, \\ \ddot{y} &= -\omega^2 y - \dot{\theta}^2 y + (R\cos\theta\sin\varphi)\ddot{\theta} + 2R\omega\dot{\theta}\cos\theta\cos\varphi, \\ \ddot{z} &= -z\dot{\theta}^2 + (R\sin\theta)\ddot{\theta}. \end{aligned} \tag{1.6.5}$$

[5] This is a "law of friction" that is more like a viscous fluid friction than a sliding friction in which ν is the ratio of the tangential force to the normal force; in any actual experimental setup (e.g., involving rolling spheres) a realistic modeling of the friction is not a trivial task; see, for example, **?**.

If $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, denote unit vectors along the x, y, and z axes, respectively, we have the easily verified relation

$$\mathbf{e}_\theta = (\cos\theta\cos\varphi)\mathbf{i} + (\cos\theta\sin\varphi)\mathbf{j} + \sin\theta\mathbf{k}. \tag{1.6.6}$$

Now consider the vector equation $\mathbf{F} = m\mathbf{a}$, where $\mathbf{F}$ is the sum of the three forces described earlier and

$$\mathbf{a} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k}. \tag{1.6.7}$$

The $\mathbf{e}_\varphi$ and $\mathbf{e}_r$ components of $\mathbf{F} = m\mathbf{a}$ tell us only what the constraint forces must be; the equation of motion comes from the $\mathbf{e}_\theta$ component:

$$\mathbf{F}\cdot\mathbf{e}_\theta = m\mathbf{a}\cdot\mathbf{e}_\theta. \tag{1.6.8}$$

Using (1.6.6), the left side of (1.6.8) is

$$\mathbf{F}\cdot\mathbf{e}_\theta = -\nu R\dot{\theta} - mg\sin\theta, \tag{1.6.9}$$

while from (1.6.5), (1.6.6), and (1.6.7), the right side of (1.6.8) is

$$\begin{aligned} m\mathbf{a}\cdot\mathbf{e}_\theta &= m\{\ddot{x}\cos\theta\cos\varphi + \ddot{y}\cos\theta\sin\varphi + \ddot{z}\sin\theta\} \\ &= m\{\cos\theta\cos\varphi[-\omega^2 x - \dot{\theta}^2 x + (R\cos\theta\cos\varphi)\ddot{\theta} - 2R\omega\dot{\theta}\cos\theta\sin\varphi] \\ &\quad + \cos\theta\sin\varphi[-\omega^2 y - \dot{\theta}^2 y + (R\cos\theta\sin\varphi)\ddot{\theta} + 2R\omega\dot{\theta}\cos\theta\cos\varphi] \\ &\quad + \sin\theta[-z\dot{\theta}^2 + (R\sin\theta)\ddot{\theta}]\}. \end{aligned}$$

Using (1.6.4), this simplifies to

$$m\mathbf{a}\cdot\mathbf{e}_\theta = mR\{\ddot{\theta} - \omega^2\sin\theta\cos\theta\}. \tag{1.6.10}$$

Comparing (1.6.8), (1.6.9), and (1.6.10), we get

$$\ddot{\theta} = \omega^2\sin\theta\cos\theta - \frac{\nu}{m}\dot{\theta} - \frac{g}{R}\sin\theta, \tag{1.6.11}$$

which agrees with our earlier equations (1.6.3).

Hamilton's Principle. Next, we will show the equivalence of Hamilton's variational principle (1830) and the Euler Lagrange equation:

$$\delta\int_a^b L(q,\dot{q}) = 0 \quad \Leftrightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial\dot{q}^i} - \frac{\partial L}{\partial q^i} = 0.$$

The variational principle tells us that the *action* integral is stationary to perturbations of the curve going from a to b, as in Figure 1.6.3.

What Hamilton's Principle means precisely is that the curve $q(t)$ is such that for any family of curves $q(t,\epsilon)$ satisfying the conditions

$$\begin{aligned} q(t,0) &= q(t) \\ q(a,\epsilon) &= q(a) \\ q(b,\epsilon) &= q(b), \end{aligned}$$

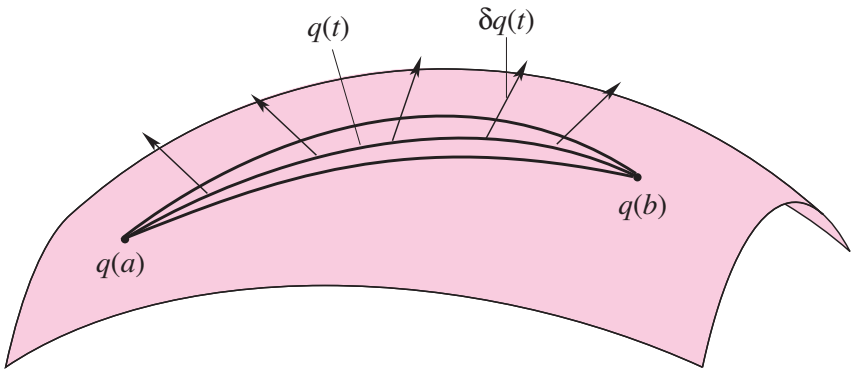

FIGURE 1.6.3. The Euler–Lagrange equations are equivalent to Hamilton's Principle: the action integral is stationary under variations of the curve $q(t)$.

we have

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \int_a^b L(q_\epsilon(t), \dot{q}_\epsilon(t)) = 0.$$

As indicated in the preceding Figure, we write:

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} q(t,\epsilon) = \delta q(t).$$

Using this notation, and differentiating under the integral sign and using integration by parts, Hamilton's Principle becomes

$$\begin{aligned} 0 = \frac{d}{d\epsilon}_{|_{\epsilon=0}} \int_a^b L(q_\epsilon(t), \dot{q}_\epsilon(t)) &= \int_a^b \frac{\partial L}{\partial q}\delta q + + \frac{\partial L}{\partial \dot{q}}\delta\dot{q}\, dt \\ &= \int_a^b \left[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial q}(\dot{\delta q})\right] dt, \end{aligned}$$

where we used equality of mixed partials in interchanging the time derivative and the ϵ derivative (that is, interchanging the overdot and the δ operations. Integration by parts then gives

$$0 = \int_a^b \left[\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i}\right] \delta q\, dt = 0,$$

where we notice that there is no boundary term since δq vanishes at the endpoints because of the conditions $q(a,\epsilon) = q(a)$ and $q(b,\epsilon) = q(b)$. Since this is zero for arbitrary $\delta q(t)$, the integrand must vanish. In summary, this argument shows indeed that *Hamilton's Principle is equivalent to the Euler–Lagrange equations.*

External Forces. We have shown that *Hamilton's Principle is equivalent to the Euler–Lagrange equations.* In the case of external forces F_{ext} (such as frictional forces or control forces), we modify Hamilton's principle to the Lagrange-d'Alembert principle:

$$\delta \int L(q, \dot{q}) dt + \int F_{\text{ext}} \cdot \delta q \, dt = 0$$

Forces that come from a potential are conventionally put into the Lagrangian. Hence, by (external) we mean forces that are not derived from a potential (such as friction or the propulsion force of a rocket). However, if potential forces were to be included as external forces, then the system remains consistent.

The same argument that was used to show that Hamilton's Principle is equivalent to the Euler–Lagrange equations shows that the Lagrange-d'Alembert principle is equivalent to the Euler–Lagrange equations with external forces:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = F_{\text{ext}}(q, \dot{q})$$

For instance in the rocket example, these equations give:

$$\frac{d}{dt}(m\dot{q}) = -\nabla V + F_{\text{ext}}$$

These are also the correct equations when the mass of the rocket is allowed to change with time. Note that one cannot just pull the m out of the derivative sign as one might guess if one naively uses $F = ma$.

For the ball in the hoop example, if we add a friction term that is proportional to the velocity (as was explained in the $F = ma$ derivation, this is a bit ad hoc; in fact modeling friction realistically is a subtle business), we get

$$mR^2\ddot{\theta} = mR^2\omega^2 \sin\theta\cos\theta - mgR\sin\theta - \nu R\dot{\theta},$$

where we regard ν as the coefficient of friction.

The Energy Equation. Next we discuss the energy equation for a mechanical system. First suppose that there are no external forces, so that the Euler–Lagrange equations hold. Define the ***energy*** to be

$$E(q, \dot{q}) = \sum_{k=1}^{n} \frac{\partial L}{\partial \dot{q}^k}\dot{q}^k - L(q^i, \dot{q}^i),$$

or, for short,

$$E(q, \dot{q}) = \frac{\partial L}{\partial \dot{q}}\dot{q} - L(q, \dot{q}).$$

Using the Euler–Lagrange equations, we compute the time derivative of E with the help of the product rule and the chain rule as follows.

$$\begin{aligned}\frac{d}{dt}E &= \sum_{k=1}^{n}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k}\dot{q}^k + \frac{\partial L}{\partial \dot{q}^k}\ddot{q}^k\right) - \sum_{j=1}^{n}\frac{\partial L}{\partial q^j}\dot{q}^j - \sum_{j=1}^{n}\frac{\partial L}{\partial \dot{q}^j}\ddot{q}^j \\ &= \sum_{k=1}^{n}\dot{q}^k\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^k} - \frac{\partial L}{\partial q^k}\right) \\ &= 0.\end{aligned}$$

Thus, A similar calculation shows that when there is an external force present, then

$$\frac{d}{dt}E = F_{\text{ext}} \cdot \dot{q} = \text{ Power of external forces.}$$

Notice that for the simple system with Lagrangian $L(q, v) = \frac{1}{2}m\|v\|^2 - V(q)$, the energy switches the sign of V so we get $E(q, v) = \frac{1}{2}m\|v\|^2 + V(q)$.

Ball in the Hoop and the Simple Pendulum. We now look at the ball in the hoop a bit more closely. First consider the case when the hoop is not rotating (that is, $\omega = 0$) and there is no friction (that is, $\nu = 0$). In this case, the equation of motion becomes that of the *simple pendulum*:

$$\ddot{\theta} + \frac{g}{R}\sin\theta = 0$$

Note that for small oscillations, in which case $\sin\theta \approx \theta$, the equation of motion simplifies to $\ddot{\theta} + \frac{g}{R}\theta = 0$ which is a simple harmonic oscillator whose angular frequency is

$$\omega_{\text{pend}} = \sqrt{\frac{g}{R}}.$$

The phase portrait of the simple pendulum is shown in Figure 1.6.4.

Phase Portraits for the Ball in the Hoop. We write the ball in the hoop as usual as a first order dynamical system as follows.

$$\begin{aligned}\dot{\theta} &= v \\ \dot{v} &= \frac{g}{R}(\alpha\cos\theta - 1)\sin\theta - \beta v,\end{aligned}$$

where $\alpha = (R/g)\omega^2$ and $\beta = \nu/m$.

The equilibrium points of this system are obtained by setting $\dot{x}$ and $\dot{v}$ equal to zero. Thus, the equilibrium points correspond to zeros of $(\alpha\cos\theta - 1)\sin\theta$. If $\sin\theta = 0$, then either $\theta = 0$ or $\theta = \pi$ (plus multiples of 2π). The other equilibrium points occur when $\alpha\cos\theta = 1$. There are no other equilibria if $1/\alpha > 1$, 2 solutions if $1/\alpha < 1$ and one solution if $\alpha = 1$.

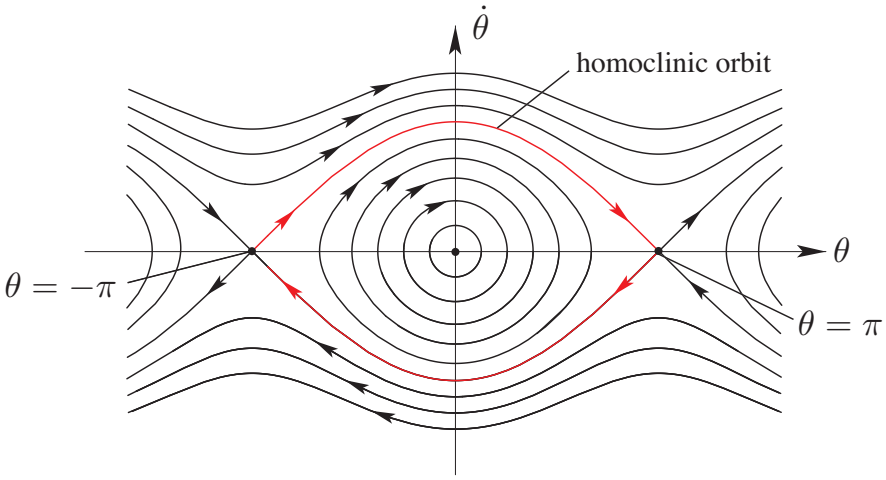

FIGURE 1.6.4. Phase portrait of the simple pendulum.

Therefore a critical value is when $\alpha = 1$; i.e., $R\omega^2 = g$. Thus, a bifurcation occurs when $\omega = \sqrt{g/R}$; i.e., interestingly, when the hoop rotates with the same angular velocity as frequency of oscillations of the *simple* pendulum. The change in the phase portrait as α crosses the critical value $\alpha = 1$ is shown in Figure 1.6.5. Adding a bit of dissipation does n(t)change this picture too much from a large scale perspective, in the sense that the bifurcation from one to three equilibria still occurs at the same critical value even in the presence of dissipation, but it does turn the centers into sinks.

Off-centered Hoop. So far we have considered the problem when the axis of rotation of the hoop is concentric with its axis of symmetry. When, the rotation axis is slightly shifted by an ϵ the phase portrait will be as shown in Figure 1.6.6. Like the symmetric case, one changes from one equilibrium inside the "pendulum homoclinic loop", but unlike the symmetric case, the one equilibrium does not splt into three, but rather two new equilibria appear through a center-saddle bifurcation as ω increases.

The Legendre Transformation. Now we show how to rewrite the Euler–Lagrange equations into what is called ***Hamiltonian form***. To do this, we introduce the Legendre transformation; namely we define the ***conjugate momentum*** by $p_i = \frac{\partial L}{\partial \dot q^i}$, introduce the change of variables

$$(q^i, \dot q^i) \mapsto (\dot q^i, p_i)$$

and assume (for example, via the implicit function theorem) that this is a legitimate change of variables; that is, that it defines, implicitly $\dot q^i$ as a (smooth) function of (q^i, p_i).

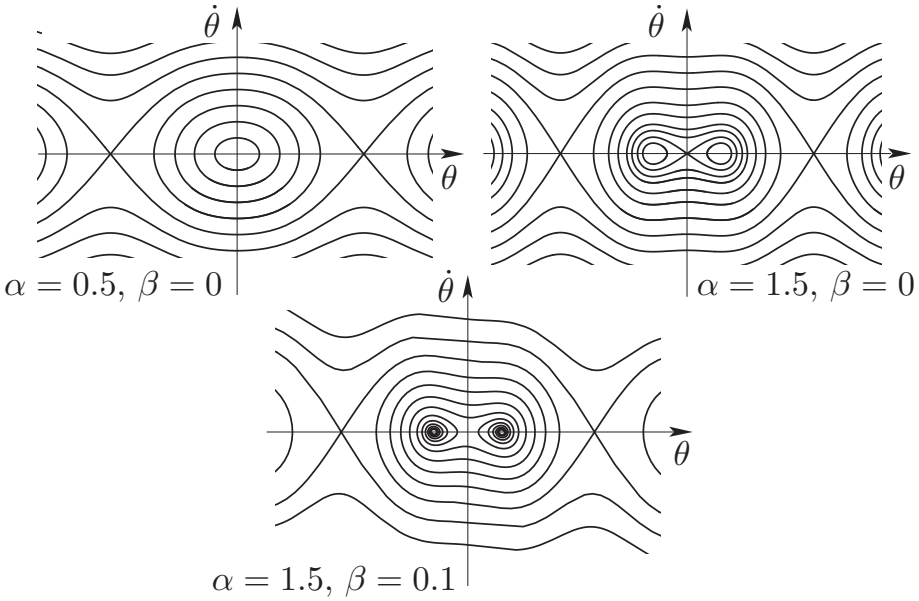

FIGURE 1.6.5. Phase portraits for the ball in the hoop. No damping, but increasing rotation rate as α increases from $\alpha = .05$ to $\alpha = 1.5$ and $\beta = 0$ and adding a bit of damping when $\beta = 0.1$.

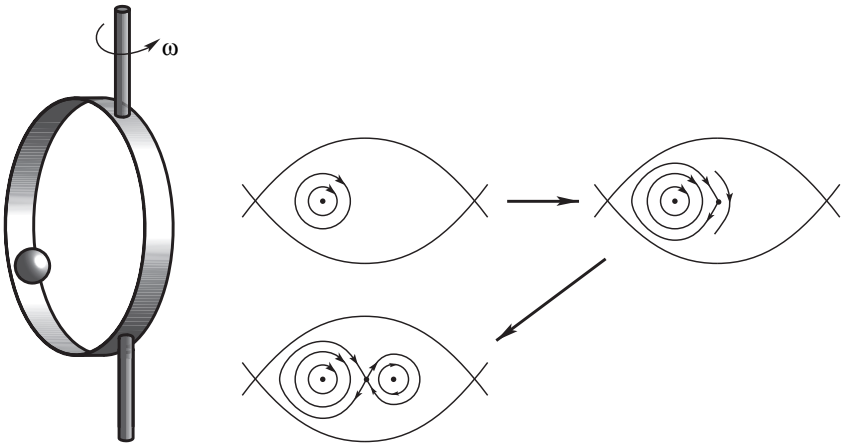

FIGURE 1.6.6. The ball in the off centered hoop and the changes in its phase portrait as the angular velocity is increases.

Using this change of variables, introduce the Hamiltonian by

$$H(q^i, p_i) = \sum_{i=1}^{n} p_i \dot{q}^i - L(q^i, \dot{q}^i)$$

Hamilton's equations for a given Hamiltonian are

$$\frac{d}{dt}\dot{q}^i = \frac{\partial H}{\partial p_i}$$
$$\frac{d}{dt}\dot{p}_i = -\frac{\partial H}{\partial \dot{q}_i}$$

The key link between Lagrangians and Hamiltonians is given in the following theorem:

Theorem. *The Euler–Lagrange equations for* $(q^i, \dot{q}^i)$ *are equivalent to Hamilton's equations for* q, p.

Proof. First assume that the Euler–Lagrange equations hold and we will show that Hamilton's equations hold (the converse is shown in a similar manner). We compute *carefully* using the chain rule as follows:

$$\begin{aligned}\frac{\partial H}{\partial p_i} &= \dot{q}^i + \sum_{j=1}^{n}\left(p_j\frac{\partial \dot{q}^i}{\partial p_j} - \frac{\partial L}{\partial \dot{q}^j}\frac{\partial \dot{q}^j}{\partial p_i}\right)\\ &= \dot{q}^i,\end{aligned}$$

the two terms cancelling by virtue of the definition of the momentum. Similarly we calculate the other partial derivative of H as follows

$$\begin{aligned}\frac{\partial H}{\partial q^i} &= \sum_{j=1}^{n}\left(p_j\frac{\partial \dot{q}^j}{\partial q_i} - \frac{\partial L}{\partial \dot{q}^j}\frac{\partial \dot{q}^j}{\partial q^i} - \frac{\partial L}{\delta q^i}\right)\\ &= -\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}\\ &= -\frac{dp_i}{dt}.\end{aligned}$$

In going from the first line to the second, we again used the definition of the conjugate momentum along with the Euler–Lagrange equations and in going to the last equality we again used the definition of the momentum. Thus we have established Hamilton's equations. ■

Example (Ball in the hoop) Recall that for the ball in the hoop,

$$L(\theta, \dot{\theta}) = \frac{1}{2}mR^2(\dot{\theta}^2 + \omega^2\sin^2\theta) + mgR\cos\theta$$

From this we calculate $p = \partial L/\partial\dot{\theta} = mR^2\dot{\theta}$ and so

$$\begin{aligned}H(\theta, p) &= p\dot{\theta} - L\\ &= mR^2\dot{\theta}^2 - L\\ &= \frac{p^2}{2mR^2} - mgR\cos\theta - \frac{1}{2}mR^2\omega^2\sin^2\theta\end{aligned}$$

From this Hamiltonian one can check directly that Hamilton's equations give the same equations that we had before.

Note, however, that this Hamiltonian is *not* the kinetic plus potential energy ! If one used that, then one would get the incorrect equations.

Conservation of Energy. Lets have another look at conservation of energy from a Hamiltonian point of view. Assuming Hamilton's equations hold, we get—using the Chain rule,

$$\begin{aligned}\frac{d}{dt}H &= \sum_{i=1}^{n}\left(\frac{\partial H}{\partial q_i}\dot{q}^i + \frac{\partial H}{\partial p_i}\dot{p}_i\right)\\ &= \sum_{i=1}^{n}\left(\frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q^i}\right)\\ &= 0\end{aligned}$$

Dirichlet's Stability Theorem. Because of conservation of energy, one can sometimes use the energy as a Liapunov function.

Theorem. *Consider a Hamiltonian $H(q^i, p_i)$ and the corresponding Hamiltonian dynamical system:*

$$\begin{aligned}\frac{d}{dt}\dot{q}^i &= \frac{\partial H}{\partial p_i}\\ \frac{d}{dt}\dot{p}_i &= -\frac{\partial H}{\partial \dot{q}_i}\end{aligned}$$

Let $(\overline{q}^i, \overline{p}^i)$ be an equilibrium point; i.e., a critical point of H. If the second derivative of H as a symmetric matrix, evaluated at this equilibrium point, is positive definite, then the equilibrium is Liapunov stable.

This result follows from using conservation of H: H has a strict minimum at the equilibrium and is conserved. We will return to stability results using Liapunov functions more generally later on.

Eigenvalue Theorem. We have seen via Liapunov's spectral theorem the important role played by the distribution of eigenvalues of the linearization of a system $\dot{x} = f(x)$ at an equilibrium point. For Hamiltonian (and also Lagrangian) systems, there is a severe restriction on how the eigenvalues can appear. In fact, the next theorem shows that the eigenvalues must be distributed symmetrically not only with respect to the real axis (always true for any real system), but also with respect to the imaginary axis.

Theorem. *As in the previous theorem, consider a Hamiltonian system with an equilibrium point. Then,the spectrum of the linearization at $(\overline{q}^i, \overline{p}^i)$ is symmetric with respect to the imaginary axis. In particular, if the conditions of the preceding theorem hold, that is, if this critical point is a*

nondegenerate minimum of H, then the spectrum of the linearization must lie on the imaginary axis.

For example, this means that in a Hamiltonian system, the spectrum at an equilibrium point can never be totally in the left half plane. The student should calculate the eigenvalues for the ball in the rotating hoop to verify that this symmetry does hold in that case.

Proof. Let $z = (q^i, p_i)$ and write Hamilton's equations in matrix form as follows

$$\dot{z} = JDH(z)$$

where

$$\dot{z} = \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix}$$

and the $2n \times 2n$ matrix J is given by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where the "ones" stand for the $n \times n$ identity matrix. Also,

$$DH(z) = \begin{pmatrix} \frac{\partial H}{\delta q^i} \\ \\ \frac{\partial H}{\delta p^i} \end{pmatrix}$$

Note these easily verified properties of J:

1. First of all, $J^T J = I$ (that is, J is an orthogonal matrix; in particular, in the case $n = 1$, it is a rotation by 90^o).
2. Second, $J^2 = -1$.
3. Thirdly, note that $\det J = 1$.

The linearization of the equations $\dot{z} = JDH(z)$ at a critical point $\overline{z} = (\overline{q}, \overline{p})$ is given by

$$\dot{z} = JD^2H(\overline{z})z =: Az,$$

where $A = JD^2H(\overline{z})$ and where $D^2H(\overline{z})$ is the matrix of second partial derivatives of H evaluated at the equilibrium point $\overline{z}$.

We claim that the identity $JAJ = A^T$ holds. To see this, we use the property $J^2 = -I$ to give

$$\begin{aligned} JAJ &= J\left(JD^2H(\overline{z})\right)J = -D^2H(\overline{z})J \\ &= \left(JD^2H(\overline{z})\right)^T = A^T \end{aligned}$$

where we have also used the fact that $D^2H(\overline{z})$ is symmetric due to equality of mixed partials, and skew symmetry of J.

The characteristic polynomial p of the matrix A is given by

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \det(J(A - \lambda I)J) \\ &= \det(JAJ + \lambda I) \end{aligned}$$

where in going from the first line to the second, we used the fact that $\det J = 1$ and in going from the second line to the third we used the fact that $J^2 = -I$. From our calculation above, $JAJ = A^T$ and so

$$\det(JAJ + \lambda I) = \det(A^T + \lambda I) = \det(A + \lambda I) = p(-\lambda).$$

Thus, $p(\lambda) = p(-\lambda)$; that is, p is an even function of λ. Thus, if λ is an eigenvalue, so is $-\lambda$, which proves the result. ■

Homoclinic Orbit of the Simple Pendulum. Finally we return to the simple pendulum. Here our goal is to calculate the equation of the hetroclinic orbit. In general the trajectories can be found using elliptic functions. However, it is interesting that the homoclinic orbit can be written in terms of elementary functions. Here is the procedure:

First of all, recall that the equations (with $g/R = 1$) are

$$\ddot{\theta} + \sin\theta = 0$$

which have the energy integral

$$E = \frac{1}{2}\dot{\theta}^2 - \cos\theta$$

which we can rewrite as

$$\dot{\theta} = \sqrt{2(E + \cos\theta)}$$

and so

$$\int \frac{d\theta}{\sqrt{2(E + \cos\theta)}} = \int dt = t + c$$

The value of E on the homoclinic trajectory equals the value of E at the saddle point $(\pi, 0)$, namely $E = 1$; in this case, the integral is available (eg, in tables of integrals) and is given by

$$\int \frac{d\theta}{\sqrt{2(1 + \cos\theta)}} = \frac{1}{2}\log\left(\frac{1 + \sin(\theta/2)}{1 - \sin(\theta/2)}\right)$$

We can also fix the integration constant by choosing $t = 0$ to correspond to the point $\theta = 0$ (note that the point $(0, 1)$, which has energy $E = 1$ lies on the homoclinic orbit); from these calculations, one concludes that the homoclinic orbit is given by

$$\theta(t) = \pm 2\tan^{-1}(\sinh t).$$