Optimal Motion of an Articulated Body in a Perfect Fluid

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Abstract—An articulated body can propel and steer itself in a perfect fluid by changing its shape only. Our strategy for motion planning for the submerged body is based on finding the optimal shape changes that produce a desired net locomotion; that is, motion planning is formulated as a nonlinear optimization problem.

I. INTRODUCTION

We study the locomotion of an articulated body that can undergo shape changes and is immersed in a perfect fluid, as shown in Figure 1. Starting from rest, the articulated body can propel and steer itself in an irrotational fluid by changing its shape. The goal of this work is to investigate the optimal shape changes that produce a desired net locomotion.

Early efforts in developing mathematically-sound models of swimming can be attributed to the work of Gray, Chilress, Lighthill, Taylor and Wu; see [3], [10], [21] and [23]. Interest re-emerged over the past few years to understand the mechanics of fish swimming and thereby enable novel engineering applications such as the design of biologically-inspired vehicles that move and steer by changes of shape rather than by direct propulsion. For recent experimental studies of the shape kinematics of biological fish and its interaction with the surrounding fluid, see, for example, [9], [14], and [22]. See also [7] and [18] for their fundamental work on the mathematical formulation of aquatic locomotion using tools from geometric mechanics.

a) Swimming in Potential Flow: In [6], we modeled the fish as an articulated body made of three rigid links, and formulated the equations governing its motion in potential flow using tools from geometric mechanics; namely, we established the trajectories of the net rigid motion $g(t)$ as geometric phases, or holonomy, over closed curves in the shape space (the space of allowable relative rotations $\theta_1(t)$ and $\theta_2(t)$ between the links). We showed under these idealized conditions, i.e., in the absence of a vortex shedding mechanism, that the fish can propel and steer itself by changing its shape only. This result is important because, contrary to some common beliefs, it demonstrates that the forces and moments applied on the fish body by shed vortices are not solely responsible for the net locomotion. The net locomotion in potential flow occurs due to the transfer of momentum between the solid and the fluid: starting from rest, the articulated body changes its shape by applying internal torques at its joints. This shape actuation sets the surrounding fluid into motion, and the coupling between the shape dynamics and the surrounding fluid causes a net locomotion of the solid.

b) Motion Planning: There is a need to establish a rigorous foundation for the selection of patterns of shape changes that produce a desired net locomotion. Motion planning and control of self-propelled underwater robotic vehicles have been the subject of several recent studies, as in [17], [2], [13] and [16]; see also references therein. These studies however address local motion planning and make the restrictive and unrealistic assumption of small shape changes, hence the need for alternative global methods. In the present work, the assumption of small shape changes is not required, and the problem of motion planning is formulated as an optimization problem that maximizes/minimizes certain cost functions. That is, we ask the question: “what are the optimal shape changes that achieve a desired net locomotion?” (see also [12] for analogous ideas for the falling cat problem). This approach is capable of capturing the complex behavior of biological fish. Indeed, it is well known that biological fish change their behavior depending on the conditions in which they swim. When swimming peacefully, their concern is to minimize their energy cost but, if attacked by a predator, their energy concerns become secondary as they speed up to escape.

c) Discrete Optimal Control: We use a new approach proposed by [5] for the optimal control of mechanical systems. The main idea is to discretize the Lagrange-d’Alembert principle directly instead of the associated forced Euler-Lagrange equations. The resulting discrete equations then serve as constraints for the optimization of the given cost functional. This approach respects, by construction, the conserved quantities in the mechanical system, and is particularly useful for the fish problem because the total momentum is conserved. The setting of the discrete optimal control method within the general optimization framework is discussed in [5]. For a different approach to motion planning of the three-link fish, see [11].

d) Organization of the Paper: First, §II describes the general setting of the problem. We write the kinetic energy of the solid-fluid system in §III, and we formulate the dynamics using the Lagrange-D’Alembert variational principle in §IV.
Motion planning is established as a problem in discrete optimal control in §V. The implementation and numerical results are presented in §VI.

II. PROBLEM DESCRIPTION

e) Setting: Consider an articulated body formed of rigid links and immersed in an infinitely large volume of an incompressible fluid which is at rest at infinity, see Figure 1. Assume that the fluid particles may slip along the boundaries of the solid but do not allow cavities to form in the fluid nor at the interface. It is well-known in fluid mechanics that, under these conditions, the equations governing the motion of the solid in an irrotational fluid can be written without explicitly incorporating the ambient fluid (see, for example, [6] for details). That is, the configuration space of the solid-fluid system can be identified with that of the submerged solid only.

\[ \begin{align*}
   & \text{Rigid Motion} \\
   & (\theta_1, \theta_2) \\
   & \text{Reduced Trajectory} \\
   & \text{Net Locomotion} \\
   & (\beta, x, y)
\end{align*} \]

This geometric picture is very convenient to address the locomotion problem; namely, the net rigid motion \((\beta, x, y)\) achieved as a result of \((\theta_1, \theta_2)\) tracing a closed trajectory in the shape space. In this work, we investigate the problem of motion planning or finding the most efficient trajectories in the shape space that produce a desired net locomotion.

i) Velocities of the Solid Links: Let \(\Omega_i\) and \(v_i\) be, respectively, the angular and translational velocities of \(B_i\) expressed relative to \(B_i\)-fixed frame. For conciseness, we introduce \(\xi_i\) such that \(\xi_i^T = (\Omega_i, v_i)^T\). For example, one has \(\xi_0^T = (\Omega_0, v_0)^T\) where

\[ \Omega_0 = \beta, \quad v_0 = \left( \begin{array}{c} \dot{x} \cos \beta + \dot{y} \sin \beta \\ -\dot{x} \sin \beta + \dot{y} \cos \beta \end{array} \right) \]  

This notation is consistent with the group theoretic notation. For a brief review of the planar rigid motion group SE(2) and its Lie algebra se(2), see, e.g., [7, Chapter 2].

\[ j) \text{The Dynamics:} \] The equations governing the dynamics of the submerged three-link fish (summarized in (10) and (11)) are derived in [6] and recalled in §III and §IV. Equations (10) and (11) can be written explicitly in terms of the configuration variables \((\theta_1, \theta_2, \beta, x, y)\) and the forcing torques (also called actuators) \(\tau_1\) and \(\tau_2\) to give a system of five second-order differential equations. The resulting second-order system can be transformed into a system of ten first-order equations of the form \(\dot{z} = f(z, u)\) in terms of the state variables \(z = (\theta_1, \theta_2, \beta, x, y, \theta_1, \theta_2, \beta, \dot{x}, \dot{y})\) and the control variables \(w = (0, 0, 0, 0, 0, \tau_1, \tau_2, 0, 0, 0)\). That is, Equations (10) and (11) could be written in the well-known state-space representation \(\dot{z} = f(z, u)\) and \(g = (\beta, x, y)\) (the net locomotion \(g\) is the output). However, the discrete
optimal control approach does not require such rewriting of the equations.

III. KINETIC ENERGY

The kinetic energy $T$ of the solid-fluid system can be written as the sum of the energies of the solid links $T_B$, and the energy of the fluid $T_F$; namely,

$$T = \sum_{i=0}^{2} T_B_i + T_F . \quad (2)$$

k) Kinetic Energy of the Solid Ellipses: The kinetic energy $T_B$ can be written in the form

$$T_B_i = \frac{1}{2} \xi_i^T \Pi_i^f \xi_i , \quad i = 0, 1, 2 . \quad (3)$$

Here, $\Pi_i^f$ is a $3 \times 3$ diagonal matrix with diagonal entries $(I_i, m_i, m_i)$ where $I_i = m_i(a_i^2 + b_i^2)/4$ is the moment of inertia of $B_i$, and $m_i = \rho_s \pi a_i b_i$ is its mass. It is important to recall that the links are neutrally buoyant, that is, $\rho_s = \rho_f$ and that the body-fixed frames are placed at the respective mass centers.

l) Kinetic Energy of the Fluid: The kinetic energy of the fluid $T_F$ is given in spatial representation by

$$T_F = \frac{1}{2} \int_{\mathbb{R}^2} \rho_f |u|^2 \, da , \quad (4)$$

where $u$ is the spatial velocity field of the fluid and $da$ is the standard area element on $\mathbb{R}^2$. For potential flow, the fluid velocity can be written as the gradient of a potential function $u = \nabla \phi$, where the potential $\phi$ is the solution to Laplace’s equation $\Delta \phi = 0$ subject to the boundary conditions

$$\left\{ \begin{array}{ll}
\nabla \phi \cdot n_i &= (v_i + \Omega_i \times X_i) \cdot n_i & \text{on } \partial B_i \\
\n\nabla \phi &= 0 & \text{at } \infty
\end{array} \right. \quad (5)$$

Here, $n_i$ is a unit normal to $\partial B_i$, and $X_i$ is the position vector of a point on $\partial B_i$ relative to the respective mass center. Under these conditions, one can show following a standard procedure (see, for example, [6] and references therein) that $T_F$ of (4) can be rewritten as

$$T_F = \frac{1}{2} \xi_i^T \Pi_i^f \xi_j , \quad i, j = 0, 1, 2 . \quad (6)$$

The $3 \times 3$ added inertia matrices $\Pi_i^f$ depend on the geometry and relative configurations $(\theta_1, \theta_2)$ of the submerged ellipses and are of the form

$$\Pi_i^f = \left( J_{ij} \right) \left[ \begin{array}{c}
\frac{\partial T_{1}}{\partial \xi_{1}} \\
\frac{\partial T_{2}}{\partial \xi_{1}} \\
\frac{\partial T_{3}}{\partial \xi_{1}}
\end{array} \right] = \frac{1}{2} \xi_i^T \Pi_i^f \xi_j , \quad i, j = 0, 1, 2 . \quad (6)$$

III. Kinetic Energy of the Solid-Fluid System: By virtue of (3) and (5), the kinetic energy $T$ in (2) can be rewritten as

$$T = \frac{1}{2} \xi_i^T \Pi_i^f \xi_j , \quad i, j = 0, 1, 2 . \quad (7)$$

where $\Pi_{ij} = \Pi_i^f + \Pi_j^f$ and $\Pi_{ij} = \Pi_{ji}^f$ for $i \neq j$. Note that, although there is an analogy between $\Pi_i^f$ and $\Pi_{ij}^f$, they are fundamentally distinct. For example, in translation, unlike the body’s actual mass, the added mass depends on the direction of the motion.

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IV. THE LAGRANGE-D’ALEMBERT PRINCIPLE

For the neutrally buoyant articulated body, the Lagrangian function $L$ is equal to the kinetic energy $T$ given in (7), and the Lagrange-d’Alembert variational principle requires that

$$\delta \int_{t_0}^{t_1} L \, dt + \int_{t_0}^{t_1} \left( \tau_1 \cdot \delta \theta_1 + \tau_2 \cdot \delta \theta_2 \right) = 0 , \quad (8)$$

for all variations $(\delta \theta_1, \delta \theta_2, \delta x, \delta y)$ that vanish at the end points $t_0$ and $t_1$. The associated equations of motion can be written concisely as in (10-11). To see this, we first introduce a momentum-like quantity $\mu_i$ associated to each $B_i$ and expressed relative to $B_i$-fixed frame

$$\mu_i = \sum_{j=1}^{3} \Pi_{ij} \xi_j . \quad (9)$$
We also define the momentum \( \mu_s \) of the solid-fluid system relative to the \( B_0 \)-fixed frame as follows
\[
\mu_s = \mu_0 + \tilde{\mu}_1 + \tilde{\mu}_2,
\]
where \( \tilde{\mu}_\alpha, \alpha = 1, 2 \) correspond to the transformed \( \mu_\alpha \) from their respective body-fixed frames to the \( B_0 \)-fixed frame.\(^1\) We then rewrite \( \mu_s \) as
\[
\mu_s (t) = (\Pi_s, P_\alpha)^T .
\]
This suggests that the problem is controllable or, at least, controllable in some finite regions of the configuration space. For a rigorous proof of controllability, one needs to appeal to the Ambrose-Singer theorem (11) which gives sufficient conditions for every net motion to be realized (this theorem is a restatement of a theorem of [4], now familiar to people in control theory). Such undertaken, although very important, is beyond the scope of the present paper.

p) The Optimization Problem: The shape variables \((\theta_1, \theta_2)\) are controlled by the input torques \((\tau_1, \tau_2)\). Therefore, we consider \((\tau_1, \tau_2)\) to be the control variables and \((\theta_1, \theta_2, \beta, x, y)\) to be the state variables which we denote by \( \eta \) for brevity. The optimization problem can then be stated as follows. Given the boundary conditions \( q(t_0) = q_0 \), and \( q(t_1) = q_1 \), find \((\tau_1, \tau_2)\) that minimize the cost function
\[
\int_{t_0}^{t_1} C(\eta, \dot{\eta}, \tau_1, \tau_2) \, dt
\]
such that
\[
\delta \int_{t_0}^{t_1} L(\eta, \dot{\eta}) \, dt + \int_{t_0}^{t_1} (\tau_1 \cdot \delta \theta_1 + \tau_2 \cdot \delta \theta_2) \, dt + p_0 \cdot \delta q_0 - p_1 \cdot \delta q_1 = 0 ,
\]
for all arbitrary variations \( \delta q \). That is, the Lagrange-d’Alembert principle (8) is restated in (13) without the apriori assumption that the variations vanish at the end points \( t_0 \) and \( t_1 \). Rather, this condition is imposed using the boundary constraints
\[
\delta q_0 = q(t_0) - q_0 = 0 , \quad \delta q_1 = q(t_1) - q_1 = 0 ,
\]
and their associated Lagrange multipliers
\[
p_0 = \frac{\partial L}{\partial \dot{q}}|_{t_0} , \quad p_1 = \frac{\partial L}{\partial \dot{q}}|_{t_1} .
\]

q) The Discrete Optimization Problem: Traditional methods in optimal control such as the multiple shooting (see, e.g., [19]) or the collocation methods (see [20]) rely on a direct integration or fulfillment of (10)-(11) at certain grid points. The corresponding solutions do not respect, in general, the conservation laws that the equations of motion satisfy, such as the conservation of total momentum in the present problem. To circumvent this difficulty, we use a novel method devised by [5] where the main idea is to discretize the cost function (12) and the variational principle (13) directly using global discretization of the states and the controls. To this end, a path \( q(t) \), where \( t \in [t_0 = 0, t_1 = 1] \), is replaced by a discrete path \( q_d : \{0, h, 2h, \ldots, Nh = 1\} \), \( N \in \mathbb{N} \). Here, \( q_d(nh) := q_n \) is viewed as an approximation to \( q(t_n = nh) \), \( n \in \mathbb{N} \) and \( n \leq N \). Similarly, the continuous torques \( \tau_\alpha, \alpha = 1, 2 \) are approximated by discrete torques \( \tau_\alpha d \) such that \( \tau_\alpha n = \tau_\alpha d(nh) \).

The cost function (12) is approximated on each time interval \([nh, (n+1)h]\) by
\[
C_d(q_n, q_{n+1}, \tau_\alpha n, \tau_\alpha n+1) \approx \int_{nh}^{(n+1)h} C(q, \dot{q}, \tau_\alpha) \, dt ,
\]
which yields the discrete cost function
\[
J_d(q_d, \tau_\alpha d) = \sum_{n=0}^{N-1} C_d(q_n, q_{n+1}, \tau_\alpha n, \tau_\alpha n+1) . \quad (15)
\]

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\(^1\)One cannot assert that \( \mu_s \) is the total momentum of the system, which in this problem is indeterminate because the fluid has an infinite domain. Traditionally, \( \mu_s \) was known as the “impulse” (see [8, Chapter 6]).
The action integral (13) is approximated on each time interval \([nh, (n+1)h]\) by a discrete Lagrangian

\[
L_d(q_n, q_{n+1}) \approx \int_{nh}^{(n+1)h} L(q, \dot{q}) \, dt.
\]

We also approximate

\[
\int_{t_n}^{t_{n+1}} \tau_\alpha \cdot \delta \theta_\alpha \cong \tau_{\alpha n} \cdot \delta \theta_{\alpha n} + \tau_{\alpha n+1}^+ \cdot \delta \theta_{\alpha n+1}, \quad \alpha = 1, 2,
\]

where \(\tau_{\alpha n}\) and \(\tau_{\alpha n+1}^+\) are called left and right discrete torques, respectively. The discrete version of (13) requires one to find paths \(\{q_n\}_{n=0}^N\) such that for all variations \(\{\delta q_n\}_{n=0}^N\), one has

\[
\delta \sum_{n=0}^{N-1} L_d(q_n, q_{n+1}) + \sum_{n=0}^{N-1} \sum_{\alpha=1}^2 \tau_{\alpha n} \cdot \delta \theta_{\alpha n} + \tau_{\alpha n+1}^+ \cdot \delta \theta_{\alpha n+1} + p_0 \cdot \delta q_0 + p_1 \cdot \delta q_1 = 0.
\]

The discrete variational principle (16) yields the following equality constraints

\[
D_2 L_d(q_{n-1}, q_n) + D_1 L_d(q_n, q_{n+1}) + \tau_{\alpha n+1}^+ - \tau_{\alpha n} = 0, \\
p_0 + D_1 L_d(q_0, q_1) + \tau_0 = 0, \\
- p_1 + D_2 L_d(q_{N-1}, q_N) + \tau_{\alpha N}^+ = 0,
\]

where \(D_1\) and \(D_2\) denote the derivatives with respect to the first and second argument, respectively, and 

\[
q_n = (\theta_{1n}, \theta_{2n}, \beta_n, x_n, y_n), \quad \tau_n = (\tau_{1n}, \tau_{2n}, 0, 0, 0), \quad n = 1, \ldots, N.
\]

r) Summary and Remarks: The discrete nonlinear optimization problem can be stated as follows: find the discrete torques \(\{\tau_{\alpha n}\}_{n=0}^N\) and paths \(\{q_n\}_{n=0}^N\) that minimize the discrete cost function (15) subject to the nonlinear constraints (17) and the boundary conditions (14). We note the following:

1) Additional constraints such as actuators limits can be easily handled by this method: one would add to the list of constraint equations (14) and (17) the inequality constraints corresponding to the limits on \(\tau_n\).

2) The method presented here is suitable for optimization over a fixed time interval \([t_0, t_1]\). Work on time optimal control is under development and would be interesting for the fish problem.

VI. Numerical Results

For simplicity, assume that the three ellipses are identical (see Figure 2) and let \(a = 10\), \(b = 1\) \(c = 2\) and \(\rho_f = 1/\pi\). Further, assume that the added inertias associated with a given ellipse are not affected by the presence of the other ellipses. \(^2\) This assumption is capable of capturing qualitatively the correct dynamics (as demonstrated in [6]). To this end, the inertia matrices are given by \(I_{11} = I_{22} = I_{33} = I\) and \(I_{ij} = 0\), for \(i \neq j\). In addition, \(I\) is a diagonal matrix with non-zero diagonal entries

\[
j = I + I_f, \quad m_1 = m + m_1^f, \quad m_2 = m + m_2^f,
\]

where the body moment of inertia \(I\) and mass \(m\) are given in §III, while the added inertias \(I_f, m_1^f\) and \(m_2^f\) due to the fluid effects are given by (see, for example, [15, Chapter 4])

\[
I_f = \frac{1}{8} \rho_f \pi (a^2 - b^2)^2, \quad m_1^f = \rho_f \pi b^2, \quad m_2^f = \rho_f \pi a^2.
\]

s) Direct Numerical Integration: Starting from rest, we prescribe periodic shape changes \((\theta_1(t), \theta_2(t))\) and compute the resulting \((\beta(t), x(t), y(t))\) by integrating (10) using a standard 4\textsuperscript{th} order Runge-Kutta integration scheme with constant time steps.

Figure 4 shows a net forward motion of the three-link fish in the \((e_1, e_2)\)-plane due to shape changes \(\theta_1 = -\cos(t)\) and \(\theta_2 = \sin(t)\). In Figure 5, the three-link fish is shown to turn counterclockwise in the \((e_1, e_2)\) plane due to shape changes \(\theta_1 = 1 - \cos(t)\) and \(\theta_2 = -1 + \sin(t)\). Snapshots of the turning maneuvers over one period \(T = 2\pi\) of shape changes are shown in Figure 6.

1) Implementation of the Discrete Optimization Scheme:

To approximate the relevant integrals in (12) and (13), we use the midpoint rule; that is, in (15), we set

\[
C_d(q_n, q_{n+1}, \tau_n, \tau_{n+1}) = h C(q_n + q_{n+1}/2, q_{n+1} - q_n, \tau_n + \tau_{n+1}/2, \tau_{n+1} + \tau_{n+1}/2), \quad \alpha = 1, 2,
\]

and, in (16), we use

\[
L_d(q_n, q_{n+1}) = h L(q_n + q_{n+1}/2, q_{n+1} - q_n)/h.
\]

\(^2\)Note that this assumption is accurate when the ellipses \(B_i\) are placed a large distance apart.
where the nonlinear constraints correspond to the equations itself by changing its shape, and we investigate the optimal show that the submerged body is able to propel and steer shape changes and is submerged in a perfect fluid. We periods). This observation leads to the interesting question obtain a repeated pattern of shape changes when optimizing over several time periods (as shown in Figure 8 over two small perturbations on these portions. Likewise, the initial Figure 4 over the time intervals [0, T]. One does not repeat a obtained pattern of shape changes when optimizing over several time periods (as shown in Figure 8 over two periods). This observation leads to the interesting question whether to optimize over one period of shape changes or over the whole time interval of the desired locomotion.

VII. CLOSING REMARKS

This paper considers an articulated body that can undergo shape changes and is submerged in a perfect fluid. We show that the submerged body is able to propel and steer itself by changing its shape, and we investigate the optimal shape changes that produce a desired net locomotion. This problem is formulated as a constrained optimization problem where the nonlinear constraints correspond to the equations governing the motion of the solid-fluid system and are defined via the Lagrange-d’Alembert variational principle.

while

\[
\tau_{\alpha n}^- = \tau_{\alpha n}^+ = \frac{h}{4} (\tau_{\alpha n} + \tau_{\alpha n+1}) \quad \alpha = 1, 2 .
\]

We solve the discrete optimization problem of finding \( \{\tau_{\alpha n}\}_{n=0}^N \) and \( \{q_n\}_{n=0}^N \) that minimize (15) subject to the nonlinear constraints (17) and the boundary conditions (14) using a built-in Matlab function for sequential quadratic programming. This optimization method is only local: based on the choice of an initial guess, the optimal solution is determined by means of infinitesimal variations.

u) Optimization Results: In the examples presented in this section, the goal is to minimize the control effort, hence the cost function (12) is taken to be

\[
J(q, \tau_1, \tau_2) = \int_{t_0}^{t_1} (\tau_1^2 + \tau_2^2) \, dt .
\]

Figure 7 shows the locally optimal solution that produces a net forward motion from \((x_0, y_0)\) to \((x_1, y_1)\) in the \((e_1, e_2)\)-plane. Interestingly, the optimal shape changes trace a trajectory of non-regular shape because of the nonlinear nature of the problem. It is worth noting that in order to obtain the initial guesses in Figure 7, we take the trajectories of Figure 4 over the time intervals \([0, T]\), \(T = 2\pi\), and impose small perturbations on these portions. Likewise, the initial guess in Figure 8 corresponds to an arbitrary perturbation on the trajectories of Figure 5 over \([0, 2T]\). One does not obtain a repeated pattern of shape changes when optimizing over several time periods (as shown in Figure 8 over two periods). This observation leads to the interesting question whether to optimize over one period of shape changes or over the whole time interval of the desired locomotion.

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