Invariant Manifolds and Liapunov Functions

Invariant Manifolds

The motivation for invariant manifolds comes from the study of critical elements of linear differential equations of the form

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n.$$

Let E^s, E^c, and E^u be the (generalized) real eigenspaces of A associated with eigenvalues of A lying on the open left half plane, the imaginary axes, and the open right half plane, respectively. As we have seen in the section on linear systems, each of these spaces is invariant under the flow of $\dot{x} = Ax$ and represents, respectively, a stable, center, and unstable subspace.

Let us call a subset $S \subset \mathbb{R}^n$ a ***k-manifold*** if it can be *locally* represented as the graph of a smooth function defined on a k-dimensional affine subspace of $\mathbb{R}^n$. As in the calculus of graphs, k manifolds have well defined tangent spaces at each point and these are independent of how the manifolds are represented as graphs. Although the notion of a manifold is much more general, this will serve our purposes.

A k-manifold $S \subset \mathbb{R}^n$ is said to be ***invariant*** under the flow of a vector field X if for $x \in S$, $F_t(x) \in S$ for small $t > 0$; i.e., X is tangent to S. One can thus say that *an invariant manifold is a union of (segments of) integral curves.*

Invariant manifolds are intuitively "nonlinear eigenspaces." A little more precisely, we may define invariant manifolds S of a *critical element* γ; that is, γ is a fixed point or a periodic orbit, to be ***stable*** or ***unstable*** depending on whether they are comprised of orbits in S that wind toward γ with increasing, or with decreasing time.

Let us focus on fixed points, say, x_e to begin. In a neighborhood of x_e, the tangent spaces to the stable and unstable manifolds are provided by the generalized eigenspaces E^s, E^c, and E^u of the linearization $A = DX(x_e)$. We are going to start with ***hyperbolic points***; that is, points where the linearization has no center subspace. Let the dimension of the stable subspace be denoted k.

Theorem (Local Invariant Manifold Theorem for Hyperbolic Points)**.** *Assume that X is a smooth vector field on $\mathbb{R}^n$ and that x_e is a hyperbolic equilibrium point. There is a k- manifold $W^s(x_e)$ and a $n-k$-manifold $W^u(x_e)$ each containing the point x_e such that the following hold:*

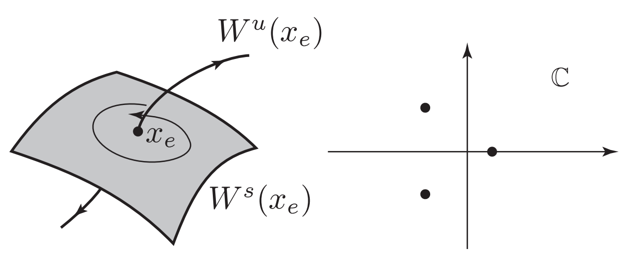
i. *Each of $W^s(x_e)$ and $W^u(x_e)$ is locally invariant under X and contains x_e.*

ii. *The tangent space to $W^s(x_e)$ at x_e is E^s and the tangent space to $W^u(x_e)$ at x_e is E^u.*

iii. *If $x \in W^s(x_e)$, then the integral curve with initial condition x tends to x_e as $t \to \infty$ and if $x \in W^u(x_e)$, then the integral curve with initial condition x tends to x_e as $t \to -\infty$.*

iv *The manifolds $W^s(x_e)$ and $W^u(x_e)$ are (locally) uniquely; they are determined by the preceding conditions.*

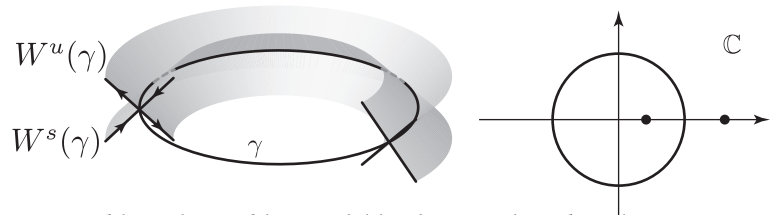
A rough depiction of stable and unstable manifolds of a fixed point are shown in the next figure.



stable and unstable manifolds of a critical point with one eigenvalue of the linearization in the right half plane and two in the left

In this case of hyperbolic fixed points we only have the locally unique manifolds $W^s(x_e)$ and $W^u(x_e)$. These can be extended to *globally unique, immersed submanifolds* by means of the flow of X. This is the ***Global Stable Manifold Theorem of Smale***.

Invariant Manifolds for Periodic Orbits. We mention that there is a similar result for invariant manifolds of periodic orbits γ. We indicate the idea of this result in the following figure.



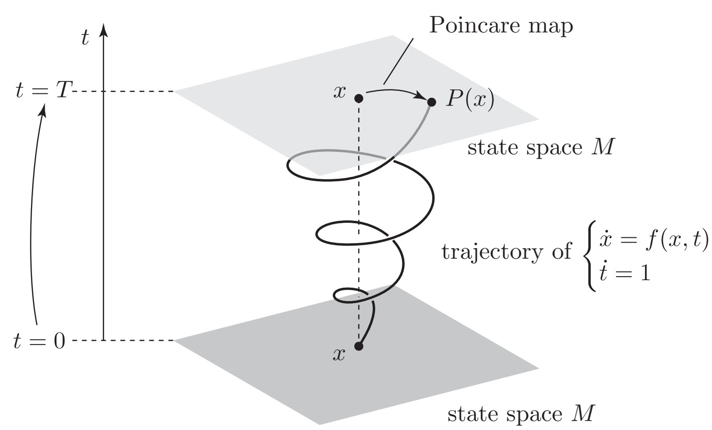
stable and unstable manifolds of a periodic orbit whose Poincare map has one eigenvalue inside and one outside the unit circle.

Invariant Manifolds for Mappings Recall that mappings rather than flows arise in at least three basic ways:

(a) Many systems are directly described by discrete dynamics: $x_{n+1} = f(x_n)$. For example, the standard map, the Henon map, many integration algorithms for dynamical systems, and many population problems may be understood this way. Delay and difference equations can be viewed in this category as well.

(b) The Poincaré map of a closed orbit has already been discussed in lecture.

(c) Suppose we are interested in nonautonomous systems of the form $\dot{x} = f(x,t)$ where f is T-periodic in t. Then the map P that advances solutions by time T, also called the Poincaré map, is basic to a qualitative study of the orbits. (See the following Figure.) This map is often used in the study of forced oscillations.



The Poincaré map of a time-periodic dynamical system.

The Center Manifold Theorem

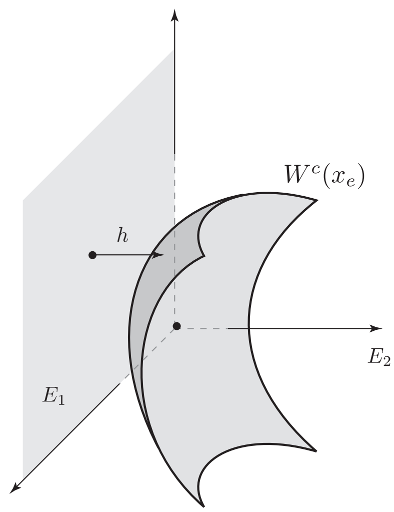
First we state the Center Manifold Theorem. For simplicity of exposition, let us assume we are dealing with an equilibrium point at the origin.

Theorem (Local Center Manifold Theorem for Flows). *Let X be a C^k vector field on $\mathbb{R}^n$ ($k \geq 1$) such that $X(0) = 0$. Let $\phi_t(x)$ denote the corresponding flow. Assume that the spectrum of $\mathbf{D}X(0)$ is of the form $\sigma = \sigma_1 \cup \sigma_2$ where σ_1 lies on the imaginary axis and σ_2 lies off the imaginary axis. Let $E_1 \oplus E_2$ be the corresponding splitting of $\mathbb{R}^n$ into generalized eigenspaces.*

Then there is a neighborhood U of 0 in $\mathbb{R}^n$ and a C^k submanifold $W^c \subset U$ of dimension d passing through 0 and tangent to E_1 at 0 such that

i. *If $x \in W^c$ and $\phi_t(x) \in U$ for all $t \in [0, t_0]$, then $\phi_{t_0}(x) \in W^c$.*

ii. *If $\phi_t(x) \in U$ for all $t \in \mathbb{R}$, then $\phi_t(x) \in W^c$. The manifold W^c is locally the graph of a C^k map $h : E_1 \to E_2$ with $\xi(0) = 0$ and $\mathbf{D}h(0) = 0$. (See the following Figure.)*



The center manifold $W^c(x_e)$ of a fixed point.

The manifold W^c is called a ***center manifold***. Property **i** says that W^c is locally invariant under the flow ϕ_t, and **ii** means that all orbits of ϕ_t that are globally defined and contained in U for all t are actually contained in W^c. Letting $d = \dim E_1$, the problem of finding orbits of x that remain near 0 reduces to the discussion of a d dimensional vector field H that is obtained by restricting the original vector field X to W^c and then pulling it back to E_1. In coordinates, we have

$$H(x_1) = \mathbb{Q}X(x_1 + h(x_1)),$$

where $\mathbb{Q}$ is the projection of $\mathbb{R}^n$ onto E_1 relative to the splitting $\mathbb{R}^n = E_1 \oplus E_2$.

`Go to in class notes for further discussion and examples` ToDo

Proofs of the Center Manifold Theorem (Optional Discussion)

This is a technical job, but the technicalities can lead to (and historically did lead to) fundamental advances and new ideas. After giving an overview of the main methods that have been used to prove the theorem, we give the details of each of three approaches. Following this, further properties of smoothness and attractivity are given.

In this section we will discuss some of the main techniques that are available to prove the center manifold theorem.

The first main division is that between maps and flows. One can take the approach of first proving the invariant manifold theorems for maps and then, using the time t map associated to any flow, deduce the center manifold theorems for flows. This approach is certainly useful since the invariant manifold theorems for maps are important in their own right. However, in our introductory approach, we have chosen to proceed *directly* with proofs for differential equations. By consulting the references cited, the reader will have no trouble tracking down the corresponding theorems for maps, should they require that.

There are several approaches in the literature to proving the invariant manifold theorems. We shall not attempt to survey them all here, but rather we shall focus on three main ideas:

1. The invariance equation approach.
2. The trajectory selection method (sometimes called the Liapunov-Peron method).
3. The normal form method.

Each of these methods sets up the problem in a different way, but once the problem is set up, there is a nonlinear equation to solve. To solve it, there are two main approaches that can be used:

A. The contraction mapping approach.
B. The deformation method.

Thus, in principle, one can follow six general lines of proof to the end. Each line has its own merits, as we shall see.

The contraction mapping principle is a familiar method for solving nonlinear equations. One formulates the equation as a fixed point problem on an appropriate complete metric space (often a Banach space) and then applies the contraction mapping principle.

As we learn in elementary analysis, one can often replace the contraction mapping argument by the inverse function theorem. Irwin [1970, 1970] has shown, this is indeed the case for the *stable* and *unstable* manifold theorems. However, it does *not* seem possible for the center manifold case. (Although a Lipschitz version of the inverse function following Pugh and Shub [1970] might be appropriate). We shall give an idea of the difficulties involved below.

The deformation method is a powerful and general method that was developed in singularity theory that has been applied to prove sharp versions of various normal form theorems, including the Morse lemma (Golubitsky and Marsden [1983]) and the Darboux theorem in mechanics (Moser [1965]). The general idea is to join the nonlinar problem to a simpler (often linear) one by a parameter, and then to flow out, using an ordinary differential equation, the solution of the simpler problem, to one for the desired problem. We shall give an abstract context for the method below.

Let us now go into the various approaches in a bit more detail. We start with equations of the form

$$\dot{x} = Ax + f(x, y) =: \phi_1(x, y) \tag{0.0.1}$$

$$\dot{y} = By + g(x, y) =: \phi_2(x, y) \tag{0.0.2}$$

where x and y belong to real Banach spaces X and Y (for us, these will generally be $X = \mathbb{R}^k$ and $Y = \mathbb{R}^l$), A and B are linear operators on X and Y respectively and f and g are nonlinear maps of a neighborhood of $(0, 0)$ in $X \times Y$ to X and Y, respectively. We assume that:

A1. *The spectrum of A is on the imaginary axis and the spectrum of B lies at a positive distance from the imaginary axis, as in Figure 0.0.1, for example.*

A2. *The mappings f and g are of class C^k, $k \geq 2$ or of class C^k_{lip}, $k \geq 1$. (C^k_{lip} denotes the functions of class C^k whose k^{th} derivative is Lipschitz.)*

A3. $f(0, 0) = 0$, $\mathbf{D}f(0, 0) = 0$, $g(0, 0) = 0$, *and* $\mathbf{D}g(0, 0) = 0$.

Remarks.

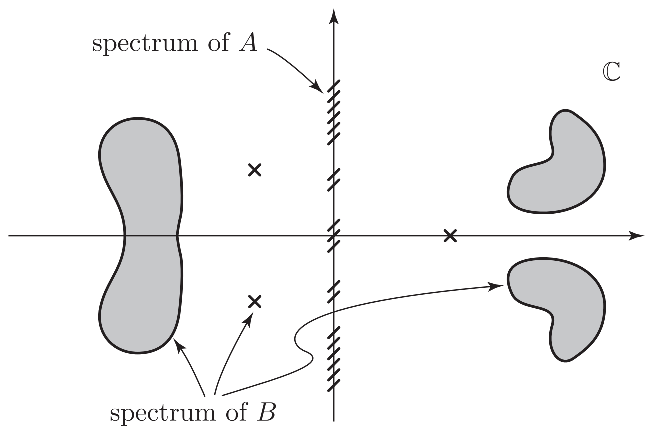


FIGURE 0.0.1. The spectrum at a fixed point can have a stable, and unstable and a center part.

1. If we begin with a differential equation $\dot{z} = F(z)$ on a Banach space Z and $F(0) = 0$, we divide $\mathbf{D}F(0)$ by spectral theory into parts with spectrum on the imaginary axis and the rest, then this defines the linear operators A and B and the functions f and g are the remainder terms after subtracting the linear terms. This is how a general system produces one of the form (5.1.1) and (5.1.2).

2. One can modify A1-A3 to allow the possibility of dependence on parameters. For example, one then asks that the spectrum of A lie *near* the imaginary axis and that $\mathbf{D}f(0)$ and $\mathbf{D}g(0)$ are *small*. However, this is mainly useful for the most technically sharp theorems that are needed when PDE's are considered. For this book we are concentrating on the finite dimensional case and then A1-A3 suffice by using the suspension trick.

♦

Now comes an important point. The next three sections will put assumptions on f and g in addition to the above that involve their behavior as $(x, y) \to \infty$. In this *global setting* one proves that *the center manifold is unique*. However, without these assumptions, which one does *not* want to make in general, *the center manifold* (unlike the stable and unstable manifolds) *is not unique, nor need it be smooth, even if f and g are*. We will give some examples of this below.

One gets the local theorem stated from the global one in a very simple way. One simply multiplies f and g by a function φ that vanishes outside a neighborhood U of $(0, 0)$, and is 1 on a smaller neighborhood V. The new system has a center manifold (depending on φ!) that is a valid center manifold for the original system on V.

If the spectrum of B lies in the strict left hand plane, then the center manifold is an attracting set (unless trajectories leave the neighborhood where it is defined) and moreover, trajectories approach orbits *on* the center manifold in the strong sense of an asymptotic phase: A trajectory $z(t)$ is said to *converge ot a trajectory $z_0(t)$ with an asymptotic phase* if there is a number t_∞ such that $\|z(t) - z_0(t + t_\infty)\| \to 0$ as $t \to \infty$. These dynamic properties, along with smoothness results for center manifolds, are proved in the last two sections of the chapter.

Next we describe the general idea of each of the methods 1, 2 and 3.

1. ***The Invariance Method***

Here we search for an invariant manifold of the form $y = h(x)$, as in Figure 0.0.2.

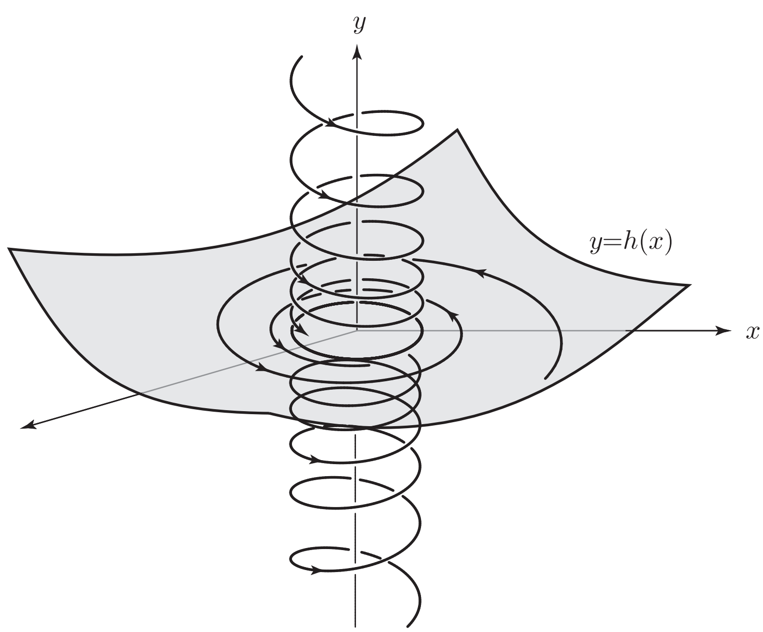


FIGURE 0.0.2. The idea of the invariance method.

The condition that $y = h(x)$ be invariant under the flow is obtained by differentiating it in time: $\dot{y} = \mathbf{D}h(x)\dot{x}$, or

$$\phi_2(x, h(x)) = \mathbf{D}h(x) \cdot \phi_1(x, h(x)). \tag{0.0.3}$$

This, together with the tangency requirement $h(0) = 0$, $\mathbf{D}h(0) = 0$ can be regarded as the equation we have to solve.

An immediate difficulty with (0.0.3) is the loss of derivatives in h due to the term $\mathbf{D}h(x)$. Second, h occurs in a nonlinear way due to the composition in both ϕ_1 and ϕ_2.

To understand the difficulties with solving (0.0.3), consider a simple example. Let $A = 0$ (so the spectrum is at zero) and $X = \mathbb{R}$, $Y = \mathbb{R}$, so (5.1.1) and (5.1.2) read

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= By + g(x, y) \end{aligned}$$

and (0.0.3) reads

$$Bh(x) + g(x, h(x)) = f(x, h(x))h'(x). \tag{0.0.4}$$

As an ode for h, this equation is singular since the coefficient (and even its derivative) of $h'(x)$ vanishes at $x = 0$! This is an *essential difficulty* that has to be overcome.

At this point, there are two techniques we shall consider to solve (0.0.4). The first is to reformulate it as a fixed point problem and, on a suitable space C^k_{lip}, apply the contraction mapping theorem. To formulate it as a fixed point problem, one proceeds in two steps.

Step 1. The second method is the deformation method. We insert a parameter ε in (5.1.1) and (5.1.2):

$$\dot{x} = Ax + \varepsilon f(x, y) =: \phi_1(x, y), \tag{0.0.5}$$

$$\dot{y} = By + \varepsilon g(x, y) =: \phi_2(x, y). \tag{0.0.6}$$

For $\varepsilon = 0$ there is a solution of (5.1.3), namely $h_0(x) = 0$. We then seek a solution $h_\varepsilon(x)$ for the above system. The procedure is to differentiate (5.1.3) in ε to obtain an equation for $dh_\varepsilon/d\varepsilon$ which can be solved as an evolution equation in the "time" ε. We get what we want at $\varepsilon = 1$.

`Remarks to be added here` ToDo

2. ***The Trajectory Selection Method***

If one looks at Figure 0.0.2, it is reasonable to think of the center manifold as the "slow manifold". For example, trajectories near, but not on the center manifold appear to spiral out, away from the origin as $t \to -\infty$ at an exponential rate (depending on the distance of the spectrum of B to the imaginary axis). Points *on* the center manifold are characterized by the fact that they either linger on the center manifold, or if they do leave a neighborhood of the origin, they do so at a slower rate.

Thus, in this method, one sets up function spaces with growth rates built in as $t \to \pm\infty$ and initial conditions are sought with "slow" growth rates. Gluing these together produces the center manifold.

More on eq'n one has to solve?

3. ***The Normal Form Method***

The idea here, borrowed from normal form theory (some easy cases of which are given in Chapter **6**), is to seek a certain change of variables of the form

$$u = x + \chi(x, y) \tag{0.0.7}$$

$$v = y + \psi(x, y) \tag{0.0.8}$$

where χ and ψ vanish, along with their derivaties at $(0, 0)$. Thus, this is a *near identity* change of variables near the origin. The equations (5.1.1) and (5.1.2) now become

$$\dot{u} = Au + \tilde{f}(u, v) \tag{0.0.9}$$

$$\dot{v} = Bu + \tilde{g}(u, v) \tag{0.0.10}$$

for new functions $\tilde{f}$ and $\tilde{g}$ that depend on χ and ψ. What we seek is to choose χ and ψ so that

$$\tilde{g}(u, 0) = 0. \tag{0.0.11}$$

This is an implicit equation for χ and ψ which, in principle, can be solved by either the contraction mapping argument or the deformation method. Once it is done, the invariant manifold is simply

$$v = 0$$

which implicitly defines the center manifold as $y = h(x)$ through the change of variables.

Examples

We now give some examples of center manifolds that show the delicacy of the situation.

A. Both this example and the next will be systems with parameters and exhibiting an interesting bifurcation. *This first example shows the non-uniqueness of the center manifold.* We consider the system

$$\begin{aligned} \dot{x} &= -x^2 + \alpha \\ \dot{y} &= -y \\ \dot{\alpha} &= 0 \end{aligned} \tag{0.0.12}$$

The phase protraits for $\alpha < 0$, $\alpha = 0$ and $\alpha > 0$ are shown in Figure 0.0.3.

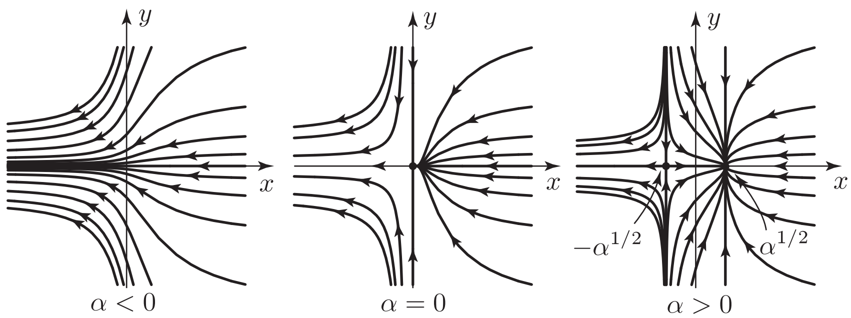


FIGURE 0.0.3. Missing Caption

Center manifolds in (x, y, α)-space are obtained by gluing together one of the curves tending to $(0, 0)$ at $\alpha = 0$ as $t \to \infty$ from $x > 0$ with the negative x-axis and with their counterparts for $\alpha < 0$ and $\alpha > 0$. One of these choices is highlighted in the figure.

As $\alpha = 0$, notice that the curves from the right half plane are given by

$$x = \frac{1}{t - t_0}, \quad y = y_0 e^{t - t_0}$$

for any t_0 and y_0; i.e., $y = y_0 e^{1/x}$. Notice that this curve is tangent to the x-axis to all orders. This is a general property of all center manifolds, as was proved by Wan [198?]. ♦

Remark. Note that the center manifold is unique at $\alpha = 0$ in the half plane $x < 0$ and for $\alpha > 0$ *between* the two fixed points created in the bifurcation. Features like this in fact are true generally when *unstable* manifolds are created by a bifurcation in an *attracting* center manifold, as follows from uniqueness of the *unstable* manifold of the bifurcated fixed point. These are *part* of the *center* manifold for the suspended system. ♦

B. Next we give an example showing that the center manifold *need not be* C^∞. It will be, for any $k \geq 0$, of class C^k on some neighborhood of the origin, but as $k \to \infty$, this neighborhood shrinks to a point.

We consider

$$\begin{aligned} \dot{x} &= -x^3 - \varepsilon x, \\ \dot{y} &= -y + x^2, \\ \dot{\varepsilon} &= 0. \end{aligned} \tag{0.0.13}$$

The phase portraits for $\varepsilon < 0$, $\varepsilon = 0$ and $\varepsilon > 0$ are shown in Figure 0.0.4.

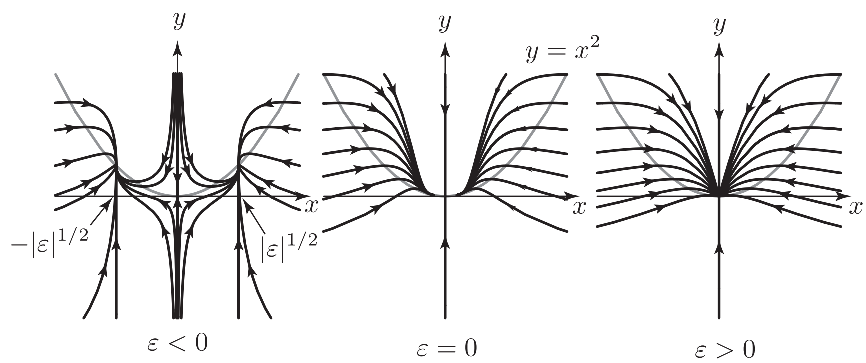


FIGURE 0.0.4. Missing Caption

In this example we can see, as in Example **A**, that the center manifold is not unique. One such choice is emphasized in the figure. (The portion containing the unstable manifold of the origin for $\varepsilon < 0$ is unique.) Let us now investigate the smoothness of this manifold.

First, *we claim that at $\varepsilon = 0$, it is not analytic*. Represent it by $y = h(x)$. If it were analytic, we could write

$$y = h(x) = \sum_{n=2}^{\infty} a_n x^n. \tag{0.0.14}$$

The invariance condition is obtained by differentiating: $\dot{y} = h'(x)\dot{x}$, or $-y + x^2 = h'(x)(-x^3)$, or

$$x^2 - \sum_{n=2}^{\infty} a_n x^n = -\sum_{n=2}^{\infty} a_n x^{n+2}. \tag{0.0.15}$$

Solving this recursively determines a_n and hence h. We get $a_2 = 1$, $a_3 = 0$ and $a_n = (n-2)a_{n-2}$ for $n \geq 4$. Thus, the odd coefficients vanish, while the even ones are $a_{2m} = 2^{m-1}(m-1)!$. In particular, the radius of convergence of this series is zero, so it proves our claim.

Second, we claim that for $\varepsilon > 0$, the center manifold loses its differentiability of class C^k on a neighborhood of the origin that shrinks to a point as $k \to \infty$.

Consider the invariant manifold for $\varepsilon > 0$ in parametrized form as $y = h_\varepsilon(x)$. The invariance condition is

$$-y + x^2 = h'_\varepsilon(x)(-x^3 - \varepsilon x). \tag{0.0.16}$$

If h_ε is of class C^{2m+1} in a neighborhood of $x = 0$, then

$$h_\varepsilon(x) = \sum_{i=1}^{2m} a_i x^i + O(x^{2m+1}) \tag{0.0.17}$$

and

$$h'_\varepsilon(x) = \sum_{i=1}^{2m} a_i i x^{i-1} + O(x^{2m}). \tag{0.0.18}$$

Substituting these in the preceding equation gives

$$\begin{aligned} &-a_1 x - (a_2 - 1)x^2 - \sum_{i=3}^{2m} a_i x^i + O(x^{2m+1}) \\ &= \left[\left(\sum_{i=1}^{2m} i a_i x^{i-1}\right) + O(x^{2m})\right](-x^3 - \varepsilon x). \end{aligned}$$

Thus, $a_1 = 0$, $a_2 = 1/(1 - i\varepsilon)$ and $(1 - i\varepsilon)a_i = (i-2)a_{i-2}$ and so

$$a_i = \frac{i-2}{1-i\varepsilon} a_{i-2}.$$

For $1 - 2m\varepsilon = 0$, or $\varepsilon = 1/2m$, $a_{2m} \to \infty$, so h can't be C^{2m+1} on a neighborhood of 0 if $\varepsilon = 1/2m$. Therefore, the neighborhood on which h is C^k shrinks as $k \to \infty$. ♦

Liapunov Functions

Besides the Liapunov spectral theorem, there is another method of proving stability that is a generalization of the energy method we have seen in the basic examples.

Definition 0.0.1. *Let X be a C^r vector field on $\mathbb{R}^n$, $r \geq 1$, and let m be an equilibrium point for X, that is, $X(m) = 0$. A **Liapunov function** for X at m is a continuous function $L : U \to \mathbb{R}$ defined on a neighborhood U of m, differentiable on $U \backslash \{m\}$, and satisfying the following conditions:*

(i) $L(m) = 0$ *and* $L(m') > 0$ *if* $m' \neq m$;

(ii) $X[L] \leq 0$ *on* $U \backslash \{m\}$;

(iii) *there is a connected chart* $\varphi : V \to \mathbf{E}$ *where* $m \in V \subset U$, $\varphi(m) = 0$, *and an* $\varepsilon > 0$ *satisfying* $B_\varepsilon(0) = \{ x \in \mathbf{E} \mid \|x\| \leq \varepsilon \} \subset \varphi(V)$, *such that for all* $0 < \varepsilon' \leq \varepsilon$,

$$\inf\{ L(\varphi^{-1}(x)) \mid \|x\| = \varepsilon' \} > 0.$$

The Liapunov function L *is said to be* ***strict****, if* (ii) *is replaced by* (ii)′ $X[L] < 0$ *in* $U \backslash \{m\}$.

Conditions (i) and (iii) are called the ***potential well hypothesis***. In finite dimensions, (iii) follows automatically from compactness of the sphere of radius ε' and (i). By the Chain Rule for the time derivative of V along integral curves, condition (ii) is equivalent to the statement: L is decreasing along integral curves of X.

Theorem. *Let* X *be a* C^r *vector field on* $\mathbb{R}^n$, $r \geq 1$, *and let* m *be an equilibrium point for* X, *that is,* $X(m) = 0$. *If there exists a Liapunov function for* X *at* m, *then* m *is stable.*

Proof. Since the statement is local, we can assume $m = 0$. By the local existence theory, there is a neighborhood U of 0 in $\mathbf{E}$ such that all solutions starting in U exist for time $t \in [-\delta, \delta]$, with δ depending only on X and U, but not on the solution. Now fix $\varepsilon > 0$ as in (iii) such that the open ball $D_\varepsilon(0)$ is included in U. Let $\rho(\varepsilon) > 0$ be the minimum value of L on the sphere of radius ε, and define the open set $U' = \{ x \in D_\varepsilon(0) \mid L(x) < \rho(\varepsilon) \}$. By (i), $U' \neq \varnothing$, $0 \in U'$, and by (ii), no solution starting in U' can meet the sphere of radius ε (since L is decreasing on integral curves of X). Thus all solutions starting in U' never leave $D_\varepsilon(0) \subset U$ and therefore by uniformity of time of existence, these solutions can be extended indefinitely in time. This shows 0 is stable. ■

Theorem. *Let* X *be a* C^r *vector field on* $\mathbb{R}^n$, $r \geq 1$, *and let* m *be an equilibrium point for* X, *that is,* $X(m) = 0$. *Suppose that* L *a strict Liapunov function for* X *at* m. *Then* m *is asymptotically stable.*

Proof. We can assume $m = 0$. By the preceding Theorem, 0 is stable, so if t_n is an increasing sequence, $t_n \to \infty$, and $x(t)$ is an integral curve of X starting in U', it lies in a bounded set and so the sequence $\{x(t_n)\} \in \mathbb{R}^n$ has a convergent subsequence. Thus, there is a sequence $t_n \to +\infty$ such that $x(t_n) \to x_0 \in D_\varepsilon(0)$, some ϵ disk. We shall prove that $x_0 = 0$. Since $L(x(t))$ is a *strictly* decreasing function of t by (ii)′, $L(x(t)) > L(x_0)$ for all $t > 0$. If $x_0 \neq 0$, let $c(t)$ be the solution of X starting at x_0, so that $L(c(t)) < L(x_0)$, again since $t \mapsto L(x(t))$ is strictly decreasing. Thus, for any solution $\tilde{c}(t)$ starting close to x_0, $L(\tilde{c}(t)) < L(x_0)$ by continuity of L. Now take $\tilde{c}(0) = x(t_n)$ for n large to get the contradiction $L(x(t_n + t)) < L(x_0)$. Therefore $x_0 = 0$ is the only limit point of $\{ x(t) \mid t \geq 0 \}$ if $x(0) \in U'$, that is, 0 is asymptotically stable. ■

The same method can be used to detect the instability of equilibrium solutions.

Theorem. *Let X be a C^r vector field on $\mathbb{R}^n$, $r \geq 1$, and let m be an equilibrium point for X, that is, $X(m) = 0$. Assume there is a continuous function $L : U \to \mathbb{R}$ defined in a neighborhood of U of m, which is differentiable on $U \backslash \{m\}$, and satisfies $L(m) = 0$, $X[L] > 0$ (respectively, $\leq a < 0$) on $U \backslash \{m\}$. If there exists a sequence $m_k \to m$ such that $L(m_k) > 0$ (respectively, < 0), then m is unstable.*

Proof. We need to show that there is a neighborhood W of m such that for any neighborhood V of m, $V \subset U$, there is a point m_V whose integral curve leaves W. By local compactness, we can assume that $X[L] \geq a > 0$. Since m is an equilibrium, there is a neighborhood $W_1 \subset U$ of m such that each integral curve starting in W_1 exists for time at least $1/a$. Let $W = \{ m \in W_1 \mid L(m) < 1/2 \}$. We can assume as usual that $m = 0$. Let $c_n(t)$ denote the integral curve of X with initial condition $m_n \in W$. Then

$$L(c_n(t)) - L(m_n) = \int_0^t X[L](c_n(\lambda))\, d\lambda \geq at$$

so that

$$L(c_n(1/a)) \geq 1 + L(m_n) > 1,$$

that is, $c_n(1/a) \notin W$. Thus all integral curves starting at the points $m_n \in W$ leave W after time at most $1/a$. Since $m_n \to 0$, the origin is unstable. ■

LaSalle's Invariance Principle

A key ingredient in proving more general global asymptotic stability results is the LaSalle invariance principle. It allows one to prove attractivity of more general invariant sets than equilibrium points.

Theorem 0.0.2. *Consider the smooth dynamical system on an n-manifold M given by $\dot{x} = X(x)$ and let Ω be a compact set in M that is (positively) invariant under the flow of X. Let $V : \Omega \to \mathbb{R}$, $V \geq 0$, be a C^1 function such that*

$$\dot{V}(x) = \frac{\partial V}{\partial x} \cdot X \leq 0$$

in Ω. Let S be the largest invariant set in Ω where $\dot{V}(x) = 0$. Then every solution with initial point in Ω tends asymptotically to S as $t \to \infty$. In particular, if S is an isolated equilibrium, it is asymptotically stable.

In the statement of the theorem, $V(x)$ need not be positive definite, but rather only semidefinite, and that if in particular S is an equilibrium point, the theorem proves that the equilibrium is asymptotically stable. The set Ω

in the LaSalle theorem also gives us an estimate of the region of attraction of an equilibrium. This is one of the reasons that this is a more attractive methodology than that of spectral stability tests, which could in principle give a very small region of attraction.

Examples.

A. The vector field on the plane with components

$$X(x,y) = (-y - x^5, x - 2y^3)$$

has the origin as an isolated equilibrium. The eigenvalues of the linearization of X at $(0,0)$ are $\pm i$ and so Liapunov's Spectral Stability Criterion does not give any information regarding the stability of the origin. If we suspect that $(0,0)$ is asymptotically stable, we can try searching for a Liapunov function of the form $L(x,y) = ax^2 + by^2$, so we need to determine the coefficients $a, b, \neq 0$ in such a way that $X[L] < 0$. We have

$$X[L] = 2ax(-y - x^5) + 2by(x - 2y^3) = 2xy(b - a) - 2ax^6 - 4by^4,$$

so that choosing $a = b = 1$, we get $X[L] = -2(x^6 + 2y^4)$ which is strictly negative if $(x,y) \neq (0,0)$. Thus the origin is asymptotically stable.

B. Consider the vector field in the plane with components

$$X(x,y) = (-y + x^5, x + 2y^3)$$

with the origin as an isolated critical point and characteristic exponents $\pm i$. Again Liapunov's Stability Criterion cannot be applied, so that we search for a function $L(x,y) = ax^2 + by^2$, $a, b \neq 0$ in such a way that $X[L]$ has a definite sign. As above we get

$$X[L] = 2ax(-y + x^5) + 2by(x + 2y^3) = 2xy(b - a) + 2ax^6 + 4by^4,$$

so that choosing $a = b = 1$, it follows that $X[L] = 2(x^6 + y^4) > 0$ if $(x,y) \neq (0,0)$. Thus, the origin is unstable.

These two examples show that if the spectrum of X lies on the imaginary axis, the stability nature of the equilibrium is determined by the nonlinear terms.

C. Consider Newton's equations in $\mathbb{R}^3$, $\ddot{\mathbf{q}} = -(1/m)\nabla V(\mathbf{q})$ written as a first order system $\ddot{\mathbf{q}} = \mathbf{v}, \ddot{\mathbf{v}} = -(1/m)\nabla V(\mathbf{q})$ and so define a vector field X on $\mathbb{R}^3 \times \mathbb{R}^3$. Let $(\mathbf{q}_0, \mathbf{v}_0)$ be an equilibrium of this system, so that $\mathbf{v}_0 = \mathbf{0}$ and $\nabla V(\mathbf{q}_0) = \mathbf{0}$. In previous lectures we have seen that the total energy

$$E(\mathbf{q}, \mathbf{v}) = \frac{1}{2} m \|\mathbf{v}\|^2 + V(\mathbf{q})$$

is conserved, so we try to use E to construct a Liapunov function L. Since $L(\mathbf{q}_0, \mathbf{0}) = 0$, define

$$L(\mathbf{q}, \mathbf{v}) = E(\mathbf{q}, \mathbf{v}) - E(\mathbf{q}_0, \mathbf{0}) = \frac{1}{2}m\|\mathbf{v}\|^2 + V(\mathbf{q} - V(\mathbf{q}_0),$$

which satisfies $X[L] = 0$ by conservation of energy. If $V(\mathbf{q}) > V(\mathbf{q}_0)$ for $\mathbf{q} \neq \mathbf{q}_0$, then L is a Liapunov function. Thus we have proved

The Dirichlet-Lagrange Stability Theorem: *An equilibrium point* $(\mathbf{q}_0, \mathbf{0})$ *of Newton's equations for a particle of mass* m*, moving under the influence of a potential* V*, which has a local absolute minimum at* $\mathbf{q}_0$*, is stable.*

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