1

Basic Theory of Dynamical Systems

1.1 Introduction and Basic Examples

Dynamical systems is concerned with *both* quantitative and qualitative properties of evolution equations, which are often ordinary differential equations and partial differential equations. In these notes we shall focus on the case of ordinary differential equations (ODE), and start off thinking of these equations in Euclidean space $\mathbb{R}^n$ as equations of the form

$$\dot{x} = f(x, t) \tag{1.1.1}$$

where f is a map of an open set in $\mathbb{R}^n \times \mathbb{R}$ to $\mathbb{R}^n$ with some regularity properties to be examined. For now, lets assume that f is smooth. One is to find solutions $x(t)$ to this equation satisfying some initial conditions, say $x(t_0)$ is given. To further simplify things, let us assume for now that f is ***autonomous***; that is, f does not depend explicitly on t. Then the equation becomes

$$\dot{x} = f(x) \tag{1.1.2}$$

However, in many examples, f can depend on parameters. We shall see concrete examples of this as we proceed. If we denote these parameters by $\mu \in \mathbb{R}^p$, then equation 1.1.1 becomes

$$\dot{x} = f(x, \mu) \tag{1.1.3}$$

and we think of solving this equation for each fixed μ and then consider how things change as μ varies.

A Simple Example. Let us start off by examining a simple system that is mechanical in nature. We will have much more to say about examples of this sort later on. Basic mechanical examples are often grounded in Newton's law, $F = ma$. For now, we can think of a as simply the acceleration, given in $\mathbb{R}^n$ by $a = \ddot{x}$, the second time derivative.[1]

Often the forces F are derived from a potential; namely $F(x) = -\nabla V(x)$ for some real valued function V, the *potential energy.* In this case, the equations take the form

$$m\ddot{x} = -\nabla V(x). \tag{1.1.4}$$

Here is a simple example in one dimension; choosing

$$V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$$

and $m = 1$, we get the equation

$$\ddot{x} = x - x^3$$

Intuitively, think of a particle moving in the potential field given by V, as in Figure 1.1.1.

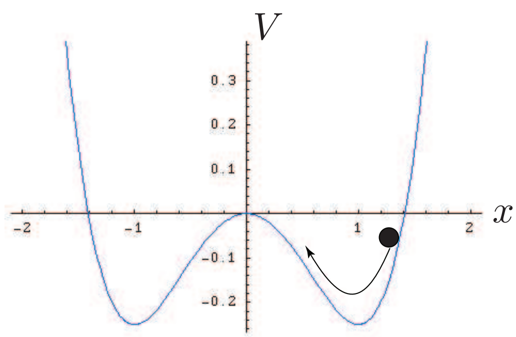

FIGURE 1.1.1. Particle at position x on the line, moving in a potential field V.

To analyze this system, some basic observations are useful. First of all, we can put the equation in first order form (1.1.2) by introducing the velocity v as a separate variable, so what we called x before becomes the pair (x, v):

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= x - x^3 \end{aligned} \tag{1.1.5}$$

Now let us now pause for a basic observation:

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[1] As we shall see later, the acceleration does not take this simple form in coordinates more general than Euclidean coordinates and also is more subtle when one is considering systems that are in motion, such as a pendulum on a rotating Earth.

First Order Form and Equilibria. Suppose we have an equation of the form (1.1.4). We first write that equation in first order form as we did in the preceding example:

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= \frac{1}{m}\left(-\nabla V(x)\right) \end{aligned} \tag{1.1.6}$$

By definition, ***equilibrium points*** of an autonomous system are points where the right hand side of equation (1.1.2) vanish. Finding and analyzing such points is a useful thing to do. Notice that if x_0 is such a point, then the solution curve through that point is simply the constant curve $x(t) = x_0$, since in this case (1.1.2) is satisfied, both sides being zero. For our equation (1.1.6), it is clear that equilibrium points are points for which $v = 0$ and x is a critical point of V. For our specific example above, these points are clearly at $x = 0$ and $x = \pm 1$.

Conservation of Energy. We claim that the kinetic energy plus the potential energy is conserved; that is, the expression

$$E(x, v) = \frac{1}{2}mv^2 + V(x) \tag{1.1.7}$$

is constant in time. To verify this simply differentiate in time, making use of equation (1.1.6):

$$\begin{aligned} \frac{d}{dt}E(x, v) &= mv\dot{v} + \nabla V(x)\dot{x} \\ &= v\left(m\dot{v} + \nabla V(x)\right) \\ &= 0 \end{aligned}$$

as claimed.

Return to the Example. We now return to the example (1.1.5). What we have shown is quite remarkable. Namely, the solution trajectories of this equation must lie on level sets of the energy function for this example, namely

$$E(x, v) = \frac{1}{2}v^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4. \tag{1.1.8}$$

A graph of this energy function (drawn using Mathematica) is shown in Figure 1.1.2 and the corresponding level sets are shown in Figure 1.1.3.

We return to this example after another short interlude.

Finding a Formula for Solution Trajectories. Using conservation of energy, we can find a formula for solutions. Start with the expression (1.1.7) and an initial condition (x_0, v_0). Use this initial condition to evaluate

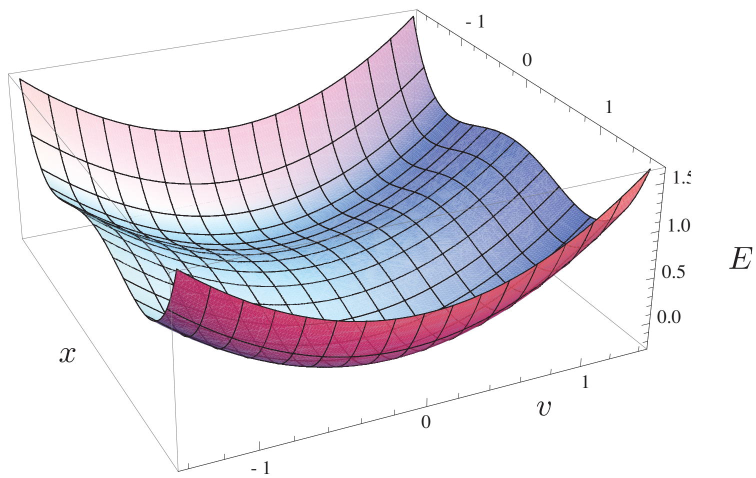

FIGURE 1.1.2. Graph of the energy function $E(x, v) = \frac{1}{2}v^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4$.

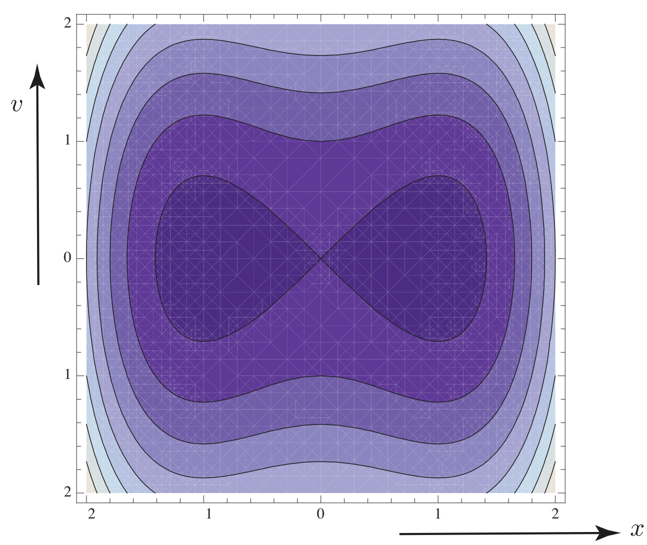

FIGURE 1.1.3. Level contours of the energy function $E(x, v) = \frac{1}{2}v^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4$.

the constant $C = E(x_0, v_0)$. Then by conservation of energy, we have the identity along the trajectory $(x(t), v(t))$ of the system:

$$C = \frac{1}{2}mv(t)^2 + V(x(t))$$

Solve for v:

$$v(t) = \frac{dx}{dt} = \pm\sqrt{2\left(C - V(x(t))\right)}$$

Note that the quantity under the square root is non-negative because the kinetic energy is positive. However, also note that the sign used in this expression depends on the sign of v. The sign can certainly change as a trajectory crosses the x axis. When it does so, the kinetic energy becomes momentarily zero and is called a ***turning point***. For instance, when a pendulum reaches the highest point of its oscillation and changes direction, this happens. With this sign in mind, we can write the solution implicitly in terms of integrals, as follows:

$$\int \frac{dx}{\sqrt{2\left(C - V(x(t))\right)}} = t + \text{constant}.$$

While such expressions certainly can be useful at times, it is often more insightful to directly simulate a system.

Simulations. On the other hand, one can draw the trajectories in the ***phase plane***, that is, the (x, v) plane by using software available over the internet, such as 3D Xplore Math, PPlane, etc (see http://www.cds.caltech.edu/~marsden/cds140a-08/computing for some examples. This gives Figure 1.1.4:

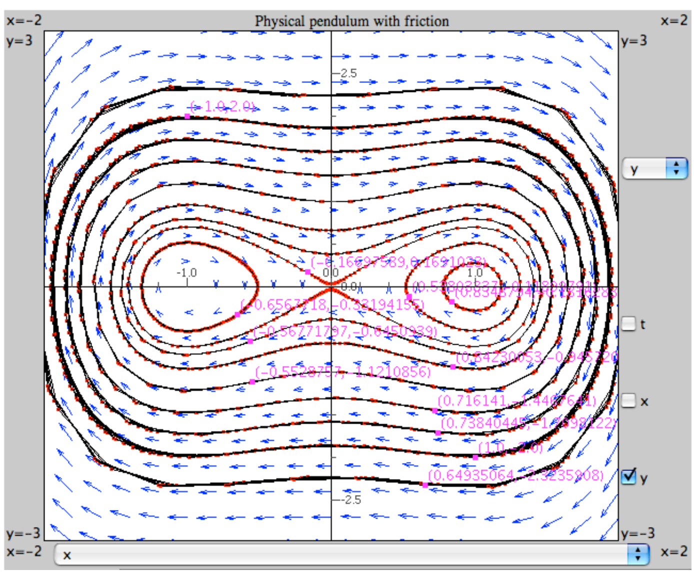

FIGURE 1.1.4. Trajectories for the equation (1.1.5).

Symmetries. Motivated by the preceding figure, we note the following:

> *If $(x(t), v(t))$ is a solution of (1.1.5), then so is $(\tilde{x}(t), \tilde{v}(t))$, where $\tilde{x}(t) = -x(-t)$ and $\tilde{v}(t) = v(-t)$. This trajectory is obtained by reflecting in the v-axis and then running time backwards. This statement is easily verified and rests on the fact that $V(x)$ is an even function of x.*
>
> *Likewise, $(\bar{x}(t), \bar{v}(t))$ is a solution, where $\bar{x}(t) = x(-t)$ and $\bar{v}(t) = -v(-t)$. This second symmetry is also verified to hold for any potential V.*

Dissipation. A simple (and naive) model of dissipative forces is to add a term $-\nu\dot{x}$ to the force. This represents a force that opposes the motion and it proportional to the velocity. It is an instance of what is often called ***Rayleigh dissipation.*** In this case, our example becomes

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= \frac{1}{m}\left(-\nabla V(x)\right) - \nu v \end{aligned} \tag{1.1.9}$$

Where $m = 1$ and $V(x)$ is as in our example. Let us check what happens to conservation of energy in this case:

$$\begin{aligned} \frac{d}{dt}E(x, v) &= mv\dot{v} + \nabla V(x)\dot{x} \\ &= v\left(m\dot{v} + \nabla V(x)\right) \\ &= -\nu v^2 \leq 0. \end{aligned}$$

Note that this holds for general V. Thus, apart from turning points, E is always decreasing, and so it is *plausible* that trajectories (at least most, but *not all* of them) will go to a minimum of E. This is verified in our example by drawing the phase portrait, as in Figure 1.1.5

Stability and Instability. We will treat these important notions informally here and illustrate them with the preceding example. First of all, notice that adding dissipation does not change the equilibrium points. Note that in the preceding figure, initial conditions near the equilibrium points $(\pm 1, 0)$ stay near the equilibrium for all future time and even tend to it as $t \to \infty$; we call such a point ***asymptotically stable***. In the case dissipation is absent, solutions with initial conditions close to one of these points still stays near the point; in fact, in this case, they move along periodic orbits, namely level sets of the energy. Such points are called *stable*. Some orbits with initial points near the origin, on the other hand, move away from the origin as $t \to \infty$ and so the origin is called an ***unstable*** equilibrium.

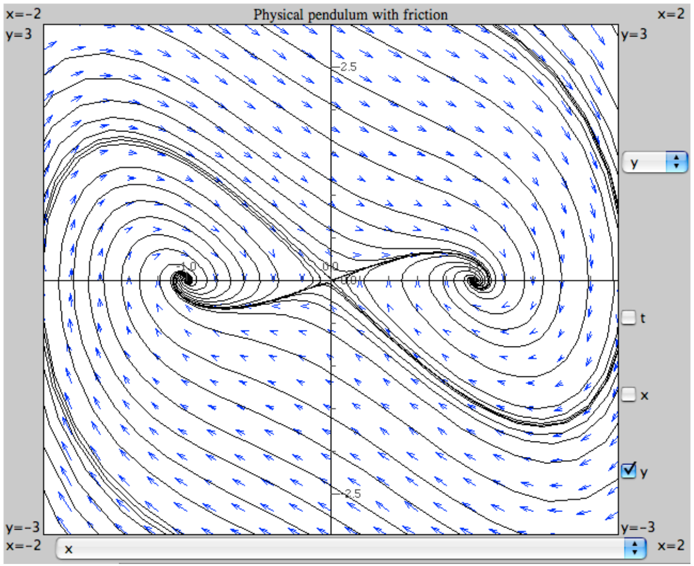

FIGURE 1.1.5. Trajectories for the equation (1.1.9) for $\nu = 1$.

Poincaré-Hopf Example. One of the most fascinating phenomena in dynamical systems is when a system starts oscillating as a parameter changes. One of the most interesting examples is in chemical reactions, the Belousov-Zhabotinsky reaction reaction, which is beautifully described in Strogatz' book. Another example is when wind blows past power lines and they begin to sing as a parameter (in this case the wind speed) is increased. Other examples of oscillating systems abound in biology, from neurons to heartbeats and many of them involve the same fundamental phenomenon, call the ***Poincaré-Hopf bifurcation.*** We illustrate here with the following simple example of this phenomenon that can be analyzed "by hand". Consider this planar system

$$\begin{aligned} \dot{x} &= -y + x\left(\mu - x^2 - y^2\right) \\ \dot{y} &= x + y\left(\mu - x^2 - y^2\right) \end{aligned} \tag{1.1.10}$$

where μ is a parameter.

To analyze this system, first note that if we leave off the nonlinear terms, what remains is a small variant of the harmonic oscillator with energy

$$E_{\text{harmonic oscillator}}(x, y) = \frac{1}{2}\left(x^2 + y^2\right).$$

This, together with the fact that the phase portrait of the harmonic oscillator consist of points moving in circles (which are level sets of this energy), *motivates* the introduction of polar coordinates. Thus, introduce polar coordinates $(r\theta)$ in the usual way:

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Differentiating the relation $r^2 = x^2 + y^2$ and using the equations (1.1.10), we see that

$$\begin{aligned} r\dot{r} &= x\dot{x} + y\dot{y} \\ &= x\left(-y + x\left(\mu - x^2 - y^2\right)\right) + y\left(x + y\left(\mu - x^2 - y^2\right)\right) \\ &= r^2\left(\mu - r^2\right) \end{aligned}$$

Thus, as long as r is not zero,

$$\dot{r} = r\left(\mu - r^2\right).$$

Similarly, by differentiating $x = r\cos\theta$ and making use of the equations for $\dot{x}$ and $\dot{r}$, we find that

$$\dot{\theta} = 1$$

This system is now easy to consider as the equations decouple. In fact, by considering the sign of $\dot{r}$, we see that the origin is stable for $\mu < 0$ and for $\mu > 0$ the origin is unstable. Note that for $\mu > 0$, there is a fixed point in the r-dynamics, namely at $r = \sqrt{\mu}$. This corresponds to a periodic orbit in the (x, y)-plane. Note that as μ crosses from negative to positive, a periodic orbit is "born" out of the origin and its amplitude grows as $\sqrt{\mu}$ as μ increases.

A movie of this basic phenomenon for this example is available at `https://www.cds.caltech.edu/help/cms.php?op=wiki&wiki_op=view&id=162`.

Bifurcation to Oscillation. As the preceding example illustrates, the Poincaré-Hopf bifurcation is a general dynamic bifurcation in which, roughly speaking, a periodic orbit born when an equilibrium looses stability. As we have mentioned, an everyday example of a Hopf bifurcation we all encounter is ***flutter***. For example, when venetian blinds flutter in the wind or a television antenna "sings" in the wind, there is probably a Hopf bifurcation occurring. The general idea is shown in Figure 1.1.6.

A related example that is physically easy to understand is that of a pipe conveying a fluid.[2] One considers a straight vertical rubber tube conveying fluid. The lower end is a nozzle from which the fluid escapes. This is called a ***follower-load*** problem since the water exerts a force on the free end of the tube which follows the movement of the tube. Those with any experience in a garden will not be surprised by the fact that the hose will begin to oscillate if the water velocity is high enough.

[2]This system has been analyzed by a large number of authors; see, for instance Bou-Rabee, Romero, and Salinger [2002], de Langre, Païdoussis, Doaré, and Modarres-Sadeghi [2007] and references therein.

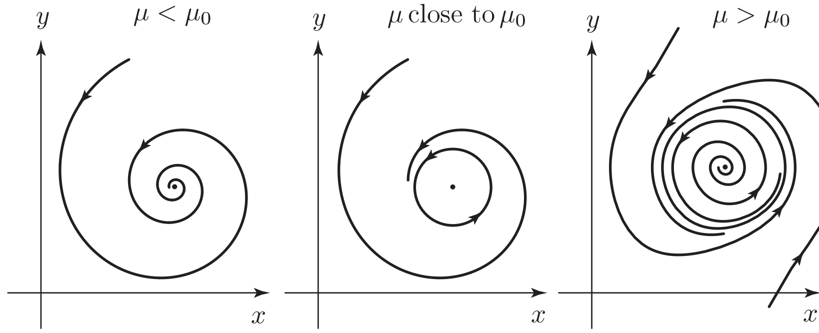

FIGURE 1.1.6. A periodic orbit appears for μ close to μ_0.

Ball in the Hoop: An Example of a Bifurcation of Equilibria. Another example that illustrates many of the concepts of dynamical systems is the ball in a rotating hoop. Refer to figure 1.1.7.

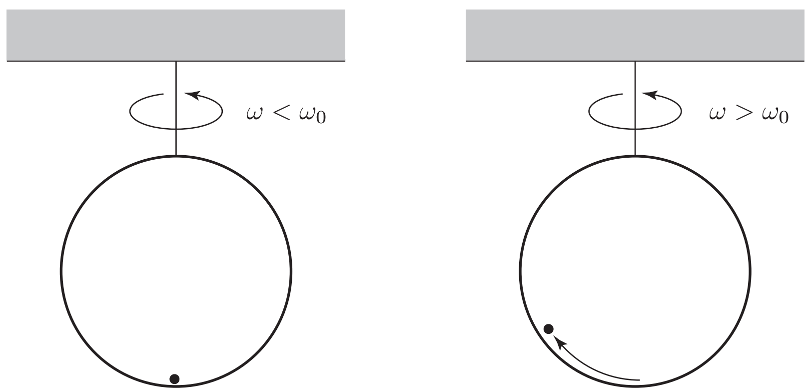

FIGURE 1.1.7. The ball in the hoop system; the equilibrium is stable for $\omega < \omega_c$ and unstable for $\omega > \omega_c$.

This system consists of a rigid hoop that hangs vertically with a small ball resting in the bottom of the hoop. The hoop rotates with frequency ω about a vertical axis through its center (Figure 1.1.9(left)).

Now we consider varying ω, keeping the other parameters fixed. For small values of ω, the ball stays in equilibrium at the bottom of the hoop and that position is stable. Accept this in an intuitive sense for the moment; we will have to define this concept carefully. However, when ω reaches a particular critical value ω_0 (which we will determine below), this point becomes unstable and the ball rolls up the side of the hoop to a new equilibrium position $x(\omega)$, which is stable. The ball may roll to the left or to the right, depending, perhaps upon the side of the vertical axis to which it was initially leaning. (Figure1.1.7(right)). The position at the bottom of the hoop is still a fixed point, but it has become *unstable*. The solutions to the initial value problem governing the ball's motion are unique for all values of ω. For $\omega > \omega_0$, we cannot predict which way the ball will roll.

Using principles of mechanics which we shall discuss in detail a bit later, one can show that the equations for this system are given by

$$mR^2\ddot{\theta} = mR^2\omega^2 \sin\theta\cos\theta - mgR\sin\theta - \nu R\dot{\theta}, \tag{1.1.11}$$

where R is the radius of the hoop, θ is the angle from the bottom vertical, m is the mass of the ball, g is the acceleration due to gravity, and ν is a coefficient of friction. To analyze this system, we use the same sort of *phase plane analysis* as was discussed above. One way to find an energy equation is to multiply each side of equation (1.1.11) by $\dot{\theta}$ and to recognize some of the terms as time derivatives; we get

$$\frac{d}{dt}\frac{1}{2}mR^2\dot{\theta}^2 = \frac{d}{dt}\frac{1}{2}mR^2\omega^2\sin^2\theta + \frac{d}{dt}mgR\cos\theta - \nu R\dot{\theta}^2$$

Thus, we get

$$\frac{d}{dt}E = -\nu R\dot{\theta}^2 \leq 0$$

where

$$E(\theta,\dot{\theta}) = \frac{1}{2}mR^2\dot{\theta}^2 - mR^2\omega^2\sin^2\theta - mgR\cos\theta$$

Thus, as with the preceding examples, note that for $\nu = 0$, trajectories must follow level curves of E and that for $\nu > 0$, it is plausible that most trajectories spiral into the minima of E. Will be evident when we write the equations in first order form, the critical points of E are exactly the equilibria. This is consistent with the phase portraits drawn below.

To analyze the system further, we write the equation as a first order system:

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= \frac{g}{R}(\alpha\cos x - 1)\sin x - \beta y\end{aligned}$$

where $\alpha = R\omega^2/g$ and $\beta = \nu/m$. This system of equations produces for each initial point in the xy-plane, a unique trajectory. That is, given a point (x_0, y_0) there is a unique solution $(x(t), y(t))$ of the equation that equals (x_0, y_0) at $t = 0$. This statement is proved by using general existence and uniqueness theory that we shall discuss later.

The equilibrium points of this system are obtained by setting the right hand side to zero. Thus, the equilibria occur in the xy-plane when $y = \dot{\theta} = 0$ and when x satisfies

$$\frac{g}{R}(\alpha\cos x - 1)\sin x = 0.$$

The solutions are when one of the factors vanishes. That is, when $x = 0, \pi$ (or, since $x = \theta$ is periodic, points of the form $\pi + 2n\pi$, where n is a (positive or negative) integer. These equilibria correspond to the particle being at the bottom or at the top of the hoop.

Other equilibria occur when $\cos x = 1/\alpha = g/R\omega^2$. For there to be a real solution, we must have $g/R\omega^2 \le 1$; that is, when $\omega \ge \sqrt{g/R}$. Referring to Figure 1.1.8, we see that for $\omega < \sqrt{g/R}$ there are no solutions, for $\omega = \sqrt{g/R}$ there is one, and for $\omega > \sqrt{g/R}$, there are two solutions (ignoring solutions that differ from these ones by a multiple of 2π). We say that there is a ***bifurcation of equilibria*** as ω passes through the critical value $w_c = \sqrt{g/R}$.

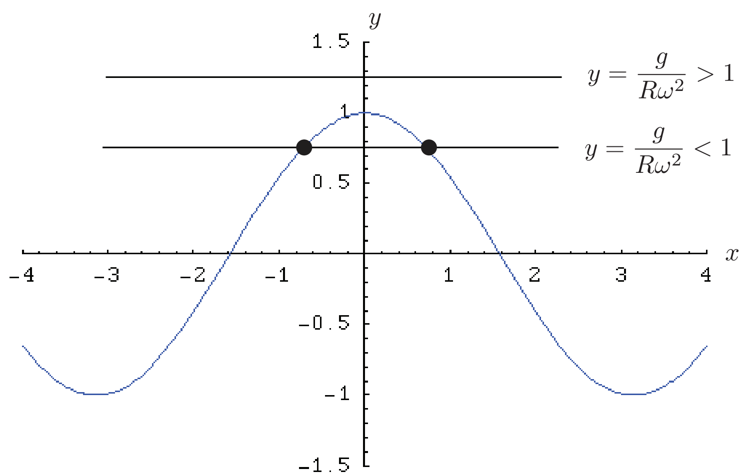

FIGURE 1.1.8. The equation $\cos x = g/R\omega^2$ has no solutions if $\omega < \sqrt{g/R}$ and two solutions if $\omega > \sqrt{g/R}$.

When we draw the phase portraits in the $(\theta, \dot{\theta})$-plane, we get figures like those shown in Figure 1.1.9.

Notice that the original stable fixed point has become *unstable* and has split into two *stable* fixed points as ω passes through its critical value $\omega_c = \sqrt{g/R}$. This is one of the simplest situations in which *symmetric problems can have non-symmetric solutions* and in which *there can be multiple stable equilibria*, so there is non-uniqueness of equilibria (even though the solution of the initial value problem is unique).

The Notion of Bifurcation. This example as well as the Poincaré-Hopf example show that in some systems the phase portrait changes as certain parameters are changed. Changes in the qualitative nature of phase portraits as parameters are varied are called ***bifurcations***. Consequently,

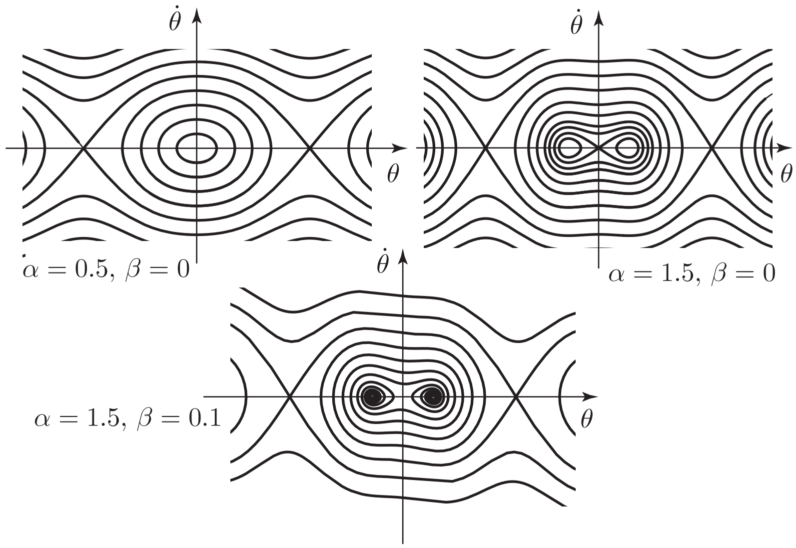

FIGURE 1.1.9. The phase portrait for the ball in the hoop before and after the onset of instability for the case $g/R = 1$.

the corresponding parameters are often called ***bifurcation parameters***. These changes can be simple such as the formation of new fixed points—these are called ***static bifurcations*** to ***dynamic bifurcations*** such as the formation of ***periodic orbits***, that is, an orbit $x(t)$ with the property that $x(t+T) = x(t)$ for some T and all t, or more complex dynamical structures.

The Role of Symmetry. We shall discuss the role of symmetry from time to time. It plays a very important role in many bifurcation problems. Already in the ball in the hoop example, one sees that symmetry plays an important role. The "perfect" system discussed so far has a symmetry in the sense that one can reflect the system in the vertical axis of the hoop and one gets an equivalent system; we say that the system has a $\mathbb{Z}_2$-symmetry in this case. This symmetry is manifested in the obvious symmetry in the phase portraits.

We say that a fixed point has symmetry when it is fixed under the action of the symmetry. The straight down solution of the ball is thus symmetric, but the solutions that are to each side are not symmetric — we say that these solutions have undergone ***solution symmetry breaking***. This simple example already shows the important point that *symmetric systems need not have symmetric solutions!*. In fact, the way solutions loose symmetry as bifurcations occur is a fundamental and important idea.

Solution symmetry breaking is distinct from the notion of ***system symmetry breaking*** in which the whole system looses its symmetry. If this

$\mathbb{Z}_2$ symmetry is broken, by setting the rotation axis a little off center, for example, then one side gets preferred, as in Figure 1.1.10.

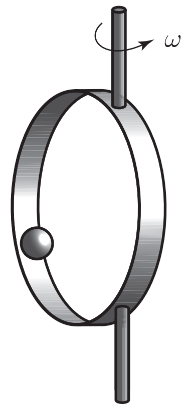

FIGURE 1.1.10. A ball in an off-center rotating hoop.

The evolution of the phase portrait for $\nu = 0$ is shown in Figure 1.1.11.

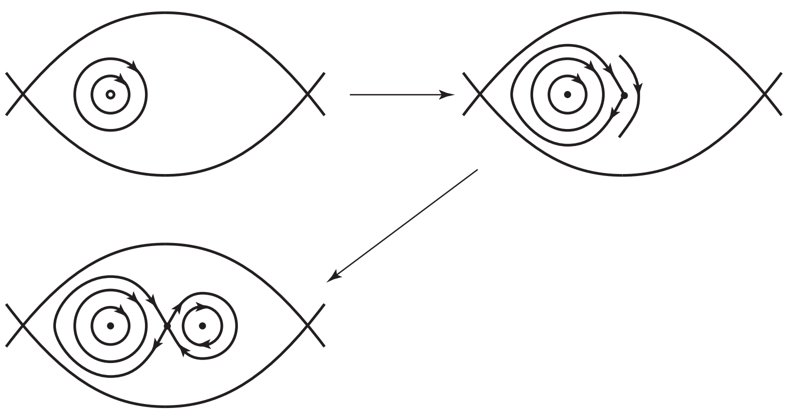

FIGURE 1.1.11. The phase portraits for the ball in the off-centered hoop as the angular velocity increases.

Modelling. We shall draw many of our models from mechanics, which is why we include some basic mechanics from a dynamical systems perspective in this book. While the modelling process is a serious issue even in mechanics (such as how should one truncate a continuous system by a discrete one), in other areas such as biological systems, the situation is even more serious–then the modelling phase of dynamical systems has to be dealt with in a significant way.

Chaos. This example is also rich in many other ways. The reader has undoubtedly heard about the concept of chaos. Dynamical systems provides a framework in which such notions can be understood and simple examples like this one provide examples of mechanical systems with chaotic solutions. As we shall see later, if one modulates the frequency periodically by, say writing $\omega = \omega^0 + \epsilon \sin \Omega t$, then the above equations can have very complex solutions; this kind of complexity is the origin of the notion of chaos, an idea going back to Poincaré about 1890. The delicacy of this concept is one of the reasons one needs a firm mathematical foundation in which to discuss it. Some history of the ideas of chaos in the context of celestial mechanics and the fundamental work of Poincaré may be found in the book of Diacu and Holmes [1996].