Linear Systems and Exponentiation of Commuting Matrices

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Proposition. *If S and N are commuting $n \times n$ matrices (that is, $SN = NS$), then*

$$e^{S+N} = e^S e^N.$$

Proof. The method is to consider a linear differential equation associated to the equality we want to prove and to invoke existence and uniqueness of solutions with given initial conditions (we will prove this in class shortly).

The strategy is to consider the differential equation

$$\dot{y} = (S+N)\,y\,. \tag{1}$$

With a given initial condition y_0.

Claim: *Both $x(t) := e^{t(S+N)}y_0$ and $z(t) := e^{tS}e^{tN}y_0$ satisfy this equation and have the same initial conditions y_0.*

If this is the case, then by existence and uniqueness of solutions, we would have $x(t) = z(t)$, or

$$e^{t(S+N)}y_0 = e^{tS}e^{tN}y_0.$$

If we then set $t = 1$ in this equality and use the arbitrariness of y_0, we would obtain the desired result.

First of all, it is clear that $x(t) = e^{t(S+N)}y_0$ satisfies equation (1) and has the correct initial conditions because, in general, the solution to $\dot{u} = Au$ with initial conditions u_0 is $u(t) = e^{tA}u_0$.

To prove the claim and hence the result, it remains to show that $z(t)$ also satisfies equation (1). It clearly satisfies the correct initial conditions. To do this, we differentiate using the product rule:

$$\frac{d}{dt}z(t) = \frac{d}{dt}\left(e^{tS}e^{tN}y_0\right) = Se^{tS}e^{tN}y_0 + e^{tS}Ne^{tN}y_0 \tag{2}$$

Now since S and N commute, we have $e^{tS}N = Ne^{tS}$. This follows easily using the power series expression for e^{tS}. Thus, equation (2) becomes

$$\begin{aligned}\frac{d}{dt}z(t) &= Se^{tS}e^{tN}y_0 + Ne^{tS}e^{tN}y_0 \\ &= (S+N)e^{tS}e^{tN}y_0 \\ &= (S+N)z\end{aligned}$$

This proves the claim.